'Jay Jagannath'

Analytical Solid Geometry

Or 3-Dimensional Co-ordinate Geometry

Or Geometry of $\mathbb{R}^3$

Any point in space can be expressed as an ordered 3-tuple of the form (x, y, z)

Consider 3 mutualey $\perp^r$ straight lines meeting at a point "O" called as origin. The 3 mutually $\perp^r$ lines are called axes of co-ordinate. They are X-axis, Y-axis and Z-axis.

These 3 axes divides the space into 8 parts. Each part is called an octant.

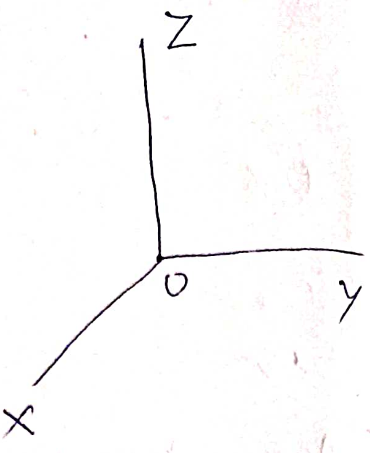
(X-axis may be +ve or -ve i-e there are two possibilities

Similarly there are two possibilities for Y-axis and Z-axis.

Hence the total no. of possibilities $= 2 \times 2 \times 2 = 8$ F.P.C)

The 8 octants can be shown as follows.

	x	y	z
✓	+	+	+
✓	+	+	−
✓	+	−	+
✓	−	+	+
✓	+	−	−
✓	−	+	−
✓	−	−	+
✓	−	−	−



The plane containing X and Y axes is called XY - plane

The plane containing Y and Z axes is called YZ - plane

The plane containing Z and X - axes is called ZX - plane.

These 3 planes are called co-ordinate planes or Fundamental planes. These 3 planes meet at origin.

Let $P(x, y, z)$ be any point in space.

The length of the $\perp$ from $P(x, y, z)$ on the XY - plane

= Z - co-ordinate of P

= z

The length of the ⊥ from $P(x, y, z)$ on the YZ-plane $= X$- Co-ordinate of P

$= x$

The length of the ⊥ from $P(x, y, z)$ on ZX plane

$= y$- Coordinate of P

$= y$

Any point on x-axis is of the form $(x, 0, 0)$

" " " y-axis " " " " $(0, y, 0)$

" " " Z-axis " " " " $(0, 0, z)$

Any pt on XY-plane " " " $(x, y, 0)$

i.e z co-ordinate $= 0$

" " " YZ-plane " " " $(0, y, z)$

i.e x- Co-ordinate $= 0$

" " " ZX-plane " " " $(x, 0, z)$

i.e y- coordinate $= 0$.

Any point on a plane // to XY-plane has Z-coordinate $=$ constant.

" " " " " to YZ-plane has x-co-ordinate $=$ constant

" " " " " to ZX-plane has y- coordinate $=$ constant.

Projection of a pt $P(x, y, z)$ on xy-plane is $(x, y, 0)$
" " " " " on yz-plane is $(0, y, z)$
" " " " " on zx-plane is $(x, 0, z)$

Image of pt $P(x, y, z)$ w.r.to xy-plane is $(x, y, -z)$

" " pt " " yz-plane is $(-x, y, z)$

" " " " " zx-plane is $(x, -y, z)$

Distance between two plane // to xy-plane
= Difference between z-co-ordinates

Distance between two planes // to yz-plane
= Difference between x-coordinates.

Distance between two planes // to zx-plane
= Difference between y-coordinates.

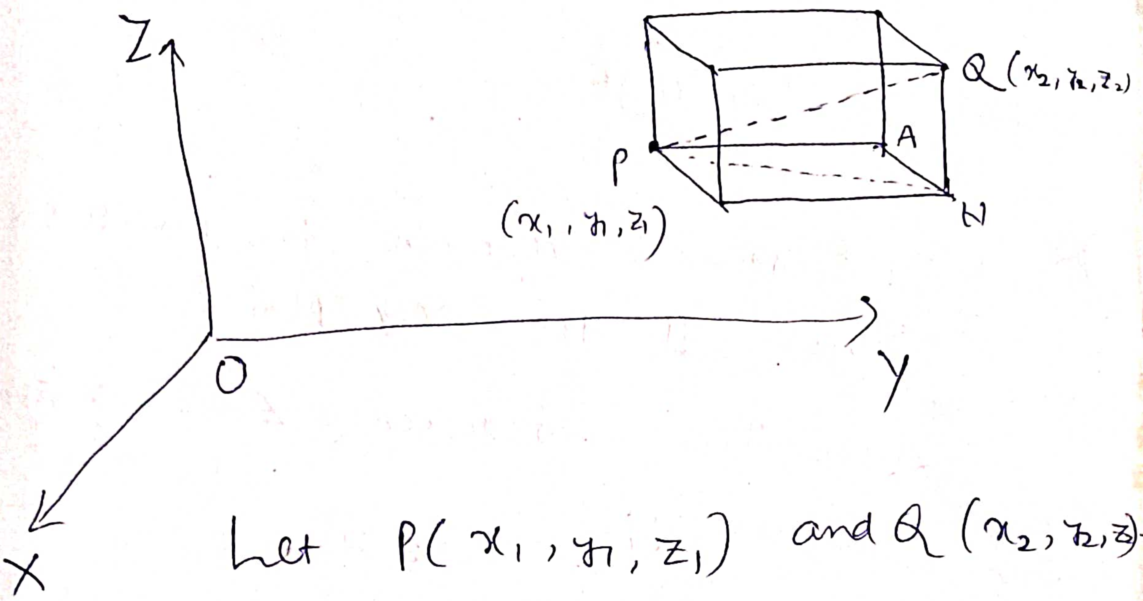
Sat-4

Distance formula

The distance between $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is

$$\sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$

Proof



Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two given points w.r.to the rectangular axes OX, OY and OZ

Through the points P and Q draw planes // to the co-ordinates so as to form a rectangular parallelopiped with one diagonal PQ.

PA = Distance between two parallel planes // to ZX-plane

= Difference between Y. co-ordinates

$= |y_2 - y_1|$

$$PA^2 = (y_2 - y_1)^2$$

AN = Distance between two parallel planes // to YZ-plane.

= Difference between x- co-ordinates

$= |x_2 - x_1|$

$AN^2 = (x_2 - x_1)^2$

NQ = Distance between two parallel planes $\parallel$ to XY-plane

= Difference between Z-co-ordinates

$= |z_2 - z_1|$

$(NQ)^2 = (z_2 - z_1)^2$

From $\triangle PAN$, we have

$PA^2 + AN^2 = PN^2$

From $\triangle PNQ$ we have

$PN^2 + NQ^2 = PQ^2$

$\Rightarrow PA^2 + AN^2 + NQ^2 = PQ^2$

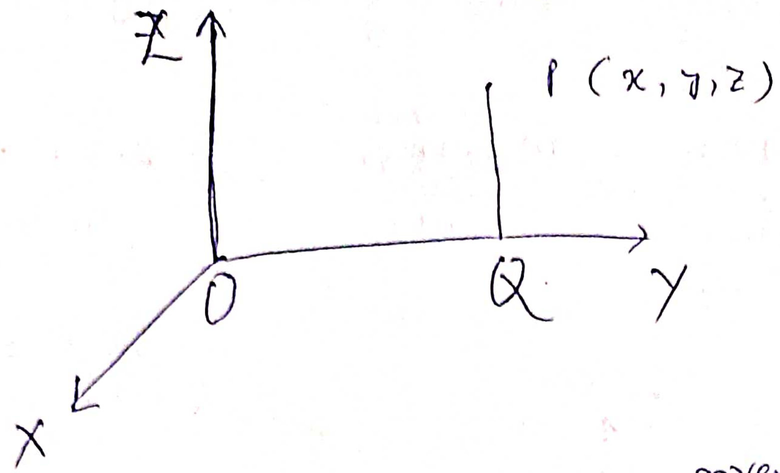
$\Rightarrow PQ^2 = (y_2 - y_1)^2 + (x_2 - x_1)^2 + (z_2 - z_1)^2$

$\Rightarrow |PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Proved

Q → Find the distance of a point $P(x, y, z)$ from X-axis, from Y-axis and from Z-axis.

Ans :



Let $P(x, y, z)$ be any given point.
Draw $\perp$ PQ on y-axis ; where
Q is foot of the $\perp$.
$\therefore$ Q is the point $(0, y, 0)$

$$\therefore |PQ| = \sqrt{(x-0)^2 + (y-y)^2 + (z-0)^2} \quad (\because \text{From y-axis})$$
$$= \sqrt{x^2 + z^2}$$

Similarly Distance of $P(x, y, z)$ from
x axis $= \sqrt{y^2 + z^2}$

~~Sim~~ Distance of $P(x, y, z)$ from
~~y~~ Z axis $= \sqrt{x^2 + y^2}$

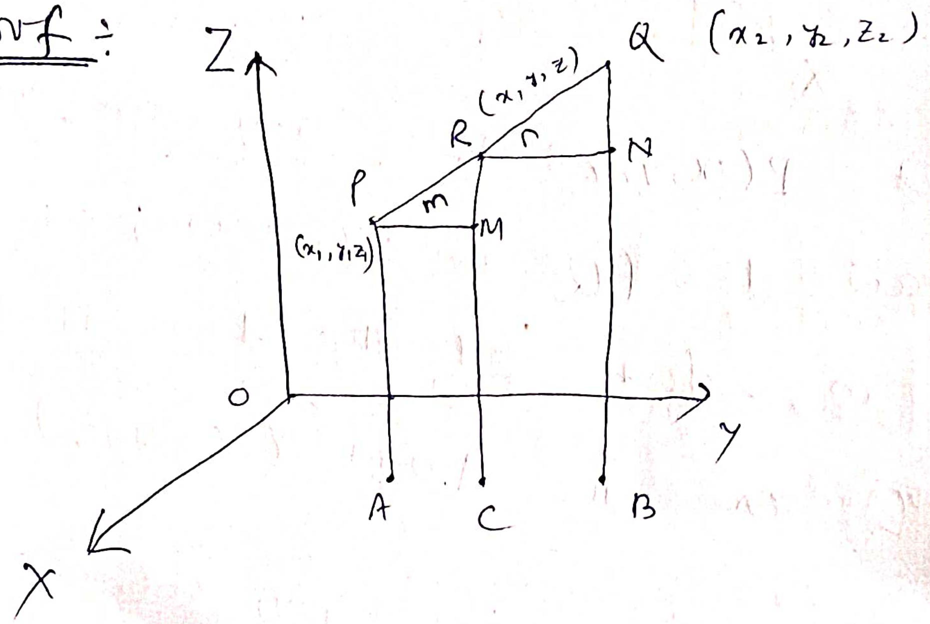
Division formula :

The point which divides the
line segment joining $P(x_1, y_1, z_1)$ and
$Q(x_2, y_2, z_2)$ in the ratio $m : n$

internally is given by

$$\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}, \frac{mz_2+nz_1}{m+n} \right)$$

Proof :-



Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two given points. Join PQ

Let $R(x, y, z)$ divides the line segment PQ ~~[illegible]~~ internally in the ratio $m:n$. Draw $\perp$ PA, QB and RC from ~~[illegible]~~ P, Q, R on XY-Plane respectively.

Draw $\perp$ PM from P on RC and $\perp$ RN from R on QB.

Now $\frac{PR}{RQ} = \frac{m}{n}$

Again the $\triangle$ PMR and $\triangle$ RNQ are similar ..

$$\therefore \frac{PR}{RQ} = \frac{RM}{QN}$$

Now $RM = RC - MC = RC - PA$

$= z - z_1$

$QN = QB - NB = QB - RC$

$= z_2 - z$

$$\therefore \frac{PR}{RQ} = \frac{RM}{QN} \Rightarrow \frac{m}{n} = \frac{z - z_1}{z_2 - z}$$

$$\Rightarrow m(z_2 - z) = n(z - z_1)$$

$$\Rightarrow mz_2 - mz = nz - nz_1$$

$$\Rightarrow mz_2 + nz_1 = (m+n)z$$

$$\Rightarrow z = \frac{mz_2 + nz_1}{m+n}$$

Similarly dropping $\perp^n$ on yz-plane and zx-plane, we get

$$x = \frac{mx_2 + nx_1}{m+n}, \quad y = \frac{my_2 + ny_1}{m+n}$$

$\therefore$ R is the point

$$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n} \right)$$

Notes:

① The mid point of line segment joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is obtained by putting $m=1, n=1$

i.e $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$

② The point which divides the line segment joining $P(x_1, y_1, z_1)$ & $Q(x_2, y_2, z_2)$ externally in the ratio $m:n$

(i.e internally in the ratio $m:-n$)

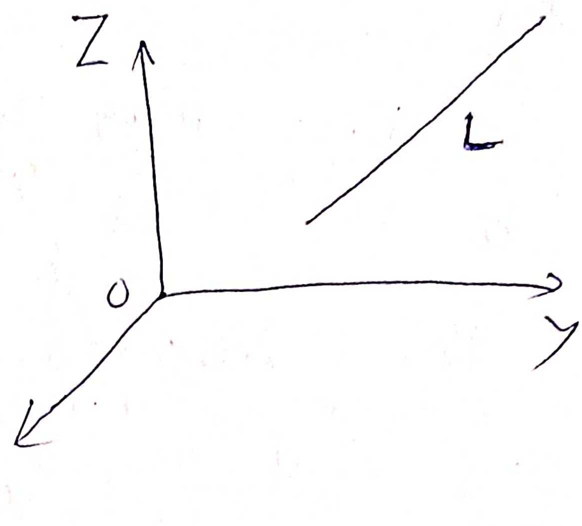
is given by

$$\left(\frac{mx_2-nx_1}{m-n}, \frac{my_2-ny_1}{m-n}, \frac{mz_2-nz_1}{m-n}\right)$$

Direction angles, Direction Cosines (d.cs) and Direction ratios (d.rs) of a line in space.

Let L be given line in space.

Let L makes angle α, β, γ with +ve directions of X, Y, Z-axes, respectively.



α, β, γ are called direction angles of line L.

$\cos\alpha, \cos\beta, \cos\gamma$ are called d.cs of line L.

We denote $l = \cos\alpha$, $m = \cos\beta$
$n = \cos\gamma$.

i.e we take l, m, n as d.cs of line L.

The quantities proportional to d.cs are called d.rs of L.

i.e if $\frac{l}{a} = \frac{m}{b} = \frac{n}{c}$ then

a, b, c are d.rs of L.

Notes :-

① A line may have infinite no of d.r.s.

② The direction angles of X-axis are $0, \frac{\pi}{2}, \frac{\pi}{2}$

The d.c.s of X-axis are 1, 0, 0

(The d.c.s of X-axis = $\langle 1, 0, 0 \rangle$)

The direction angles of Y-axis are $\frac{\pi}{2}, 0, \frac{\pi}{2}$

The d.c.s of Y-axis = $\langle 0, 1, 0 \rangle$

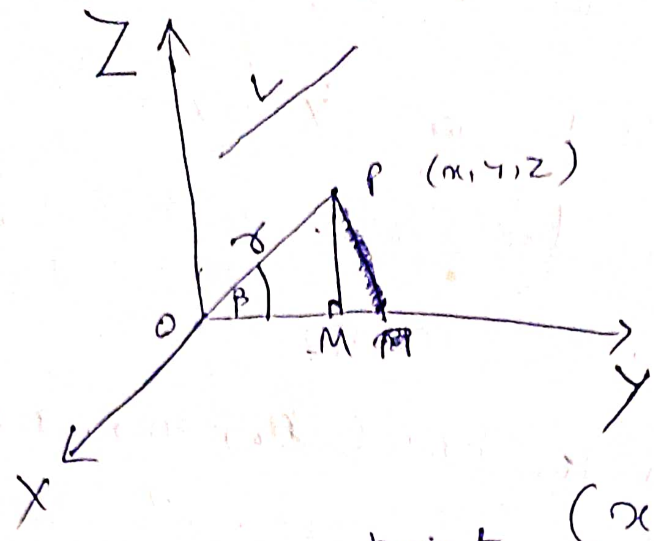
The dirn angles of Z-axis are $\frac{\pi}{2}, \frac{\pi}{2}, 0$

The d.c.s of Z-axis $\langle 0, 0, 1 \rangle$

Q-) Imp. Prove that $l^2+m^2+n^2=1$ where l, m, n are d.cs of a line

Proof : Let L be a line with d.cs. l, m, n. Draw a line

OP // to L and Passing through origin.



P is the point (x, y, z)

Now $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$

where L makes angle α, β, γ with +ve direction of X, Y, Z axes respectively.

Since OP // L

∴ OP makes angles α, β, γ with +ve direction of X, Y, Z axes respectively.

Draw a $\perp^r$ PM from P on Y-axis.

$\therefore OM = y$

Let $OP = r = \sqrt{x^2 + y^2 + z^2}$ (Distance formula)

Now from $\triangle$ PMO we have

$$\cos\beta = \frac{OM}{OP} = \frac{y}{r}$$

$\Rightarrow m = \frac{y}{r} \quad \Rightarrow y = mr$.

Dropping $\perp^{r}$ on x-axis and on Z-axis from P we get

$x = lr, \quad z = nr$

$\therefore$ P is the point (lr, mr, nr)

Now $l^2 + m^2 + n^2$

$$= \frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2}$$

$$= \frac{x^2 + y^2 + z^2}{r^2} = \frac{r^2}{r^2} = 1$$

(Proved)

Notes :-

① If l, m, n are d.c.s of a line then −l, −m, −n will also be d.c.s of the same line

② Let a, b, c be d.r.s of the line L. Let l, m, n be its d.c.s

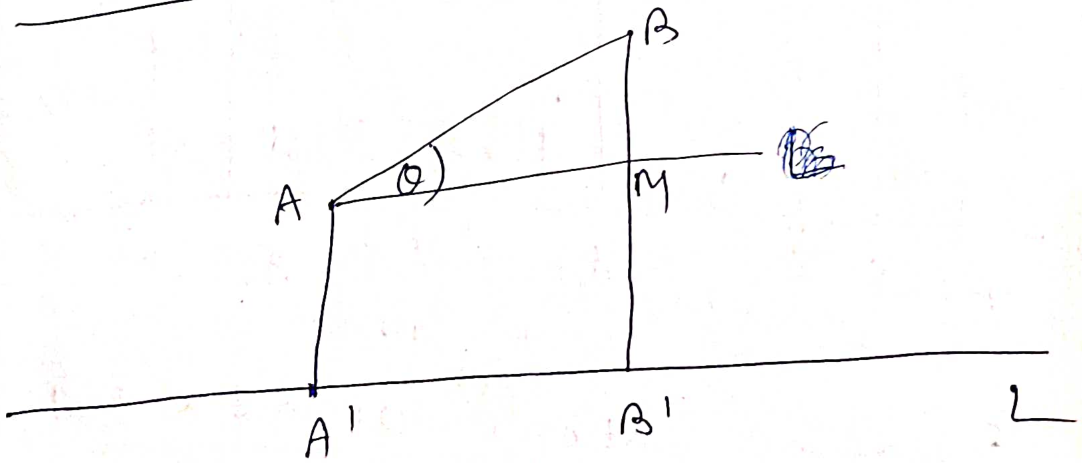
$$\therefore \frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \frac{\sqrt{l^2+m^2+n^2}}{\pm\sqrt{a^2+b^2+c^2}}$$

$$= \frac{1}{\pm\sqrt{a^2+b^2+c^2}}$$

$$\therefore \quad l = \frac{a}{\pm\sqrt{a^2+b^2+c^2}}, \quad m = \frac{b}{\pm\sqrt{a^2+b^2+c^2}}$$

$$n = \frac{c}{\pm\sqrt{a^2+b^2+c^2}}$$

Projection of a line segment on a given line.



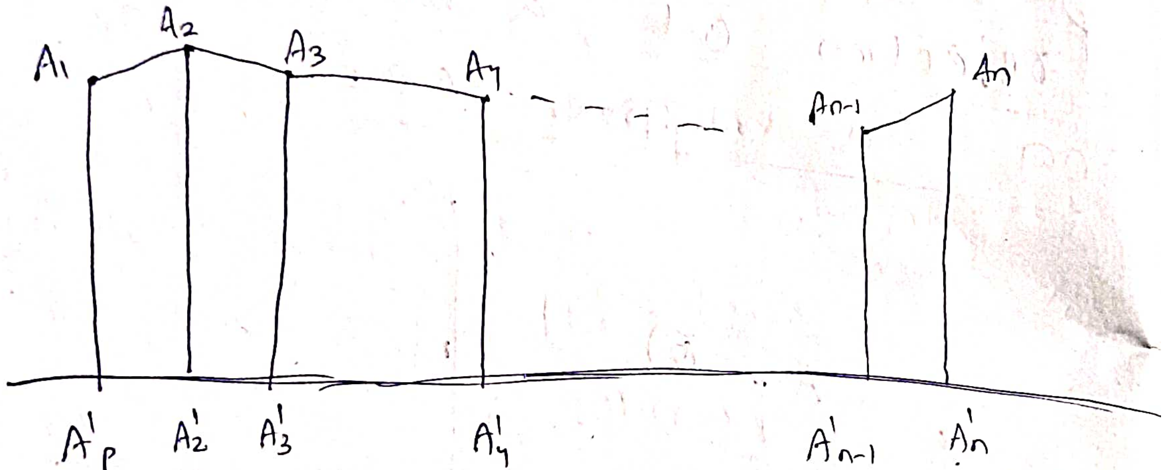
A'B' is the projection of AB on the line L.

Let θ be the angle between the line segment AB and the line L

$$\therefore \quad A'B' = AB \cos\theta.$$

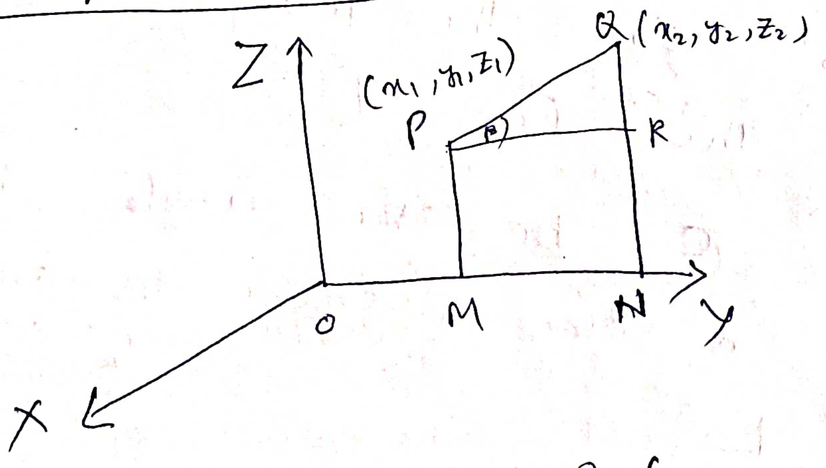
The algebraic sum of the projections of the line segments A_1A_2, A_2A_3, A_3A_4 ---------, $A_{n-1}A_n$

where A_1, A_2, A_3 ---------, A_n are any points in space on any given line is equal to the projection of A_1A_n on the same line



. i.e sum of the projections is the projection of the ~~same~~ sum.

Direction ratios of a line segment joining two given points



Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two given points.

Draw $\perp^r$ PM and QN on y-axis.

Draw $\perp^r$ PR on QN.

$OM = y_1$, $ON = y_2$

$PR = MN = ON - OM = y_2 - y_1$

Let l, m, n be d.cs of PQ

$\therefore l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$

where α, β, γ are direction angles.

$$\therefore \angle QPR = \beta$$

$$\therefore \cos\beta = \frac{PR}{PQ} = \frac{y_2 - y_1}{PQ}$$

Similarly ~~cos~~ $\cos\alpha = \dfrac{x_2 - x_1}{PQ}$

$$\cos\gamma = \frac{z_2 - z_1}{PQ}$$

$$\therefore l = \frac{x_2 - x_1}{PQ}, \quad m = \frac{y_2 - y_1}{PQ}, \quad n = \frac{z_2 - z_1}{PQ}$$

$$\Rightarrow \frac{l}{x_2 - x_1} = \frac{1}{PQ} = \frac{m}{y_2 - y_1} = \frac{n}{z_2 - z_1}$$

Now $x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$ are proportion to l, m, n. Hence $x_2 - x_1$, $y_2 - y_1$ and $z_2 - z_1$ are d.rs.

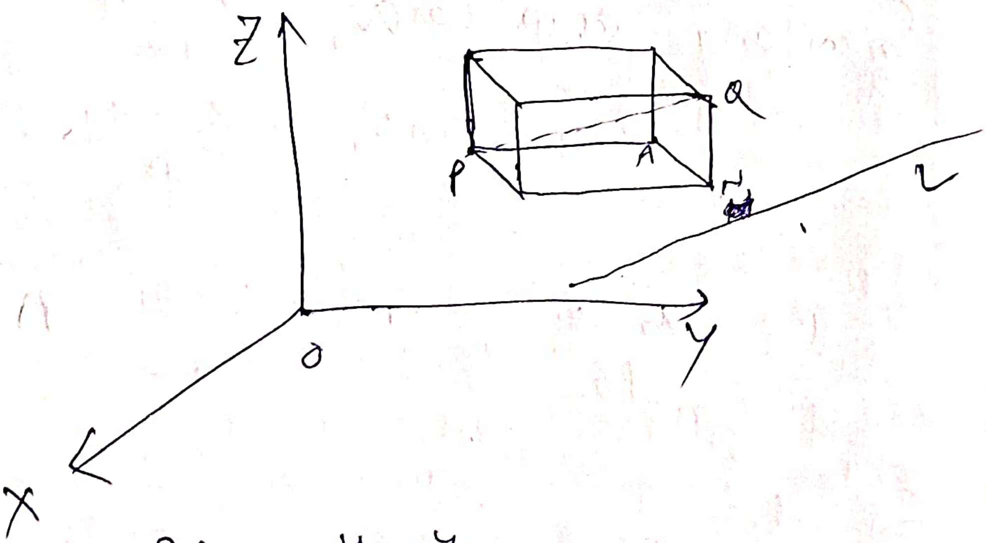
Projection of line segment on a given line

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two given points. Let L be a given line with d.cs. l, m, n

where ~~Q~~ $l = \cos\alpha$, $m = \cos\beta$,

$n = \cos\gamma$

Construct a rectangular parallelopiped with PQ as one of its diagonal ~~diagram~~ and the faces are // to co-ordinate planes.



$PA = y_2 - y_1$

$AN = x_2 - x_1$

$NQ = z_2 - z_1$

Now PA, AN and NQ are // to

Y, X and Z axes respectively.

Hence L makes angles α, β, γ with AN, PA and NQ respectively.

Projection of PA on L $= PA \cos\beta$

$= (y_2 - y_1) m$

Projection of AN on L $= AN \cos\alpha$

$= (x_2 - x_1) l$

Projection of NQ on L $= NQ \cos\gamma$

$= (z_2 - z_1) n$

Projection of PQ on L

= Projection of PA on L + Proj. of AN on L

+ Proj. of NQ on L

$= (y_2 - y_1) m + (x_2 - x_1) l + (z_2 - z_1) n$

$= l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)$

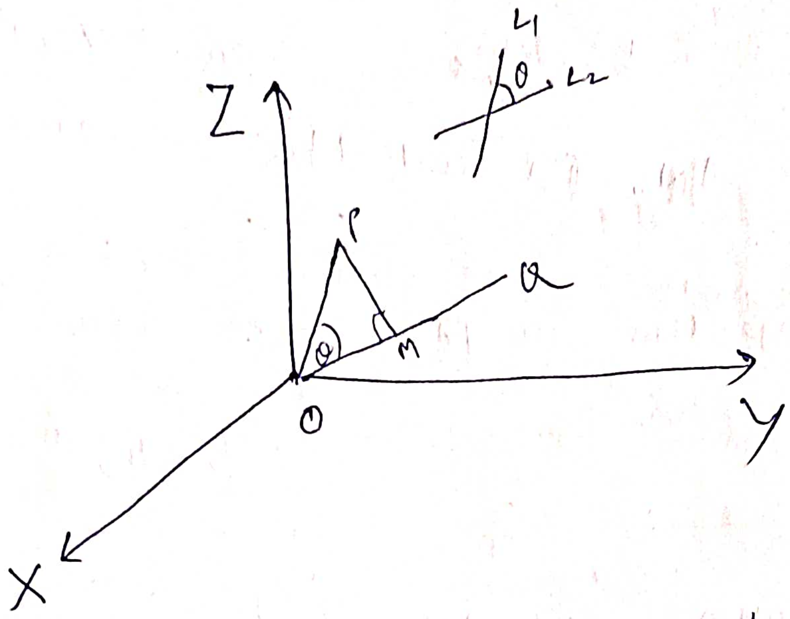
Imp.

Angle between two lines

Let L_1 and L_2 be two given lines in space.

Draw lines OP and OQ // to L

and L_2 ~~xx~~ respectively.



Let the d.cs of L_1 be

l_1, m_1, n_1

Let the d.c.s of L_2 be

l_2, m_2, n_2.

Let θ be the angle between L_1

and L_2.

i.e θ is the angle between OP

and OQ.

i.e $\angle POQ = \theta$

Draw a $\perp$ PM from P on OQ

Let $|OP| = r$

Since $OP \parallel L_1$ and d.cs. of L_1

are l_1, m_1, n_1

$\therefore$ d.cs. of OP are l_1, m_1, n_1.

Hence P is the Pt $(l_1 r, m_1 r, n_1 r)$

Now projection of OP on OQ

$= OM = r\cos\theta \quad \left(\because \frac{OM}{OP} = \cos\theta \text{ i.e } \frac{OM}{r} = \cos\theta\right)$

But projection of OP on OQ

$= l_2(l_1 r - 0) + m_2(m_1 r - 0) + n_2(n_1 r - 0)$

$= (l_1 l_2 + m_1 m_2 + n_1 n_2) r$

$\therefore r\cos\theta = (l_1 l_2 + m_1 m_2 + n_1 n_2) r$

$\Rightarrow \boxed{\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2}$

$\therefore \theta = \cos^{-1}(l_1 l_2 + m_1 m_2 + n_1 n_2)$

Which is the required angle between L_1 and L_2.

Notes :

(1) Suppose the d.r.s of L_1 are a_1, b_1, c_1 and the d.r.s of L_2

are a_2, b_2, c_2.

Let θ be the angle between them.

$$\therefore l_1 = \frac{a_1}{\pm\sqrt{a_1^2+b_1^2+c_1^2}}, \quad m_1 = \frac{b_1}{\pm\sqrt{a_1^2+b_1^2+c_1^2}}$$

$$n_1 = \frac{c_1}{\pm\sqrt{a_1^2+b_1^2+c_1^2}},$$

$$l_2 = \frac{a_2}{\pm\sqrt{a_2^2+b_2^2+c_2^2}}, \quad m_2 = \frac{b_2}{\pm\sqrt{a_2^2+b_2^2+c_2^2}}$$

$$n_2 = \frac{c_2}{\pm\sqrt{a_2^2+b_2^2+c_2^2}}$$

$$\therefore \boxed{\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2 = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\pm\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}}$$

(2). If two lines are $\perp^r$ then

$$\cos\theta = 0$$

$$\Rightarrow \boxed{l_1 l_2 + m_1 m_2 + n_1 n_2 = 0}$$

Proof in page last

$$\Rightarrow \boxed{a_1 a_2 + b_1 b_2 + c_1 c_2} = 0$$

Then eq^ns are also true ① $l_1 a_2 + m_1 b_2 + n_1 c_2 = 0$, $l_2 a_1 + m_2 b_1 + n_2 c_1 = 0$

(3) $\sin\theta = \pm\sqrt{1-\cos^2\theta}$

$= \pm\sqrt{1-(l_1l_2+m_1m_2+n_1n_2)^2}$

$= \pm\sqrt{(l_1^2+m_1^2+n_1^2)(l_2^2+m_2^2+n_2^2)-(l_1l_2+m_1m_2+n_1n_2)^2}$

= Calculate

$= \pm\sqrt{(l_1m_2-m_1l_2)^2+(m_1n_2-n_1m_2)^2+(n_1l_2-l_1n_2)^2}$

(4) If two lines are $\parallel$ then

$\sin\theta = 0$

$\Rightarrow \pm\sqrt{(l_1m_2-m_1l_2)^2+(m_1n_2-n_1m_2)^2+(n_1l_2-l_1n_2)^2} = 0$

$\Rightarrow (l_1m_2-m_1l_2)^2+(m_1n_2-n_1m_2)^2+(n_1l_2-l_1n_2)^2 = 0$

$\Rightarrow l_1m_2-m_1l_2=0$, $m_1n_2-n_1m_2=0$

$n_1l_2-l_1n_2=0$

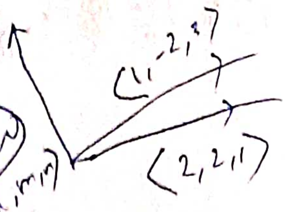
$\Rightarrow \dfrac{l_1}{l_2} = \dfrac{m_1}{m_2} = \dfrac{n_1}{n_2}$

$\Rightarrow$ D.C.s are proportional.

$\Rightarrow \dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$

$\Rightarrow$ D.r.s are proportional.

6 (a), Ex- 6, Page 162



Draw the diagram.

Ex-3 draw diagram in book (1 min)

Ax in Ex- 6 :-

D.C.S of OE are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

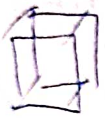
Diagonal of a face is OB.

D. r.s of OB are $a, a, 0$

D. C. S of OB ,, $\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0$.

Angle between OE and OB be θ

$$\therefore \cos\theta = \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} \cdot 0$$

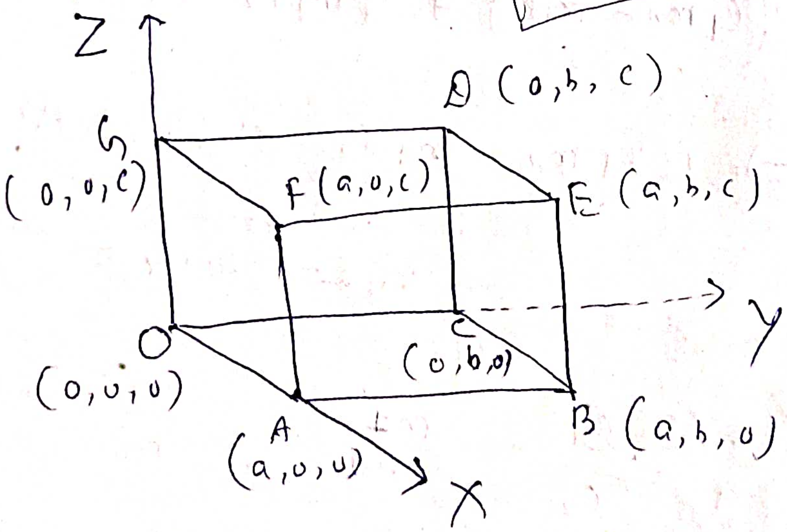


$$= \frac{1}{\sqrt{6}} + \frac{1}{\sqrt{6}} = \frac{2}{\sqrt{6}} = \sqrt{\frac{2}{3}}$$

$$\theta = \cos^{-1}\left(\sqrt{2/3}\right)$$

$\boxed{5\text{-}15\text{PM}}$

7. V.V. Imp



D. r.s of OE are a, b, c

D. C.S, of OE are $\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}}$

D.C.S of DA are $\frac{+a}{\sqrt{\Sigma a^2}}, \frac{-b}{\sqrt{\Sigma a^2}}, \frac{-c}{\sqrt{\Sigma a^2}}$

" " GB " $\frac{+a}{\sqrt{\Sigma a^2}}, \frac{+b}{\sqrt{\Sigma a^2}}, \frac{-c}{\sqrt{\Sigma a^2}}$

" " CF " $\frac{a}{\sqrt{\Sigma a^2}}, \frac{-b}{\sqrt{\Sigma a^2}}, \frac{c}{\sqrt{\Sigma a^2}}$

Let θ_1 be the angle between OE and DA

$$\cos\theta_1 = \frac{+a^2 - b^2 - c^2}{\Sigma a^2}$$

$$\Rightarrow \theta_1 = \cos^{-1}\left(\frac{+a^2 - b^2 - c^2}{\Sigma a^2}\right)$$

Let θ_2 be the angle between OE and GB.

$$\cos\theta_2 = \frac{+a^2 + b^2 - c^2}{\Sigma a^2}$$

$$\Rightarrow \theta_2 = \cos^{-1}\left(\frac{a^2 + b^2 - c^2}{\Sigma a^2}\right)$$

Let θ_3 be the angle between OE and CF

$$\cos\theta_3 = \frac{a^2 - b^2 + c^2}{\Sigma a^2} \Rightarrow \theta_3 = \cos^{-1}\left(\frac{a^2 - b^2 + c^2}{\Sigma a^2}\right)$$

Let θ_4 be the angle between DA and GB

$$\cos\theta_4 = \frac{a^2 - b^2 + c^2}{\Sigma a^2} \Rightarrow \theta_4 = \cos^{-1}\left(\frac{a^2 - b^2 + c^2}{\Sigma a^2}\right)$$

Let θ_5 be the angle between $\mathcal{D}A$ and CF

$$\theta_5 = \cos^{-1}\left(\frac{a^2+b^2-c^2}{\Sigma a^2}\right)$$

Let θ_6 be the angle between GB and CF

$$\theta_6 = \cos^{-1}\left(\frac{a^2-b^2-c^2}{\Sigma a^2}\right)$$

Hence the angles in general between six pairs are given by

$$\cos^{-1}\left(\frac{a^2 \pm b^2 \pm c^2}{a^2+b^2+c^2}\right)$$

8. V. Imp
P. 188

Let L_1 be the line with d.cs l_1, m_1, n_1

Let L_2 be the line with d.cs l_2, m_2, n_2

Let L_3 " " " d.c.s l, m, n

Given that L_1 and L_2 are mutually $\perp^r$

$$\therefore \quad l_1 l_2 + m_1 m_2 + n_1 n_2 = 0 \quad \text{———} (i)$$

Also given that L_3 is L_1 to L_2

~~$\therefore l l_2 + m m_2 + n n_2 = 0$~~

$$\therefore \quad l l_1 + m m_1 + n n_1 = 0 \quad \text{———} (ii)$$

Again given that L_3 is $\perp^r$ to L_2

$$l l_2 + m m_2 + n n_2 = 0 \quad \text{———} (iii)$$

Solving ② & ③ by cross multiplication method.

$$\frac{l}{m_1n_2 - m_2n_1} = \frac{m}{n_1l_2 - n_2l_1} = \frac{n}{l_1m_2 - l_2m_1}$$

$$= \frac{\sqrt{l^2+m^2+n^2}}{\sqrt{(m_1n_2-m_2n_1)^2+(n_1l_2-l_1n_2)^2+(l_1m_2-l_2m_1)^2}}$$

$$= \frac{1}{\sin\theta}$$ ($\because l^2+m^2+n^2 = 1$, θ is angle between L_1 and L_2 & $\sin\theta = \sqrt{\Sigma(m_1n_2 - m_2n_1)^2}$)

$= 1$ ($\because L_1$ and L_2 are $\perp^r$ $\therefore \theta = \frac{\pi}{2}$, $\sin\theta = 1$)

$$\therefore \frac{l}{m_1n_2 - m_2n_1} = \frac{m}{n_1l_2 - l_1n_2} = \frac{n}{l_1m_2 - l_2m_1} = 1$$

$\Rightarrow l = m_1n_2 - m_2n_1$, $m = n_1l_2 - l_1n_2$

$n = l_1m_2 - l_2m_1$

$\therefore$ The d.cs of L_3 are $m_1n_2 - m_2n_1$, $n_1l_2 - l_1n_2$, $l_1m_2 - l_2m_1$

(Proved)

Q → Find the d.cs of a line which is equally inclined to the axes
(gmp)

Ans: If the line makes angles α, β, γ with axes then

$$\cos\alpha = \cos\beta = \cos\gamma \quad (\because \alpha = \beta = \gamma \text{ given})$$

$$\Rightarrow l = m = n$$

$$\therefore \frac{l}{1} = \frac{m}{1} = \frac{n}{1} = \frac{\sqrt{l^2+m^2+n^2}}{\pm\sqrt{1^2+1^2+1^2}} = \frac{1}{\pm\sqrt{3}}$$

$$l = \pm\frac{1}{\sqrt{3}}, \quad m = \pm\frac{1}{\sqrt{3}}, \quad n = \pm\frac{1}{\sqrt{3}}$$

$\therefore$ d.cs are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

or $-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}$

(Actually there are 4 distinct such lines exist and 8 directed lines exist)

Q-1 (gmp) → Prove that the st. lines whose d.cs given by the relations

$$al + bm + cn = 0 \quad \text{and}$$

$$fmn + gnl + hlm = 0 \quad \text{are } \perp^r$$

if $\frac{f}{a} + \frac{g}{b} + \frac{h}{c} = 0$ and $\parallel$ if

$$\sqrt{af} \pm \sqrt{bg} \pm \sqrt{ch} = 0$$

or $a^2f^2 + b^2g^2 + c^2h^2 - 2bcgh - 2cahf - 2abfg = 0$

Q → Prove that the two lines whose d.c.s are given by $al + bm + cn = 0$ and $ul^2 + vm^2 + wn^2 = 0$ are $\perp^r$ if $a^2(v+w) + b^2(w+u) + c^2(u+v) = 0$

& parallel if $\frac{a^2}{u} + \frac{b^2}{v} + \frac{c^2}{w} = 0$

Proof :-

$$al + bm + cn = 0$$

$$\Rightarrow l = -\left(\frac{bm + cn}{a}\right)$$

Also, $fmn + gnl + hlm = 0$

$$\Rightarrow fmn - gn\left(\frac{bm+cn}{a}\right) - hm\left(\frac{bm+cn}{a}\right) = 0$$

$$\Rightarrow afmn - gbmn - gcn^2 - hbm^2 - hcmn = 0$$

$$\Rightarrow hbm^2 + mn(gb + hc - af) + gcn^2 = 0$$

$$\Rightarrow hb\frac{m^2}{n^2} + (gb + hc - af)\frac{m}{n} + gc = 0 \quad \text{—— (1)}$$

Let $\frac{m_1}{n_1}$ and $\frac{m_2}{n_2}$ be two roots

$\frac{l_1}{l_2} = \left(\frac{m_1}{m_2} = \frac{n_1}{n_2}\right)$

$\frac{m_1}{n_1} = \frac{m_2}{n_2}$

If two lines are // then their d.c.s are proportional

∴ The above eqn has equal roots.

∴ Discriminant = 0

$$\Rightarrow (gb + hc - af)^2 - 4.(hb).(gc) = 0$$

$\Rightarrow (gb + hc - af)^2 = 4bcgh$

$\Rightarrow gb + hc - af = \pm 2\sqrt{bcgh}$

$\Rightarrow gb + ch \pm 2\sqrt{bcgh} = af$

$\Rightarrow (\sqrt{gb} \pm \sqrt{ch})^2 = (\sqrt{af})^2$

$\Rightarrow \pm(\sqrt{gb} \pm \sqrt{ch}) = \sqrt{af}$

$\Rightarrow \pm\sqrt{gb} \pm \sqrt{ch} = \sqrt{af}$

$\Rightarrow \sqrt{af} \pm \sqrt{gb} \pm \sqrt{ch} = 0$

which is the required condition.

Also $(gb + hc - af)^2 = 4bcgh$

$\Rightarrow a^2f^2 + b^2g^2 + c^2h^2 - 2bcgh - 2cahf - 2abfg = 0$

Also from eqn

$$\frac{m_1}{n_1} \cdot \frac{m_2}{n_2} = \frac{gc}{hb}$$

$$\Rightarrow \frac{m_1 m_2}{gc} = \frac{n_1 n_2}{hb}$$

$$\Rightarrow \frac{m_1 m_2}{g/b} = \frac{n_1 n_2}{h/c}$$

Similarly eliminating n we get

$$\frac{l_1 l_2}{f/a} = \frac{m_1 m_2}{g/b}$$

$$\therefore \frac{l_1 l_2}{f/a} = \frac{m_1 m_2}{g/b} = \frac{n_1 n_2}{h/c} = K$$

$$\Rightarrow l_1 l_2 = \frac{Kf}{a}, \quad m_1 m_2 = \frac{Kg}{b}, \quad n_1 n_2 = \frac{Kh}{c}$$

If two lines are $\perp^r$ then

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$$

$$\Rightarrow K\frac{f}{a} + K\frac{g}{b} + K\frac{h}{c} = 0$$

$$\Rightarrow \frac{f}{a} + \frac{g}{b} + \frac{h}{c} = 0 \qquad \text{(Proved)}$$

(2) $n = -\left(\frac{al + bm}{c}\right)$

$$\therefore ul^2 + vm^2 + w\left(\frac{al+bm}{c}\right)^2 = 0$$

$$\Rightarrow ul^2 + vm^2 + \frac{w}{c^2}(a^2l^2 + b^2m^2 + 2ab\,lm) = 0$$

$$\Rightarrow c^2ul^2 + c^2vm^2 + a^2l^2w + b^2m^2w + 2ab\,lmw = 0$$

$$\Rightarrow l^2(c^2u + a^2w) + m^2(c^2v + b^2w) + 2ab\,lmw = 0$$

$$\Rightarrow (c^2u + a^2w)\frac{l^2}{m^2} + 2abw\cdot\frac{l}{m} + (c^2v + b^2w) = 0 \quad \text{—— (i)}$$

$$\Rightarrow \frac{l_1}{m_1}\frac{l_2}{m_2} = \frac{c^2v + b^2w}{c^2u + a^2w}$$

$$\Rightarrow \frac{l_1 l_2}{b^2w + c^2v} = \frac{m_1 m_2}{c^2u + a^2w}$$

Similarly, $\frac{m_1 m_2}{c^2 u + a^2 w} = \frac{x_1 x_2}{a^2 v + b^2 u}$

$\therefore \frac{l_1 l_2}{b^2 w + c^2 v} = \frac{m_1 m_2}{c^2 u + a^2 w} = \frac{n_1 n_2}{a^2 v + b^2 u} = K$

Since they are $\perp^r$

$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$

$\Rightarrow (b^2 w + c^2 v) + (c^2 u + a^2 w) + (a^2 v + b^2 u) = 0$

$\Rightarrow a^2 (v + w) + b^2 (w + u) + c^2 (u + v) = 0$

If the lines are $\parallel$ then

discriminant $= 0$

$\Rightarrow \cancel{4} a^2 b^2 w^2 = \cancel{4} (c^2 u + a^2 w)(c^2 v + b^2 w)$

$\Rightarrow \cancel{a^2 b^2 w^2} = c^4 uv + c^2 b^2 uw + a^2 c^2 wv$
$+ \cancel{a^2 b^2 w^2}$

$\Rightarrow c^2 uv + b^2 uw + a^2 wv = 0$

$\Rightarrow \frac{c^2}{w} + \frac{b^2}{v} + \frac{a^2}{u} = 0$

(Proved)

diagram

2D + any object → curve

3D → " " → surface

Plane (Defn)

rugby → ellipsoide

stool → hyperboloid.

A plane is defined as an surface such that the line joining any two points on the surface lies wholly on it.

Eqn of plane in different forms

① General form :-

Imp Every first degree eqn (linear eqn) in x, y and z of the form $Ax + By + Cz + D = 0$ represents a plane.

Proof :- The given eqn is

$$Ax + By + Cz + D = 0 \quad \text{——— (i)}$$

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be any two points on the given surface ①

∴ The co-ordinates of P and Q will satisfy eqn (i)

$$\therefore \; Ax_1 + By_1 + Cz_1 + D = 0 \quad \text{——— ②}$$

$$Ax_2 + By_2 + Cz_2 + D = 0 \quad \text{——— ③}$$

multiplying eqn ② by n and eqn ③ by m and adding

Provided ~~m+n~~ $m+n \neq 0$, we get

$$A(mx_2+nx_1) + B(my_2+ny_1) + C(mz_2+nz_1)$$
$$+ D(m+n) = 0$$

$$\Rightarrow A\left(\frac{mx_2+nx_1}{m+n}\right) + B\left(\frac{my_2+ny_1}{m+n}\right) + C\left(\frac{mz_2+nz_1}{m+n}\right)$$
$$+ D = 0$$

which shows that the point

$$\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}, \frac{mz_2+nz_1}{m+n}\right)$$

satisfies eqn ①

Hence this point lies on the given surface ①

But this point is any point on the line joining P and Q

Hence every point on the line ~~surface~~ joining P & Q lies on the surface ①

i.e the line joining P and Q lies wholly on the surface ①

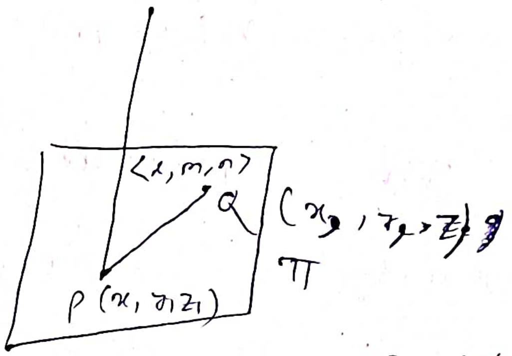
i.e the line ~~giving~~ joining any two points on the surface ① lies wholly on it (∵ P and Q are any two pts taken)

∴ The surface (1) is a plane.

(2) Eqⁿ of plane passing through a point P (x_1, y_1, z_1) with d.c.s of its normal as l, m, n is $l(x-x_1) + m(y-y_1) + n(z-z_1) = 0$

Proof

(name for plane :- π, P)



Let π be a plane passing through a given point P(x_1, y_1, z_1) and d.c.s of its normal are (l, m, n)

To find the eqⁿ of plane π

Let Q (x, y, z) be any arbitrary point on the plane π.

Join PQ.

D.r.s of PQ are $x-x_1, y-y_1, z-z_1$

Now PQ is ⊥ to the normal.

$\therefore l(x-x_1) + m(y-y_1) + n(z-z_1) = 0$

Since it is a relation between x and y and z, it is the required

Eqn of Plane (Solved)

Notes

(1) The eqn of plane passing through a point $P(x_1, y_1, z_1)$ with d.r.s of its normal is a, b, c is

$$a(x-x_1) + b(y-y_1) + c(z-z_1) = 0$$

(2) The general eqn of plane is

$$Ax + By + Cz + D = 0$$

$$\Rightarrow \frac{A}{D}x + \frac{B}{D}y + \frac{C}{D}z + 1 = 0 \quad (\text{Provided } D \neq 0)$$

$$\Rightarrow \lambda_1 x + \lambda_2 y + \lambda_3 z + 1 = 0$$

where $\lambda_1 = \frac{A}{D}$, $\lambda_2 = \frac{B}{D}$, $\lambda_3 = \frac{C}{D}$

Now there are 3 unknowns for the parameters $(\lambda_1, \lambda_2, \lambda_3)$ of eqn of plane. Hence 3 non-collinear points will determine a unique plane.

(3) The general eqn of plane passing through origin is

$$Ax + By + Cz = 0$$

origin $(0,0,0)$

$\boxed{3}$ Eqn. of plane through 3 given non-collinear points

Let the plane π passes through 3 non-collinear points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3).

Let the eqn of plane π be

$$Ax + By + Cz + D = 0 \quad \text{——— (i)}$$

Since it passes through (x_1, y_1, z_1)

$$\therefore Ax_1 + By_1 + Cz_1 + D = 0 \quad \text{——— (ii)}$$

Subtracting (2) from (1), we get

$$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0 \quad \text{——— (iii)}$$

Again since the plane passes through (x_2, y_2, z_2) and (x_3, y_3, z_3) we have

$$A(x_2 - x_1) + B(y_2 - y_1) + C(z_2 - z_1) = 0 \quad \text{——— (iv)}$$

$$A(x_3 - x_1) + B(y_3 - y_1) + C(z_3 - z_1) = 0 \quad \text{——— (v)}$$

Eliminating A, B, C from eqns (3), (4), (5), we get

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

which is the required eqn of plane.

(4) : Intercept form

Let the plane π cuts x-axis, y-axis and z-axis. Let {a, b, c be the intercepts.}

Now plane π meets x-axis at $(a,0,0)$
" " " y-axis $(0,b,0)$
" " " z-axis $(0,0,c)$.
" "

Let the eqn of plane be
$$Ax+By+Cz+D=0$$

Since it passes through $(a,0,0)$, $(0,b,0)$ and $(0,0,c)$, we have

$$Aa+D=0 \Rightarrow A=-\frac{D}{a}$$

$$Bb+D=0 \Rightarrow B=-\frac{D}{b}$$

$$Cc+D=0 \Rightarrow C=-\frac{D}{c}$$

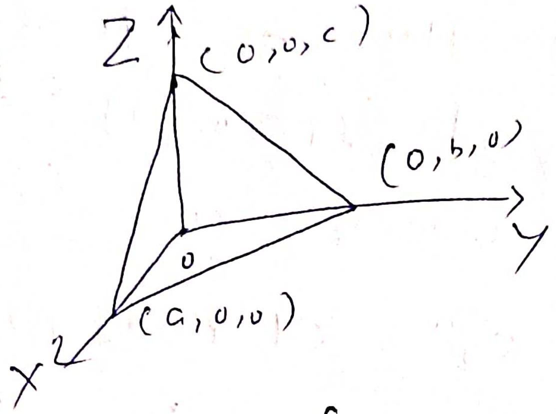
$\therefore$ Eqn of plane becomes

$$-\frac{D}{a}x-\frac{D}{b}y-\frac{D}{c}z+D=0$$

$\Rightarrow \frac{x}{a} + \frac{y}{b} + \frac{z}{c} - 1 = 0$

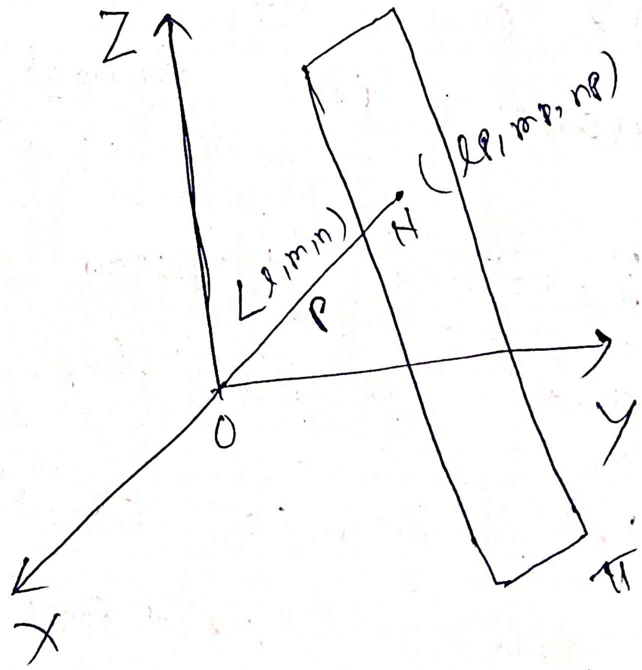
$$\Rightarrow \boxed{\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1}$$

Fig :



(5) Normal form

Let p be the length of ~~the~~ the ⊥ drawn from origin to the plane π. Let l, m, n be the d.c.s of its normal.



Now let ON be the ⊥ drawn from O to the plane π.

$|ON| = p$

D.Cs of ON are l, m, n.

The point N is (lp, mp, np)

Hence the plane π passes through the point N (lp, mp, np) and d.c.s of its normal are (l, m, n).

$\therefore$ The eqn of plane is

$$l(x - lp) + m(y - mp) + n(z - np) = 0$$

$$\Rightarrow lx + my + nz = (l^2 + m^2 + n^2)p$$

$$\Rightarrow \boxed{lx + my + nz = p} \quad (\because l^2 + m^2 + n^2 = 1)$$

It is the required eqn of plane in normal form.

Relation between Normal form and general form

Let the eqn of plane in ~~normal~~ general form be $Ax + By + Cz + D = 0$ — (i)

Let its normal form be
$lx + my + nz - p = 0$

Since (1) and (2) represents the same plane

∴ The Co-efficients are proportional.

$$\therefore \frac{A}{l} = \frac{B}{m} = \frac{C}{n} = \frac{D}{-p}$$

$$\text{i.e } \frac{D}{-p} = \frac{A}{l} = \frac{B}{m} = \frac{C}{n} = \frac{\pm\sqrt{A^2+B^2+C^2}}{\pm\sqrt{l^2+m^2+n^2}}$$

$$= \pm\sqrt{A^2+B^2+C^2}$$

$$\therefore l = \frac{A}{\pm\sqrt{A^2+B^2+C^2}}, \quad m = \frac{B}{\pm\sqrt{A^2+B^2+C^2}}$$

$$n = \frac{C}{\pm\sqrt{A^2+B^2+C^2}}, \quad p = \frac{-D}{\pm\sqrt{A^2+B^2+C^2}}$$

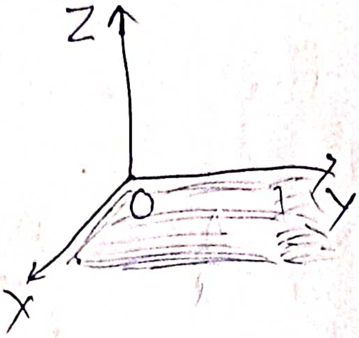
Since p is +ve, if D is +ve then we take -ve square root and if D is -ve then we take +ve square root.

Notes

(1) In general eqn of a plane i.e in $Ax+By+Cz+D=0$, A, B, C are the d.r.s of the normal to the plane.

② Eqⁿ of XY-plane

Here XY-plane passes Through O (0,0,0)



Also Z-axis is the normal whose d.c.s are (0,0,1)

Hence the eqⁿ of XY-plane is

$$0(x-0) + 0(y-0) + 1(z-0) = 0$$

$$\Rightarrow \boxed{Z = 0}$$

Similarly eqⁿ of YZ plane $x = 0$
" " " ZX plane $y = 0$

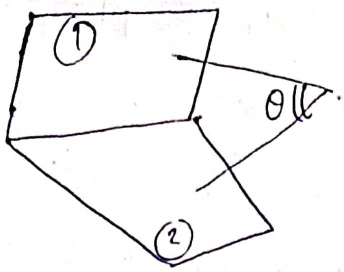
Angle between two planes –

Let the eqⁿ of two planes be

$$A_1x + B_1y + C_1z + D_1 = 0 \quad \text{— (i)}$$

$$A_2x + B_2y + C_2z + D_2 = 0 \quad \text{— (ii)}$$

The angle between two planes is the angle between their normals.



Let the angle be θ.

Now Dr.s of the normal to the 1st plane are A_1, B_1, C_1

Dr.s of the ~~dr.s~~ ~~of the~~ normal to the 2nd plane are A_2, B_2, C_2

$$\therefore \cos\theta = \frac{A_1A_2 + B_1B_2 + C_1C_2}{\pm\sqrt{A_1^2+B_1^2+C_1^2}\sqrt{A_2^2+B_2^2+C_2^2}}$$

Notes :- 1

① If the two planes are parallel then their normals are ॥.

$\therefore$ The d.r.s are proportional.

$$\therefore \frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}.$$

② If two planes are identical then the co-efficients of x, y, z and the constant terms in the two eqn.s are proportional.

i.e $\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2} = \frac{D_1}{D_2}$.

③ If two planes are $\perp$

then $\theta = \frac{\pi}{2}$, $\Rightarrow \cos\theta = 0$

$$\Rightarrow \boxed{A_1A_2 + B_1B_2 + C_1C_2 = 0}$$

System of planes or family of planes

The eq^n of system of planes passing through the intersection of two given planes

$$A_1x + B_1y + C_1z + D_1 = 0$$
$$A_2x + B_2y + C_2z + D_2 = 0$$ is

given by

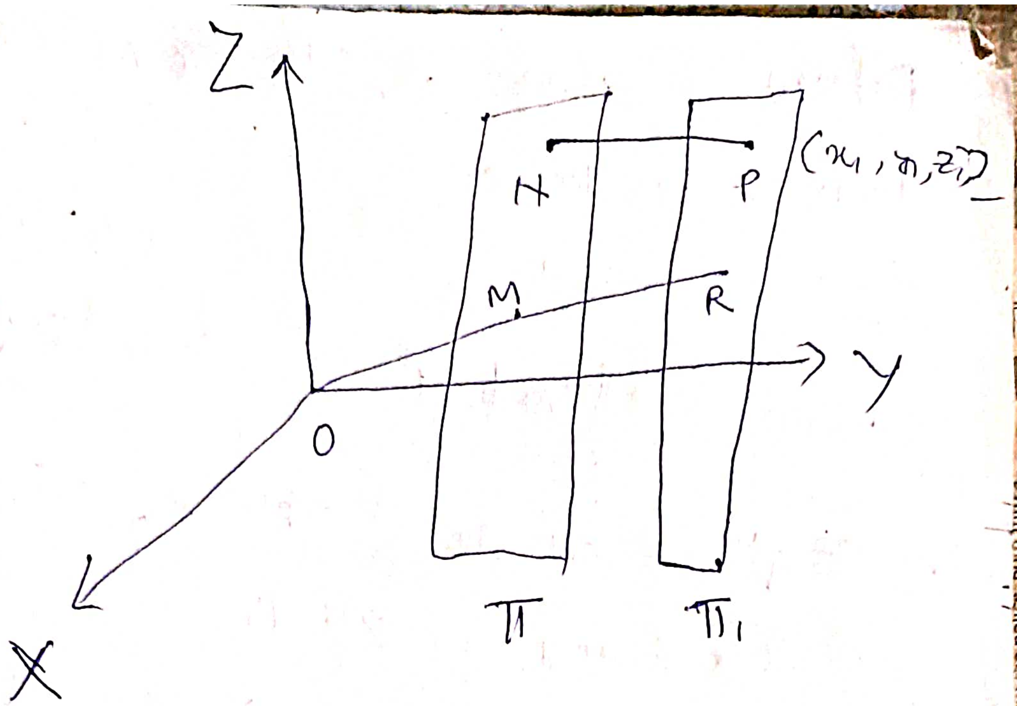
$$(A_1x + B_1y + C_1z + D_1) + K(A_2x + B_2y + C_2z + D_2) = 0$$

where K is any arbitrary constant.

Perpendicular distance of a point from a plane

Let π be a given plane with eq^n $Ax + By + Cz + D = 0$

Let its normal from be $lx + my + nz = p$



where $l = \dfrac{A}{\pm\sqrt{A^2+B^2+C^2}}$, $m = \dfrac{B}{\pm\sqrt{A^2+B^2+C^2}}$

$n = \dfrac{C}{\pm\sqrt{A^2+B^2+C^2}}$, $p = \dfrac{-D}{\pm\sqrt{A^2+B^2+C^2}}$

Let $P(x_1, y_1, z_1)$ be a given point which lies between outside the plane.

Let PN be the $\perp$ drawn from P to the plane π.

Draw a plane π_1 passing through the point P and $\parallel$ to plane π.

Draw a $\perp$ OM from origin to the plane π. Let it meets the plane π_1 at R.

$\therefore$ OR is also $\perp$ to the plane π_1 from origin

Now, d.c.s of OM are l, m, n. Hence d.c.s of OR are l, m, n

Also $|OM| = p$, let $|OR| = p_1$

$\therefore$ Eqn of the plane π_1 is

$$lx + my + nz = p_1$$

Also this plane π_1 passes through $P(x_1, y_1, z_1)$

$$\therefore \quad lx_1 + my_1 + nz_1 = p_1$$

Hence $|PM| = |RM| = |OR - OM|$

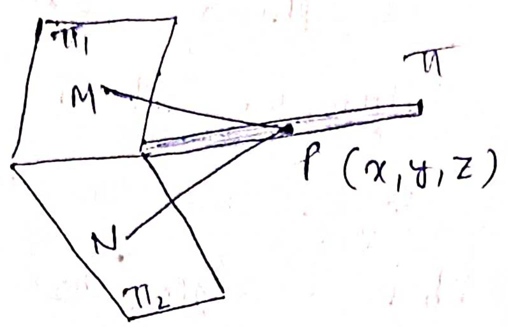
$$= |p_1 - p|$$

$$= |lx_1 + my_1 + nz_1 - p|$$

$$= \left| \frac{Ax_1 + By_1 + Cz_1 + D}{\pm\sqrt{A^2 + B^2 + C^2}} \right|$$

$$= \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

Eqⁿ of Bisector planes of angles between two given planes :



Let the eqⁿ of two planes π_1 and π_2 be

$$A_1x + B_1y + C_1z + D_1 = 0$$

and $A_2x + B_2y + C_2z + D_2 = 0$

respectively.

Let π be the bisector plane.

Let $P(x, y, z)$ be any point on the bisector plane π.

Draw $\perp$ PM and PN on π_1 and π_2 respectively.

$$\therefore \; |PM| = |PN|$$

$$\Rightarrow \frac{|A_1x + B_1y + C_1z + D_1|}{\sqrt{A_1^2 + B_1^2 + C_1^2}} = \frac{|A_2x + B_2y + C_2z + D_2|}{\sqrt{A_2^2 + B_2^2 + C_2^2}}$$

$$\Rightarrow \frac{A_1x + B_1y + C_1z + D_1}{\sqrt{A_1^2 + B_1^2 + C_1^2}} = \pm \frac{A_2x + B_2y + C_2z + D_2}{\sqrt{A_2^2 + B_2^2 + C_2^2}}$$

Since it is a relation between x, y, z and it is the required eqn of bisector planes one of the these two bisector planes bisecus the acute angle and the other bisects the obtuse angle between the two given planes.

Note:-

Eqn of plane // to xy-plane is $z =$ constant $= K$ (say)

Eqn of plane // to YZ plane is $x = K$

" " " " " ZX plane is $y = K$

Q→ Show that the four points $(0, 4, 3)$, $(-1, -5, -3)$, $(-2, -2, 1)$ and $(1, 1, -1)$ are coplanner and find the common plane.

Ans→ Eqn of plane having through $(0, 4, 3)$ is

$$A(x-0) + B(y-4) + C(z-3) = 0 \quad \text{--- (i)}$$

Since it passes through $(-1,-5,-3)$ and $(-2,-2,1)$, we get

$A(-1)+B(-9)+C(-6)=0$ ——— (2)

$A(-2)+B(-6)+C(-2)=0$ ——— (3)

$$\Rightarrow \left.\begin{array}{l} A+9B+6C=0 \\ 2A+6B+2C=0 \end{array}\right\}$$

Solving by cross multⁿ

$$\frac{A}{18-36}=\frac{B}{12-2}=\frac{C}{6-18}$$

$$\Rightarrow \frac{A}{-18}=\frac{B}{10}=\frac{C}{-12}$$

$$\Rightarrow \frac{A}{9}=\frac{B}{-5}=\frac{C}{6}$$

∴ The eqⁿ of plane through $(0,4,3)$, $(-1,-5,-3)$, $(-2,-2,1)$

is

$9x-5(y-4)+\cancel{6}6(z-3)=0$

$\Rightarrow 9x-5y+6z+2=0$

Now putting $x=1$, $y=1$, $z=1$ in the above eqⁿ we see that

$9-5-6+2=0$

i.e the pt $(1,1,-1)$ lies on the above plane.

Hence the four pts lie on the plane i.e on $9x-5y+6z+2=0$

$\therefore$ The 4 pts are coplanner and the eqn of common plane is

$$9x-5y+6z+2=0$$

12. The plane meets the co-ordinate axes at A, B, C
P-193

Let A be the pt $(a,0,0)$
B be the pt $(0,b,0)$
C be the pt $(0,0,c)$

The eqn of the plane is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

$$\Rightarrow \frac{x}{a}+\frac{y}{b}+\frac{z}{c}-1=0 \quad \text{——— (i)}$$

Eqn of plane through A and // to yz-plane is $x=a$

Eqn of plane B & // to zx plane is $y=b$

Eqn of plane through C and // xy-plane is $z = c$.

Now length of the $\perp$ drawn from $(0,0,0)$ to the plane (1)

$$= \frac{|-1|}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}} = p \quad \left(\text{By the question}\right)$$

$$\Rightarrow \frac{1}{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}} = p^2$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$$

The locus of the pt. of intersection of the planes $x=a$, $y=b$ and $z=c$ can be obtained by eliminating a, b, c from the above eqn.

Hence the required locus is

$$\frac{1}{p^2} = \frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2} \quad \text{(Proved)}.$$

13. Eqn of plane cutting 3 axes is $\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma} = 1$,

Since it passes through (a, b, c)

$\therefore$ we get $\frac{a}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma} = 1$ — (1)

The planes $\parallel$ to coordinate plane are $x = \alpha$, $y = \beta$, $z = \gamma$

Eliminating α, β, γ from (1), we get

$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 1$ which is the required locus.

(proved)

14. The plane in new position is $(4x + 7y + 4z + 81) + K(5x + 3y + 10z - 25) = 0$ — (1)

Angle between (1) and $4x + 7y + 4z + 81 = 0$ is $\frac{\pi}{2}$

$\therefore \cos\frac{\pi}{2} = 0 \Rightarrow K = -1$

15. The eqn of plane passing through intersection of the two given planes $lx + my = 0$ and $z = 0$ is

$lx + my + Kz = 0$

The d.r.s of the normal of this new plane are l, m, K

The d.r.s of the normal of the plane in old position are $l, m, 0$

But according to question, the angle between them in α.

$$\therefore \cos\alpha = \frac{l^2 + m^2 + k.0}{\sqrt{l^2+m^2+k^2}\sqrt{l^2+m^2}}$$

$$= \frac{\sqrt{l^2+m^2}}{\sqrt{l^2+m^2+k^2}}$$

$$\sin\alpha = \pm\sqrt{1-\cos^2\alpha} = \pm\sqrt{1-\frac{l^2+m^2}{l^2+m^2+k^2}}$$

$$= \frac{k}{\pm\sqrt{l^2+m^2+k^2}}$$

$$\therefore \tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \pm\frac{k}{\sqrt{l^2+m^2}}$$

$$\Rightarrow k = \pm\sqrt{l^2+m^2}\tan\alpha$$

$\therefore$ Eqn of plane in new position in $lx + my \pm \sqrt{l^2+m^2}\tan\alpha\, z = 0$

(proved)

Notes :-

(1) Two points (x_1, y_1, z_1) and (x_2, y_2, z_2) lie on the same side or opposite

side of the plane $Ax + By + Cz + D = 0$

according as $Ax_1 + By_1 + Cz_1 + D = 0$

and $Ax_2 + By_2 + Cz_2 + D$ are of same

or opposite signs.

Q. Find whether the two points $(2, 3, -5)$ and $(3, 4, 7)$ lie on the same side or opposite sides on the plane

$$x + 2y - 2z - 9 = 0$$

Ans :- Putting the coordinates of the given pts in the eqn of plane we get $+9$ and -12 which are of opp. sign.

$\therefore$ The two pts lie on the opposite sides of the given plane.

Note :- 2

How to find the distance between two // planes

1st method :- Choose any point on one of the planes and finds its

$\perp$ distance from the other.

2nd method :

Find out the $\perp$ distance of each plane from the origin and retain their signs and subtract two.

Q. Prove that the planes $3x+4y+5z+10=0$ and $9x+12y+15z+20=0$ are parallel. Then find the distance between them.

Ans: Here $\frac{3}{9} = \frac{4}{12} = \frac{5}{15}$

i.e the co-efficients of x, y and z in both eqn's are proportional and hence the planes are parallel.

To find the $\perp$ distance by 2nd method

The eqn of 1st plane is

$$3x+4y+5z+10=0$$

or $\frac{10-(\frac{20}{3})}{\sqrt{3^2+4^2+5^2}} = \frac{10}{3\cdot5\sqrt{2}}$

Its normal form is

$$\frac{3}{\pm 5\sqrt{2}}x + \frac{4}{\pm 5\sqrt{2}}y + \frac{5}{\pm 5\sqrt{2}}z = \frac{-10}{\pm 5\sqrt{2}}$$

i.e $\frac{3x}{\pm 5\sqrt{2}} + \frac{4y}{5\sqrt{2}} + \frac{z}{\pm\sqrt{2}} = \frac{-2}{\pm\sqrt{2}}$

Since $d = 10 > 0$

∵ We take −ve square roots

$$\therefore \frac{3x}{-5\sqrt{2}} + \frac{4y}{-5\sqrt{2}} + \frac{z}{-\sqrt{2}} = \sqrt{2} = p_1$$

Similarly $p_2 = -20/\pm 15\sqrt{2} = \frac{-20}{-15\sqrt{2}} = \frac{2\sqrt{2}}{3}$.

2nd method : ⊥ distance from origin to the first plane

$$= \frac{10}{\sqrt{50}} = \frac{10}{5\sqrt{2}} = \sqrt{2}$$

∴ ⊥ distance between two plane $\sqrt{2} - \frac{2\sqrt{2}}{3} = \frac{\sqrt{2}}{3}$

⊥ distance from origin to the 2nd plane $= \frac{20}{15\sqrt{2}} = \frac{2\sqrt{2}}{3}$

or $p_2 = \frac{-20}{\pm 15\sqrt{2}} = \frac{-20}{-15\sqrt{2}} = \frac{2\sqrt{2}}{3}$.

∴ ⊥ distance between two planes $= \sqrt{2} - \frac{2\sqrt{2}}{3} = \frac{\sqrt{2}}{3}$

1st method : (Trick → Take any two point as 0,0 and find the third point)

Take a point $(0,0,-2)$ on 1st plane ⊥ dist. on 2nd plane $= \frac{|-30+20|}{15\sqrt{2}} = \frac{10}{15\sqrt{2}} = \frac{\sqrt{2}}{3}$

Note 3:

To test whether origin lies in the acute angles or obtuse angle between the planes.

If the origin lies in the acute angle between the planes the angle between the normals to the planes is obtuse and hence the value of $\cos\theta$ is $-ve$.

i.e $A_1A_2 + B_1B_2 + C_1C_2$ is $-ve$.

Similarly if origin lies in the obtuse angle between the planes then $A_1A_2 + B_1B_2 + C_1C_2$ is $+ve$.

While calculating $A_1A_2 + B_1B_2 + C_1C_2$ we should note that the constant terms in both of them are $+ve$.

The eq^n of bisector plane bisecting the angle between the planes containing origin is

$$\frac{A_1x + B_1y + C_1z + D_1}{\sqrt{A_1^2 + B_1^2 + C_1^2}} = \frac{A_2x + B_2y + C_2z + D_2}{\sqrt{A_2^2 + B_2^2 + C_2^2}}$$

Hence the eqⁿ of bisector plane bisecting the angle between the planes which does not contain origin is

$$\frac{A_1x+B_1y+C_1z+D_1}{\sqrt{A_1^2+B_1^2+C_1^2}} = -\frac{(A_2x+B_2y+C_2z+D_2)}{\sqrt{A_2^2+B_2^2+C_2^2}}$$

9 (a) (i) One bisector is $11x+6y+5z+86=0$

The angle between these bisectors and $3x-6y+2z+5=0$, $\cos\theta = ?$,

$\sin\theta = ?$, $\tan\theta = \;\; > 1$

$\therefore \theta > 45°$, $2\theta > 90°$

This bisects obtuse angle.

$\therefore$ Other " acute "

(b) Adjust the eqⁿs such that the constant terms are +ve

Hence $-x-2y-2z+9=0$

$4x-3y+12z+13=0$

Hence $A_1A_2+B_1B_2+C_1C_2$

$= (-1)4 + (-2)(-3) + (-2)12$

$= -22 < 0$

∴ $\cos\theta$ is -ve where θ is angle between the normals.

∴ Angle θ between the normal is obtuse.

∴ Angle between two planes is acute.

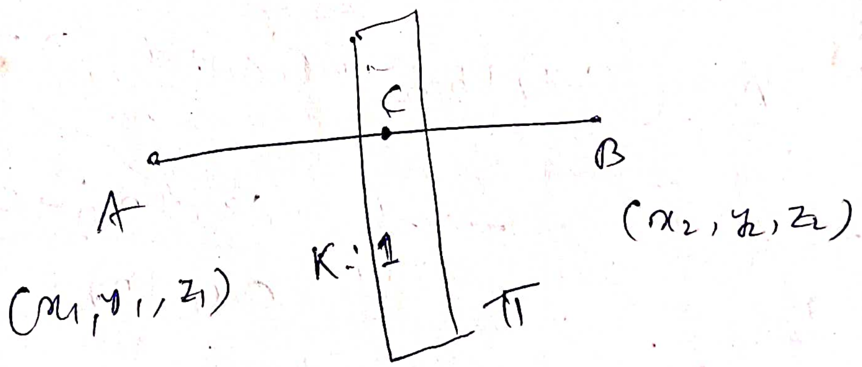
∴ origin lies ~~...~~ in acute angle

The eqn of required bisector plane is

$$\frac{-x-2y-2z+9}{3} = + \left\{\frac{4x-3y+12z+13}{13}\right\}$$

⟶ Calculate.

11. [19?] [an]



Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be two given points. The plane π intersects AB at C in the ratio K : 1 (Say)

Hence C is the point

$$\left(\frac{Kx_2+x_1}{K+1}, \frac{Ky_2+y_1}{K+1}, \frac{Kz_2+z_1}{K+1} \right)$$

Since C lies on the plane π

it satisfies the eqn of plane

Hence $a\left(\frac{Kx_2+x_1}{K+1}\right) + b\left(\frac{Ky_2+y_1}{K+1}\right)$

$$+ c\left(\frac{Kz_2+z_1}{K+1}\right) + d = 0$$

$\Rightarrow a(Kx_2+x_1) + b(Ky_2+y_1) + c(Kz_2+z_1)$

$$+ d(K+1) = 0$$

$\Rightarrow K(ax_2+by_2+cz_2+d) = -(ax_1+by_1+cz_1+d)$

$$\Rightarrow K = -\left(\frac{ax_1+by_1+cz_1+d}{ax_2+by_2+cz_2+d}\right)$$

(proved)

Q-) Find the area of the triangle, the co-ordinates of whose vertices are $(1,2,3)$ $(-2,1,-4)$ $(3,4,-2)$

Soln :- Co-ordinates of the points of projection on yz-plane of the 3 vertices are

(0, 2, 3), (0, 1, -4) (0, 4, -2)

In 2-dimensional Geometry then pts are (2, 3), (1, -4), (4, -2)

Area of then $\Delta = A_x = \frac{1}{2}\begin{vmatrix} 2 & 3 & 1 \\ 1 & -4 & 1 \\ 4 & -2 & 1 \end{vmatrix}$

$= \frac{19}{2}$

Similarly $A_y = \frac{29}{2}$, $A_z = -2$

Required Area $= \sqrt{A_x^2 + A_y^2 + A_z^2}$

$= \frac{\sqrt{1218}}{2}$

—:—

Straight line

① General form or unsymmetrical form

We know that the intersection of two planes is a st. line. Hence the eqn of st. line in space can be given by the joint eqn of two planes.

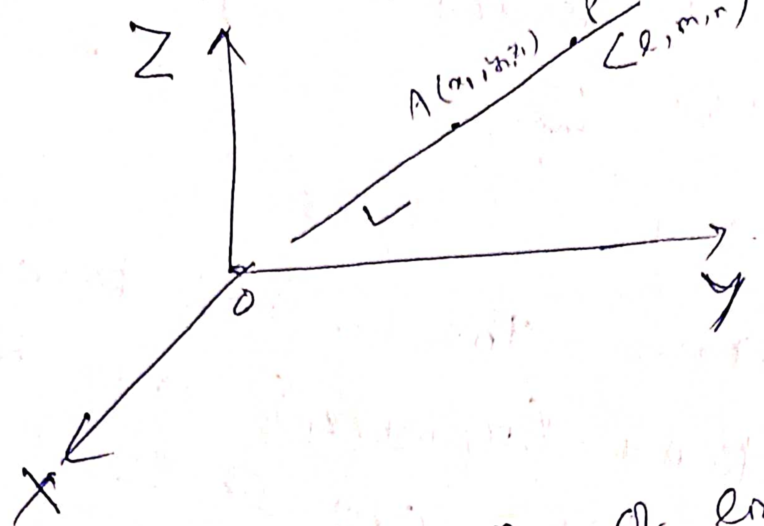
Hence the eqn of st. line in space is of the form:

$$\left.\begin{aligned} a_1x+b_1y+c_1z+d_1&=0\\ \text{and } a_2x+b_2y+c_2z+d_2&=0 \end{aligned}\right\}$$

It is called General form or unsymmetrical form or eqn of st. line or $\boxed{a_1x+b_1y+c_1z+d_1=0=a_2x+b_2y+c_2z+d_2}$

② Symmetrical form of eqn of st. line

Suppose a st. line L passes through a point $A(x_1, y_1, z_1)$ and with d.c.s l, m, n



To find the eqn of line L.

Let $P(x, y, z)$ be any point on the line L.

Now d.r.s of AP are $(x - x_1)$, $(y - y_1)$, $(z - z_1)$ of AP are the d.r.s

But d.r.s of line L

$\therefore$ d.r.s. of L are $(x - x_1), (y - y_1)$ $(z - z_1)$.

But d.c.s. of L are l, m, n.

$\therefore$ They are proportional.

$$\therefore \frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

Which is a relation between x, y, z and it is the required eqn of line L.

It is called symmetrical form of eqn L.

Notes:

1. Suppose the line L passes through the point (x_1, y_1, z_1) and has d.r.s is a, b, c, Then eqn of L is

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$$

2. The symmetrical form of eqn of L is

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} = r$$

$$= \frac{\sqrt{(x-x_1)^2+(y-y_1)^2+(z-z_1)^2}}{1}$$

where l, m, n are d.cs.

Here r is the actual distance of any pt. (x, y, z) from given point $A(x_1, y_1, z_1)$

3. The symmetrical form of eqn of line L is

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} = r$$

$\Rightarrow$ $x = x_1 + lr$, $y = y_1 + mr$ /

$z = z_1 + nr$.

Hence any point on the line L is of the form (x_1+lr, y_1+mr, z_1+nr)

(3) Two-point form

Suppose a line L passes through $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$. To find the eqn of L. The d.r.s. of L are d.r.s. of AB

Hence d.r.s of L are $x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$

Also the line L passes through $A(x_1, y_1, z_1)$.

Hence c.h eqn is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

~~Hence~~ 9.7 is called two-point form or eqn of line L.

Imp. question

Transformation of Unsymmetrical form to symmetrical form

Q→ Find in symmetrical form of the line eqn of a line

$$x+y+z+1=0=4x+y-2z+2$$ and

find its d.cs

Soln :-

Let l, m, n, be the d.cs of the given line L. Its eqn is $x+y+z+1=0$
$= 4x+y-2z+2$

i.e the line L is the intersection of the two planes $x+y+z+1=0$

and $4x+y-2z+2=0$

Hence the line L lies on both planes.

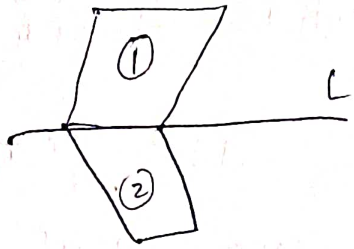
Now D.rs. of the normal to the 1st plane are $1, 1, 1$

and D.r.s of the normal to the 2nd plane are $(4, 1, -2)$

Now line L is $\perp$ to the normals to the 1st plane and 2nd plane

$\therefore \; l + m + n = 0$ —— (1)

$4l + m - 2n = 0$ —— (2)



Solving by cross-multiplication method, we get

$$\frac{l}{-3} = \frac{m}{6} = \frac{n}{-3}$$

$$\Rightarrow \frac{l}{-1} = \frac{m}{2} = \frac{n}{-1} = \frac{\sqrt{l^2+m^2+n^2}}{\sqrt{6}} = \frac{1}{\sqrt{6}}$$

$\therefore \; l = \frac{-1}{\sqrt{6}}, \quad m = \frac{2}{\sqrt{6}} = \frac{\sqrt{2}}{\sqrt{3}}, \quad n = \frac{-1}{\sqrt{6}}$

Also d r s are $-1, 2, -1$

putting $z = 0$, we have $x + y + 1 = 0$

$4x + y + 2 = 0$

Solving $3x + 1 = 0 \Rightarrow x = \frac{-1}{3}$

$y = \frac{-2}{3}$

Hence the point is $\left(-\frac{1}{3}, -\frac{2}{3}, 0\right)$

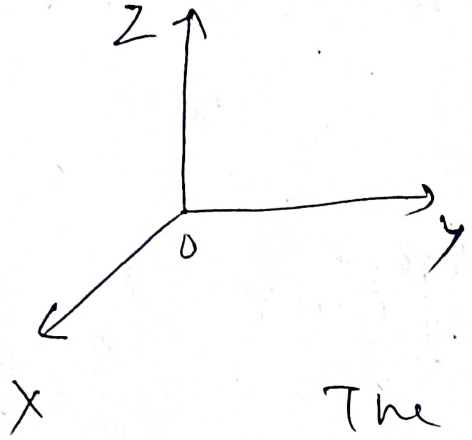
which lies on the line L

Hence the eqn of L is

$$\frac{x + \frac{1}{3}}{-1} = \frac{y + \frac{2}{3}}{2} = \frac{z}{-1}$$

Notes :-

① The eqⁿ of x-axis in unsymmetrical form is $y=0,\ z=0$



The eqⁿ of y-axis in unsymmetrical form $x=0,\ z=0$

The eqⁿ of z-axis in unsymmetrical form $x=0,\ y=0$

② The eqⁿ of x-axis in symmetrical form

X-axis passes through $(0,0,0)$ & having d.cs $1,0,0$. The eqⁿ of x-axis in symmetrical form is

$$\frac{x-0}{1} = \frac{y-0}{0} = \frac{z-0}{0}$$

i.e $\frac{x}{1} = \frac{y}{0} = \frac{z}{0}$

Similarly the eqⁿ of y-axis

$$\frac{x}{0} = \frac{y}{1} = \frac{z}{0}$$

The eqn of z axis

is $\frac{x}{0} = \frac{y}{0} = \frac{z}{1}$

Condition that a line will lie on a plane

Suppose the line L has the eqn

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$$

and the eqn of a plane π is

$$Ax + By + Cz + D = 0$$

Suppose the line L lies on the plane π.

Hence the line L is $\perp$ to the normal to plane π

D.r.s of L are l, m, n and D.r.s of normal are A, B, C

Hence $\boxed{Al + Bm + Cn = 0}$

Again (x_1, y_1, z_1) lies on the line L and L lies on plane π

$\therefore$ (x_1, y_1, z_1) lies on the plane and hence satisfies the eqn of plane.

$\therefore \boxed{Ax_1 + By_1 + Cz_1 + D = 0}$ —— (2)

(1) and (2) are the required condition.

Conditions for two lines to be Co-planner 4.30 P.M

Case-1 Suppose both lines are in symmetric form

Suppose the eqn of two lines be

$$\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_2}{n_1} \text{ and}$$

$$\frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2}$$

Let both lines will lie on a plane $Ax + By + Cz + D = 0$.

Since the 1st line lies on the plane

$\therefore Al_1 + Bm_1 + Cn_1 = 0$ —— (1)

$Ax_1 + By_1 + Cz_1 + D = 0$ —— (2)

Since 2nd line lies on the plane.

$\therefore Al_2 + Bm_2 + Cn_2 = 0$ — (3) and

$Ax_2 + By_2 + Cz_2 + D = 0$ — (4)

From (2) and (4)

$$A(x_2 - x_1) + B(y_2 - y_1) + C(z_2 - z_1) = 0$$

Eliminating A, B, C from the eqns (1); (3) and (5) we get

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$ which

is the required ~~Condi~~ Condition

Case-II Suppose one line is in form and other in unsymmetrical form.

Symmetrical form.

i.e $\dfrac{x - x_1}{l} = \dfrac{y - y_1}{m} = \dfrac{z - z_2}{n}$ and

$ax + by + cz + d = 0$, $= a'x + b'y + c'z + d'$

Now from the 1st line

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z_1 - z_2}{n} = r$$

$\Rightarrow x = x_1 + lr$, $y = y_1 + mr$, $z = z_1 + nr$

The ~~coordination~~ co-ordinates will satisfy the eq^n of 2nd line

i.e $ax+by+cz+d=0 \Rightarrow a(x_1+lr)+b(y_1+mr)$
$+c(z_1+nr)+d=0$

$$\Rightarrow r = -\frac{(ax_1+by_1+cz_1+d)}{al+bm+cn}$$

Also $a'x+b'y+c'z+d'=0$

$\Rightarrow a'(x_1+lr)+b'(y_1+mr)+c'(z_1+nr)$
$+d'=0$

$$\Rightarrow r = -\frac{(a'x_1+b'y_1+c'z_1)}{a'l+b'm+c'n}$$

Now equating the values of r we get

$$\frac{ax_1+by_1+cz_1+d}{al+bm+cn} = \frac{a'x_1+b'y_1+c'z_1+d'}{a'l+b'm+c'n}$$

which is the required condition

Case-III

Both lines are in unsymmetrical form

Suppose the lines are $a_1x+b_1y+c_1z+d_1=0$

$= a_2x + b_2y + c_2z + d_2$

and $a_3x + b_3y + c_3z + d_3 = 0 = a_4x + b_4y + c_4z + d_4$

If the two lines are Co planner then their point of intersection will satisfy all the 4 eqn's provided they will intersect at finite pt.

Then the pt. of intersection will satisfy the 4 eqns.

Eliminating x, y, z from the 4 eqn's we get the required Condition as

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = 0$$

Actually to find this 4th order determinant, it is difficult, so we convert both eqns to symmetric form and proceed as in Case I

How to find the point of intersection of two intersecting lines given in symm-form

Suppose the two lines are

$$\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1} = r_1$$

$$\Rightarrow x = x_1 + l_1 r_1, \quad y = y_1 + m_1 r_1, \quad z = z_1 + n_1 r_1$$

and

$$\frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2} = r_2$$

$$\Rightarrow x = x_2 + l_2 r_2, \quad y = y_2 + m_2 r_2,$$

$$z = z_2 + n_2 r_2$$

Since they intersects

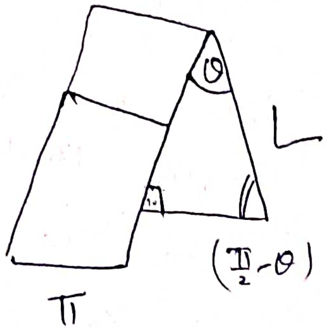
$$\therefore \quad x_1 + l_1 r_1 = x_2 + l_2 r_2$$

$$y_1 + m_1 r_1 = y_2 + m_2 r_2$$

$$z_1 + n_1 r_1 = z_2 + n_2 r_2$$

Solving the first two eqns we get r_1 and r_2. Putting the values of r_1 and r_2 in 3rd eqn, if we see that the 3rd eqn is satisfied then the two lines intersects and for the point of ~~contact~~ intersection take r_1 and r_2

Angle between a line and a plane



Let the line L be

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

and the plane π be $Ax+By+Cz+D=0$

Let the angle between them be θ.

The angle between the line and the normal to the plane is $(\pi/2-\theta)$

Now d.r.s of normal to the plane are A, B, C

D.r.s of the line are a, b, c

$$\therefore \cos\left(\frac{\pi}{2}-\theta\right) = \frac{Aa+Bb+Cc}{\pm\sqrt{A^2+B^2+C^2}\sqrt{a^2+b^2+c^2}}$$

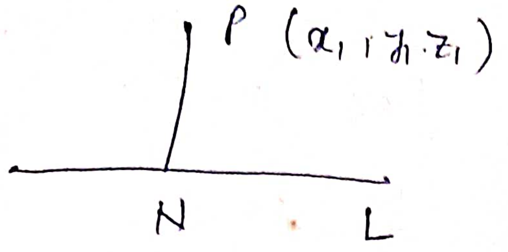
$$\Rightarrow \sin\theta = \frac{Aa+Bb+Cc}{\pm\sqrt{A^2+B^2+C^2}\sqrt{a^2+b^2+c^2}}$$

Note: If the line ∥ to the plane

$\sin\theta = 0$

$\therefore Aa+Bb+Cc=0$

Distance of a point form a line :



Let $P(x_1, y_1, z_1)$ be an external point and L be the line.

$$\frac{x-x_2}{a} = \frac{y-y_2}{b} = \frac{z-z_2}{c} = r$$

$\Rightarrow x = x_2 + ar, \quad y = y_2 + br, \quad z = z_2 + cr$

are the co-ordinates of any point on L

Suppose pt is N.

$\therefore$ Drs of PN are $x_2 + ar - x_1$,

$y_2 + br - y_1$, $z_2 + cr - z_1$

Drs of L are a, b, c.

Since PN $\perp$ line L

$\therefore a(x_2 + ar - x_1) + b(y_2 + br - y_1)$

$+ c(z_2 + cr - z_1) = 0$

which gives the value of r.

$\therefore$ The point N is $(x_2 + ar, y_2 + br,$

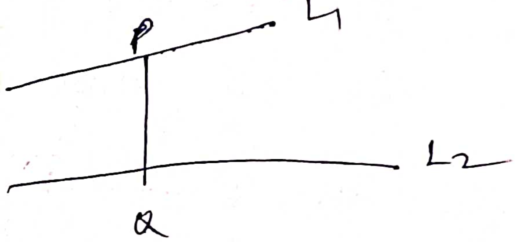
$z_2 + cr)$

Then find the $\perp$ dist. PN.

Probability and statistics Khatri

Skew lines

In space there are lines which donot intersect at all (neither at finite point or at infinite pt.) Such lines are called Skew lines and these lines are not co-planer.



PQ is the common ⊥ between the two skew lines L_1 and L_2.

PQ is called shortest distance (SD)

Here we can find the length and eqn of S.D.

Note:

How to find foot of ⊥ of a point on a line (e,f,g)

Take $\frac{x-a}{2} = \frac{y-b}{3} = \frac{z-c}{4} = r$

Find x, y, z in terms of r.

Find d.r.s subtracting x, y, z from $\boxed{e, f, g}$

$2.(d.r.s) + 3(d.r.s) + 4(d.r.s) = 0$

Find r, find x, y, z (see - 2nd sem elements)

Problems

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10. (a) Let A(1,2,3) & B(2,1,−1), C(−1,3,1) and D(3,1,5) be 4 points.

Eqn of line AB is

$$\frac{x-1}{2-1} = \frac{y-2}{1-2} = \frac{z-3}{-1-3}$$

$$\Rightarrow \frac{x-1}{1} = \frac{y-2}{-1} = \frac{z-3}{-4} = k_1$$

$$\Rightarrow x = k_1 + 1, \quad y = 2 - k_1, \quad z = 3 - 4k_1$$

Eqn of line CD is

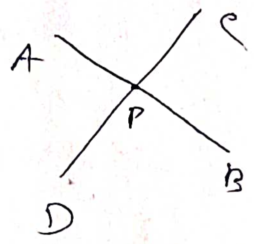
$$\frac{x+1}{3+1} = \frac{y-3}{1-3} = \frac{z-1}{5-1}$$

$$\Rightarrow \frac{x+1}{4} = \frac{y-3}{-2} = \frac{z-1}{4} = K_2$$

$$\Rightarrow x = 4K_2 - 1, \quad y = 3 - 2K_2, \quad z = 4K_2 + 1$$

If AB and CD intersect at P (say)

then for the point P,



$$K_1 + 1 = 4K_2 - 1 \quad \text{—— (1)}$$

$$2 - K_1 = 3 - 2K_2 \quad \text{—— (2)}$$

$$3 - 4K_1 = 4K_2 + 1 \quad \text{—— (3)}$$

From (1) and (2) $\quad 3 = 2K_2 + 2$

$$\Rightarrow \boxed{K_2 = \frac{1}{2}}$$

From (1) $\quad K_1 = 4K_2 - 2 = 0$

Putting $K_1 = 0$, $K_2 = \frac{1}{2}$ in eqn (3).

We see that $3 - 4K_1 = 3 = 4K_2 + 1$

i.e Eqn (3) is satisfied by $K_1 = 0$

and $K_2 = \frac{1}{2}$.

$\therefore$ The two lines AB and CD intersect.

and the point of intersection can be

obtained by taking $K_1 = 0$ i.e the point

P is $(1,2,3)$

15. The 1st line is

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$$x - mz - a = 0 = y - nz - b$$

$\Rightarrow x - a = mz$ & $y - b = nz$

$\Rightarrow \frac{x-a}{m} = \frac{z}{1}$ & $\frac{y-b}{n} = \frac{z}{1}$

$$\Rightarrow \frac{x-a}{m} = \frac{y-b}{n} = \frac{z}{1} = K_1$$

$\Rightarrow x = a + mK_1$, $y = b + nK_1$, $z = K_1$

The 2nd line is

$$x - m'z - a' = 0 = y - n'z - b'$$

$$\Rightarrow \frac{x-a'}{m'} = \frac{y-b'}{n'} = \frac{z}{1} = K_2$$

$\Rightarrow x = a' + m'K_2$, $y = b' + n'K_2$, $z = K_2$

If they will intersect at a common point P (say), then for the point P, we have

$a + mK_1 = a' + m'K_2$ ——— (1)

$b + nK_1 = b' + n'K_2$ ——— (2)

$K_1 = K_2$ ——— (3)

From (3) and (2), we have

$b + nK_1 = b' + n'K_1$

$\Rightarrow b - b' = K_1(n' - n)$

$\Rightarrow K_1 = \frac{b-b'}{n'-n} = K_2$

Since the lines intersects, eqn (1) is satisfied by the above values of K_1 and K_2

$$\therefore \quad a + \frac{m(b-b')}{n'-n} = a' + m' \frac{b-b'}{n'-n}.$$

$$\Leftrightarrow \quad a - a' = \frac{(b-b')}{(n'-n)} (m'-m)$$

$$\Leftrightarrow \quad (a-a')(n'-n) = (b-b')(m'-m)$$

$$\Leftrightarrow \quad (a-a')(n-n') = (b-b')(m-m')$$

(Proved)

7. (a) Eqn's of line passing through (a_1, b_1, c_1) and (a_2, b_2, c_2) is

$$\frac{x-a_1}{a_2-a_1} = \frac{y-b_1}{b_2-b_1} = \frac{z-c_1}{c_2-c_1}$$

Since it passes through $(0,0,0)$, we have

$$\frac{-a_1}{a_2-a_1} = \frac{-b_1}{b_2-b_1} = -\frac{c_1}{c_2-c_1}$$

$$\Rightarrow \frac{a_2-a_1}{-a_1} = \frac{b_2-b_1}{-b_1} = \frac{c_2-c_1}{-c_1}$$

$$\Rightarrow -\frac{a_2}{a_1} + 1 = -\frac{b_2}{b_1} + 1 = -\frac{c_2}{c_1} + 1$$

$$\Rightarrow \frac{a_2}{a_1} = \frac{b_2}{b_1} = \frac{c_2}{c_1}$$

$$\Rightarrow a_2 b_1 - b_2 a_1 = 0 \quad \& \quad b_2 c_1 - c_2 b_1 = 0$$

$$\& \quad a_2 c_1 - c_2 a_1 = 0$$

Now $p_1 = \perp$ distance from origin to (a_1, b_1, c_1)

$$= \sqrt{a_1^2 + b_1^2 + c_1^2}$$

$$p_2 = \sqrt{a_2^2 + b_2^2 + c_2^2}$$

$$p_1^2 p_2^2 = (a_1^2 + b_1^2 + c_1^2)(a_2^2 + b_2^2 + c_2^2)$$

$$= a_1^2 a_2^2 + a_1^2 b_2^2 + a_1^2 c_2^2 + b_1^2 a_2^2 + b_1^2 b_2^2 + b_1^2 c_2^2 + c_1^2 a_2^2 + c_1^2 b_2^2 + c_1^2 c_2^2.$$

$$= (a_2 b_1 - b_2 a_1)^2 + 2 a_1 a_2 b_1 b_2 + (b_2 c_1 - c_2 b_1)^2 + 2 b_1 b_2 c_1 c_2 + (a_2 c_1 - c_2 a_1)^2 + 2 a_1 a_2 c_1 c_2 + a_1^2 a_2^2 + b_1^2 b_2^2 + c_1^2 c_2^2$$

$$= 0 + 2 a_1 a_2 b_1 b_2 + 0 + 2 b_1 b_2 c_1 c_2 + 0 + 2 a_1 a_2 c_1 c_2 + a_1^2 a_2^2 + b_1^2 b_2^2 + c_1^2 c_2^2$$

$$= (a_1 a_2 + b_1 b_2 + c_1 c_2)^2$$

(Lagrange's identity)

$$(a_1^2 + b_1^2 + c_1^2) \cdot (a_2^2 + b_2^2 + c_2^2) = (a_1 a_2 + b_1 b_2 + c_1 c_2)^2$$

$$\Rightarrow a_1 a_2 + b_1 b_2 + c_1 c_2 = p_1 p_2$$

(Proved)

The Sphere (Defn)

A sphere is the locus of a point which moves in the space such that it is always at a constant distance from a fixed point.

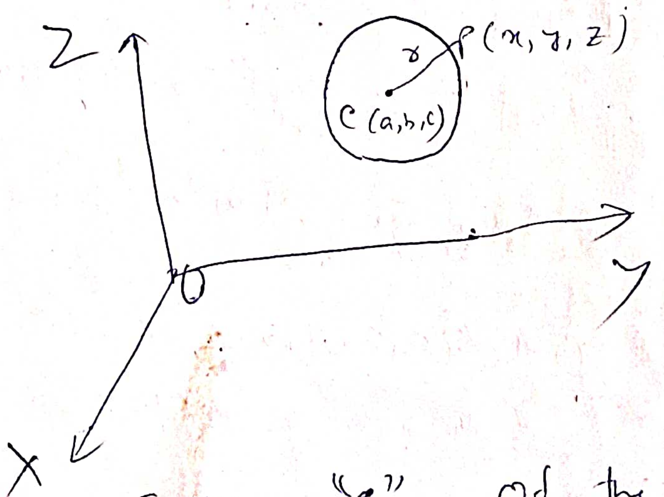
The fixed point is called centre of the sphere and the constant distance is called radius of the sphere.

Eqn of sphere in different form

① **Standard form** ÷

The eqn of sphere with centre at (a, b, c) and radius r is given by $(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$

Proof :



Let the centre "c" of the sphere be (a, b, c) and radius be "r".

Let $P(x, y, z)$ be any point on the

surface of the sphere.

Join CP. Hence $|CP| = r$

$$\Rightarrow \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} = r$$

$$\Rightarrow (x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$$

which is a relation between x, y & z and is the required eqn of sphere.

(proved)

Note:-

The eqn of sphere with centre at origin and radius r is

$$x^2 + y^2 + z^2 = r^2$$

② **General form:**

The 2nd degree of eqn in (x, y, z) of the form $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

represents a sphere.

Proof:-

The given eqn is

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \quad \text{---(i)}$$

$$\Rightarrow x^2 + 2ux + u^2 + y^2 + 2vy + v^2 + z^2 + 2wz + w^2 = u^2 + v^2 + w^2 - d$$

$\Rightarrow (x+u)^2 + (y+v)^2 + (z+w)^2 = u^2+v^2+w^2-d$

$$\Rightarrow \{x-(-u)\}^2 + \{y-(-v)\}^2 + \{z-(-w)\}^2 = \left(\sqrt{u^2+v^2+w^2-d}\right)^2$$

which is a sphere ~~centre~~ with Centre $(-u,-v,-w)$ and radius $\sqrt{u^2+v^2+w^2-d}$. (proved)

Note :-

From the general eqn of sphere we note the following observation,

① The eqn of sphere is 2nd degree eqn in x, y, z.

② Co-efficient x^2 = Co-efficient of y^2 = Co-efficient of z^2

③ The product terms xy, yz and zx are absent.

④ There are 4 unknown parameters u, v, w, d in the eqn of sphere and hence 4 non-coplanar points are required to get a unique sphere.

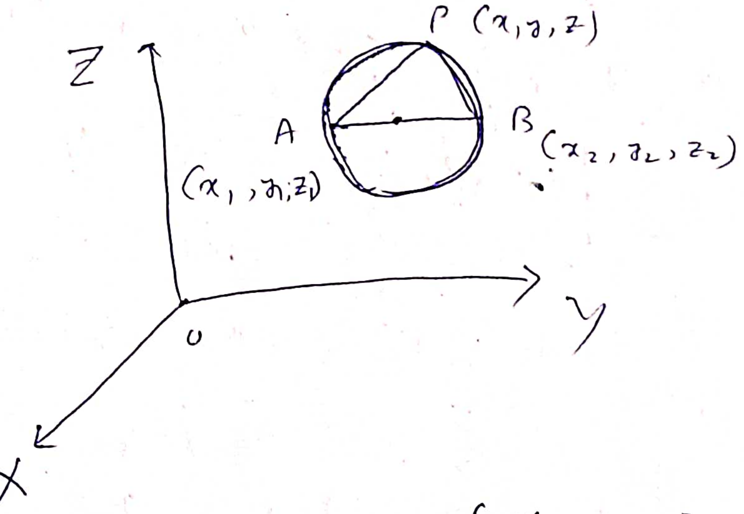
Note: If the sphere passes through origin then the constant term i.e $d = 0$ $\left(\because x^2+y^2+z^2+2ux+2vy+2wz+d=0 \right.$ $0+0+0+0+0+0+d=0$ $\Rightarrow d=0$ ($\because$ it passes through $(0,0,0)$) $\left.\right)$

③ **Diamete form**

The eqn of sphere with $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ as the end points of diameter is

$$(x-x_1)(x-x_2) + (y-y_1)(y-y_2) + (z-z_1)(z-z_2) = 0$$

Proof



Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be two end points of diameter AB. Let $P(x, y, z)$ be any point on the sphere. Join AP and BP.

d.r.s of AP are $x-x_1,\ y-y_1,\ z-z_1$

d.r.s of BP are $x-x_2,\ y-y_2,\ z-z_2$

Now $\angle APB = \frac{\pi}{2}$, ($\because$ Angle within a hemisphere is a right angle) i.e $AP \perp BP$.

i.e $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0$

which is a relation between x, y and z and hence it is the required eqn of sphere. (Proved)

(4) 4 - point form

The eqn of sphere through 4 non-coplanar points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) and (x_4, y_4, z_4) is

$$\begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ x_1^2+y_1^2+z_1^2 & x_1 & y_1 & z_1 & 1 \\ x_2^2+y_2^2+z_2^2 & x_2 & y_2 & z_2 & 1 \\ x_3^2+y_3^2+z_3^2 & x_3 & y_3 & z_3 & 1 \\ x_4^2+y_4^2+z_4^2 & x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$$

Proof: Let the eqn of sphere be $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ — (1)

Since it passes through (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) and (x_4, y_4, z_4), we get

$x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d=0$ — (2)

$x_2^2+y_2^2+z_2^2+2ux_2+2vy_2+2wz_2+d=0$ — (3)

$x_3^2+y_3^2+z_3^2+2ux_3+2vy_3+2wz_3+d=0$ — (4)

$x_4^2+y_4^2+z_4^2+2ux_4+2vy_4+2wz_4+d=0$ — (5)

Eliminating u, v, w, d from eq^ns (1), (2), (3), (4), (5), we get

$$\begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ x_1^2+y_1^2+z_1^2 & x_1 & y_1 & z_1 & 1 \\ x_2^2+y_2^2+z_2^2 & x_2 & y_2 & z_2 & 1 \\ x_3^2+y_3^2+z_3^2 & x_3 & y_3 & z_3 & 1 \\ x_4^2+y_4^2+z_4^2 & x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$$

Which is the required eq^n of sphere.

Note :-

To find the eq^n of sphere through 4 ~~[illegible]~~ non-coplanar point we are not advisible to use the above determinant form because of is difficult to expand 5×5 determinant. So we solve for u, v, w, d form eq^ns (2), (3), (4), (5) and put the values ~~[illegible]~~ in eq^n (1) to get the eq^n of sphere.

Q → Find the eq^n of sphere through the points $(0, 0, 0)$ $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$

Ans: Let the eq^n of sphere be

$x^2+y^2+z^2+2ux+2vy+2wz+d=0$

Since it passes through $(0, 0, 0)$, we get
$d=0$

Since it passes through $(a, 0, 0)$

we get $a^2+2ua=0$

$\Rightarrow u=-\frac{a}{2}$

Since it passes through $(0, b, 0)$

we get $b^2+2vb=0$
$\Rightarrow v=-\frac{b}{2}$

Since it passes through $(0, 0, c)$

we get $c^2+2wc=0$ $\Rightarrow w=-\frac{c}{2}$

Putting these values of u, v, w, d in eqn (1),

we get

$x^2+y^2+z^2-ax-by-cz=0$

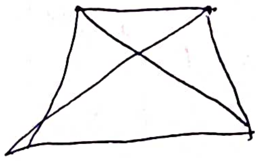
which is the required eqn of the sphere. The centre is $(\frac{a}{2}, \frac{b}{2}, \frac{c}{2})$ and radius $\sqrt{\frac{a^2}{4}+\frac{b^2}{4}+\frac{c^2}{4}}=\sqrt{a^2+b^2+c^2}/2$

Q→ Find the eqn of sphere circumscribing a tetrahedron bounded by the planes $x=0$, $y=0$, $z=0$ and $x+y+z=1$

For a vertex is intersection of 3 planes, edge intersection of 2 planes.

For a tetrahedron 4 vertices, 4 faces, 6 edges

Soln



The tetrahedron is bounded by the planes $x=0$, $y=0$, $z=0$ and $x+y+z=1$.

The vertices of the tetrahedron are obtained by solving the eqns $x=0$, $y=0$, $z=0$ and $x+y+z=1$ taken 3 at a time.

Taking $x=0$, $y=0$ & $z=0$ we get the 1st vertex as $(0,0,0)$

Taking $x=0$, $y=0$, $x+y+z=1$ we get the 2nd vertex as $(0,0,1)$

Taking $x=0$, $z=0$, $x+y+z=1$ we get the 3rd vertex as $(0,1,0)$

The last vertex is obtained as $(1,0,0)$ taking $y=0$, $z=0$ and $x+y+z=1$

The sphere circumscribing the ~~tetrad~~ ~~te~~ tetrahedron passes through these 4 vertices (0,0,0) (1,0,0) (0,1,0) (0,0,1)

The eqn of sphere is

$x^2+y^2+z^2-x-y-z=0$ (As proved in the last question)

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The faces of tetrahedron are

$x=0, \quad y=0, \quad z=0, \quad 2x+2y+z=1$

Now the length of the ⊥ from the in Centre to each face becomes the radius of the sphere. Let the radius of the sphere be "r".

The length of the ⊥ from centre to the plane $z=0$ is equal to r which is z-co-ordinate of the Centre. i.e $z=r$.

Similarly $x=r$ & $y=r$ for in Centre

i.e the centre is (r,r,r)

The length of ⊥ from (r,r,r) to the plane $2x+2y+z-1=0$ is "r" i.e $\dfrac{|2r+2r+r-1|}{\sqrt{4+4+1}} = r$

$\Rightarrow |5r-1| = 3r$

$\Rightarrow 5r-1 = \pm 3r$

$\Rightarrow 8r = 1$ or $2r = 1$

$\Rightarrow r = \frac{1}{8}$ or $r = \frac{1}{2}$

But $r \neq \frac{1}{2}$ because centre lies inside tetrahedron which has vertices $(0,0,0)$ $(0,0,1)$, $(\frac{1}{2},0,0)$ $(0,\frac{1}{2},0)$

$\therefore r = \frac{1}{8}$

Hence eqn of sphere is

$(x-\frac{1}{8})^2 + (y-\frac{1}{8})^2 + (z-\frac{1}{8})^2 = (\frac{1}{8})^2$

$\Rightarrow 32(x^2+y^2+z^2) - 8(x+y+z) + 1 = 0$.

7. Let us consider the centre of the cube as origin. Let the length of each side of cube be $2a$.

$\therefore$ The eqns of 6 ~~six~~ faces are

$x = \pm a$, $y = \pm a$, $z = \pm a$

Let the variable point be $P(x, y, z)$

Dist. of P from the plane $x = a$ is

$|x - a|$

Square of this dist $= (x-a)^2$

Similarly the square of the distances of P from the other planes are

$(x+a)^2$, $(y-a)^2$, $(y+a)^2$, $(z-a)^2$, $(z+a)^2$

But given that sum of these squares of distances $=$ Constant $= K^2$ (say)

i.e $(x-a)^2 + (x+a)^2 + (y-a)^2 + (y+a)^2$

$$+ (z-a)^2 + (z+a)^2 = K^2$$

$$\Rightarrow 2\left[x^2 + a^2 + y^2 + a^2 + z^2 + a^2\right] = K^2$$

$$\Rightarrow x^2 + y^2 + z^2 = \frac{K^2}{2} - 3a^2 = r^2$$ (Say)

Which is a relation between x, y, z and it is the required eqn of the locus.

Here this eqn is the eqn of

sphere with centre at $(0,0,0)$ and radius $r = \sqrt{\frac{K^2}{3} - 3a^2}$. Hence the locus is a sphere.

8. Let the eqn of plane be

$$\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 1 \quad \text{——— (1)}$$

It intersects the axes at

$P(\alpha, 0, 0)$, $Q(0, \beta, 0)$, $R(0, 0, \gamma)$

Since the plane passing through (a, b, c) we get, $\frac{a}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma} = 1$ —(2)

The eqn of sphere OPQR is

$$x^2 + y^2 + z^2 - \alpha x - \beta y - \gamma z = 0 \quad \left(\begin{matrix}\text{Since}\\\text{it}\end{matrix}\right)$$

(Since it passes P, Q, R & origin)

Its centre is $\left(\frac{\alpha}{2}, \frac{\beta}{2}, \frac{\gamma}{2}\right)$

Let its centre be (x, y, z)

$\therefore x = \frac{\alpha}{2}$, $y = \frac{\beta}{2}$, $z = \frac{\gamma}{2}$

$\Rightarrow \alpha = 2x$, $\beta = 2y$, $\gamma = 2z$

Eliminating α, β, γ from eqn (2),

we get $\frac{a}{2x} + \frac{b}{2y} + \frac{c}{2z} = 1$

$$\Rightarrow \frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 2$$

which is the required locus

of the centre (proved)

9. The sphere meets the co-ordinate axes at P, Q and R

Let P be the pt $(\alpha, 0, 0)$

Q " " " $(0, \beta, 0)$

R " " " $(0, 0, \gamma)$

Now the centroid of ΔPQR

is $\left(\frac{\alpha}{3}, \frac{\beta}{3}, \frac{\gamma}{3}\right)$

Now the sphere passes through O, P, Q, R.

i.e passes through $(\alpha, 0, 0)$, $(0, \beta, 0)$ $(0, 0, \gamma)$ and $(0, 0, 0)$

The eq^n of sphere is

$$x^2 + y^2 + z^2 - \alpha x - \beta y - \gamma z = 0$$

(Prove it)

Here radius $= \frac{\sqrt{\alpha^2 + \beta^2 + \gamma^2}}{2} = K$ (given)

$\Rightarrow \dfrac{\alpha^2+\beta^2+\gamma^2}{4} = k^2$

$\Rightarrow \alpha^2+\beta^2+\gamma^2 = 4k^2$

Let the centroid be (x, y, z)

where $x = \dfrac{\alpha}{3}$, $y = \dfrac{\beta}{3}$, $z = \dfrac{\gamma}{3}$

$\Rightarrow \alpha = 3x$, $\beta = 3y$, $\gamma = 3z$

Now $\alpha^2+\beta^2+\gamma^2 = 4k^2$

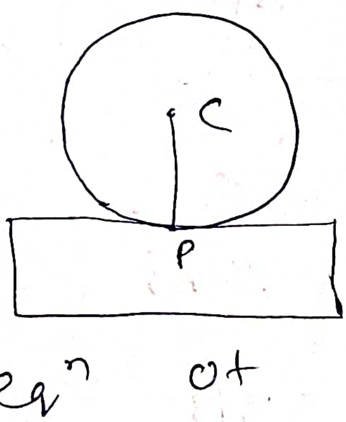
$\Rightarrow (3x)^2+(3y)^2+(3z)^2 = 4k^2$

$\Rightarrow 9(x^2+y^2+z^2) = 4k^2$

which is the required locus

(proved)

10.



The eq^n of sphere is

$x^2+y^2+z^2-2x-2y-2z-6=0$

It is of the form

$x^2+y^2+z^2+2ux+2vy+2wz+d=0$

where $u=-1$, $v=-1$, $w=-1$, $d=-6$

Centre is $(-u, -v, -w)$ i.e. $(1,1,1)$

& radius $= \sqrt{u^2+v^2+w^2-d}$

$= \sqrt{1+1+1+6}$

$= 3$

The eqn of plane is $x+y+z-a=0$

Length of $\perp$ from centre $(1,1,1)$ to the plane

$$= \frac{|1+1+1-a|}{\sqrt{3}} = \frac{|3-a|}{\sqrt{3}}$$

Since the plane touches the sphere, the length of $\perp$ from the centre to the plane becomes the ~~radi~~ radius of the sphere

i.e $\frac{|3-a|}{\sqrt{3}} = 3$

$\Rightarrow |3-a| = 3\sqrt{3}$

$\Rightarrow |a-3| = 3\sqrt{3}$

$\Rightarrow a-3 = \pm 3\sqrt{3}$

$\Rightarrow a = 3 \pm 3\sqrt{3} = 3(1 \pm \sqrt{3})$

which is the required condition.

Problems st. line

11. The 1st given line is

$$\frac{x+3}{2} = \frac{y+5}{3} = \frac{z-7}{-3} = r_1$$

$\Rightarrow x = 2r_1 - 3$, $y = 3r_1 - 5$, $z = -3r_1 + 7$

Also the 2nd line is

$$\frac{x+1}{4} = \frac{y+1}{5} = \frac{z+1}{-1} = r_2$$

$\Rightarrow x = 4r_2 - 1$, $y = 5r_2 - 1$, $z = -r_2 - 1$

If the two lines will intersect at a pt P then for the pt P we have

$2r_1 - 3 = 4r_2 - 1$ —— (1)

$3r_1 - 5 = 5r_2 - 1$ —— (2)

$-3r_1 + 7 = -r_2 - 1$ —— (3)

Now from (2) and (3) we have

$2 = 4r_2 - 2$ (adding)

$\Rightarrow 4 = 4r_2 \Rightarrow r_2 = 1$

$\therefore 3r_1 - 5 = 5 - 1 = 4$

$\Rightarrow r_1 = 3$

Putting $r_1 = 3$, $r_2 = 1$ we see that eq^n (1) is satisfied.

$\therefore$ The two lines will intersect & the pt of Intersection P is

$(3,4,-2)$ (taking $r_2 = 1$)

Hence the two lines are coplanar.

The plane containing both lines will contain the pt P $(3,4,-2)$

Let the eqn of plane be

$$A(x-3) + B(y-4) + C(z+2) = 0 \quad \text{--- (4)}$$

Now A, B, C are d.r.s of its normal.

The normal is $\perp$ to both lines.

The d.r.s of 1st line are 2, 3, −3

" " 2nd line " 4, 5, −1

$$\therefore \quad 2A + 3B - 3C = 0 \quad \text{--- (5)}$$

$$4A + 5B - C = 0 \quad \text{--- (6)}$$

From 5 & 6

$$\frac{A}{-3+15} = \frac{B}{-12+2} = \frac{C}{10-12}$$

$\Rightarrow \frac{A}{12} = \frac{B}{-10} = \frac{C}{-2}$

$\Rightarrow \frac{A}{-6} = \frac{B}{5} = \frac{C}{1}$

$\therefore$ The eqn of plane is

$(-6)(x-3) + 5(y-4) + 1(z+2) = 0$

$\Rightarrow -6x + 18 + 5y - 20 + z + 2 = 0$

$\Rightarrow -6x + 5y + z = 0$

$\Rightarrow 6x - 5y - z = 0$ (Ans)

25. Let the eqn of the plane through origin be

$lx + my + nz = 0$

It is $\parallel$ to given line

$\frac{x-1}{2} = \frac{y+3}{-1} = \frac{z+1}{-2}$

$\therefore 2l - m - 2n = 0 \Rightarrow m = 2l - 2n$ — (i)

The distance of the plane from the line is the dist. of the pt $(1, -3, -1)$ from it.

$\therefore \frac{|l - 3m - n|}{\sqrt{l^2 + m^2 + n^2}} = \frac{5}{3}$

$$\Rightarrow \frac{l-3m-n}{\sqrt{l^2+m^2+n^2}} = \pm\frac{5}{3}$$

$$\Rightarrow 9(l-3m-n)^2 = 25(l^2+m^2+n^2)$$

$$\Rightarrow 9(l-6l+6n-n)^2 = 25(l^2+4l^2+4n^2-8ln+n^2)$$

$$\Rightarrow 9(-5l+5n)^2 = 25(5l^2+5n^2-8ln)$$

$$\Rightarrow 225l^2+125n^2-200ln = 9(25l^2+25n^2-50ln)$$

$$\Rightarrow 100l^2+100n^2-250ln = 0$$

$$\Rightarrow 10l^2+10n^2-25ln = 0$$

$$\Rightarrow 2l^2+2n^2-5ln = 0$$

$$\Rightarrow (2l-n)(l-2n) = 0 \Rightarrow 2l = n \text{ or } l = 2n$$

If $2l = n$, then $m = -n$, from (1)

$\therefore 2l = n = -m$

$\Rightarrow 2l = -m = n$

$$\Rightarrow \frac{l}{1} = \frac{m}{-2} = \frac{n}{2}$$

$\therefore$ The plane is $x-2y+2z=0$

If $l = 2n$, $m = l$

$\therefore l = m = 2n \Rightarrow \frac{l}{2} = \frac{m}{2} = \frac{n}{1}$

$\therefore$ The plane is $2x+2y+z=0$

18. (a) $x = 2y = 2z$

$\Rightarrow x-2y=0$ & $y-z=0$

$3x+4y-1=0=4x+5z-2$

The eq^n of line, intersecting the ~~the~~ two given lines is given by

$$\left.\begin{aligned}(x-2y)+\lambda_1(y-z)&=0\\(3x+4y-1)+\lambda_2(4x+5z-2)&=0\end{aligned}\right\}$$

Since it passes through $(1,0,-1)$ we get $(1+\lambda_1)=0 \Rightarrow \lambda_1=-1$

$2+\lambda_2(-3)=0 \Rightarrow \lambda_2=\frac{2}{3}$

$\therefore$ The line becomes;

$x-3y+z=0$

$17x+12y+10z-7=0$

Now to express it in symmetrical form, Let α, β, γ be its d.rs

$\therefore \alpha-3\beta+\gamma=0$

$17\alpha+12\beta+10\gamma=0$

$\Rightarrow \frac{\alpha}{-42}=\frac{\beta}{7}=\frac{\gamma}{63}$

$$\Rightarrow \frac{\alpha}{-6} = \frac{\beta}{1} = \frac{\gamma}{9}$$

Taking, $y=0$ we have

$$x+z=0$$

& $17x+10z-7=0$

$\therefore x=1, \quad z=-1$

$\therefore$ The point on the line is

$(1,0,-1)$

$\therefore$ The eqn of line in symmetrical form is

$$\frac{x-1}{-6} = \frac{y-0}{1} = \frac{z+1}{9}$$

18.(b) $\frac{x}{1} = \frac{y+a}{1} = \frac{z}{1} = r_1$ (say)

$\Rightarrow x = r_1, \quad y = r_1 - a, \quad z = r_1$

$\therefore$ Any point on the 1st line is $P(r_1, r_1-a, r_1)$

2nd line is $\frac{x+a}{2} = \frac{y}{1} = \frac{z}{1} = r_2$

$\Rightarrow x = 2r_2 - a, \quad y = r_2, \quad z = r_2$

$\therefore$ Any point on 2nd line is $Q(2r_2-a, r_2, r_2)$

$\therefore$ D.r.s of PQ are resp.

$2r_2 - r_1 - a, \quad r_2 - r_1 + a, \quad r_2 - r_1$

But D.r.s. of PQ are 2, 1, 2

$$\therefore \frac{2r_2 - r_1 - a}{2} = \frac{r_2 - r_1 + a}{1} = \frac{r_2 - r_1}{2}$$

Taking $\frac{2r_2 - r_1 - a}{2} = \frac{r_2 - r_1}{2}$

we get $r_2 = a$

Taking last two

$r_2 - r_1 + a = \frac{r_2 - r_1}{2}$

$\Rightarrow a - r_1 + a = \frac{a - r_1}{2}$

$\Rightarrow r_1 = 3a$

$\therefore$ P is $(3a, 2a, a)$

Q is (a, a, a)

(c) $\frac{x-5}{3} = \frac{y-7}{-1} = \frac{z+2}{1} = r_1$

$\Rightarrow x = 3r_1 + 5, \quad y = 7 - r_1, \quad z = r_1 - 2$

$\therefore$ Any point P is $(3r_1+5, 7-r_1, r_1-2)$

2nd line is $\frac{x+3}{-3} = \frac{y-3}{2} = \frac{z-6}{4} = r_2$

$\Rightarrow x = -3 - 3r_2, \quad y = 3 + 2r_2, \quad z = 6 + 4r_2$

$\therefore Q$ is $(-3-3r_2,\ 3+2r_2,\ 6+4r_2)$

Drs. of PQ $\quad -3r_1-3r_2-8,\ 2r_2+r_1-4,$

$-r_1+4r_2+8$

But Dr.rs are $2, 7, -5$

$$\therefore \frac{-8-3r_1-3r_2}{2} = \frac{2r_2+r_1-4}{7} = \frac{4r_2-r_1+8}{-5}$$

$$\Rightarrow \frac{8+3r_1+3r_2}{2} = \frac{4-r_1-2r_2}{7} = \frac{r_1-4r_2-8}{-5}$$

$$= \frac{(8+3r_1+3r_2)+3(4-r_1-2r_2)}{2+3\times7}$$

$$= \frac{20-3r_2}{23}$$

$$= \frac{(8+3r_1+3r_2)-3(r_1-4r_2-8)}{2+(-5)(-3)}$$

$$= \frac{32+15r_2}{17} = \frac{5(20-3r_2)+(32+15r_2)}{5\times23+17\times1}$$

$$= \frac{132}{132} = 1$$

$\therefore\ 20-3r_2 = 23 \Rightarrow r_2 = -1$

Taking $\dfrac{8+3r_1+3r_2}{2} = 1$

$\Rightarrow \dfrac{8+3\gamma_1-3}{2} = 1$

$\Rightarrow \gamma_1 = -1$

pts are $P(2, 8, -3)$
$Q(0, 1, 2)$

$\therefore |PQ| = \sqrt{4+49+25} = \sqrt{78}$ (Ans)

23. The eqn of plane is
$(3x+y+z-5) + \lambda(x-2y+4z+4) = 0$
Since it passes through $(2, 0, -3)$
$\therefore -6\lambda - 2 = 0 \Rightarrow \lambda = -\frac{1}{3}$

Eqn becomes $\frac{8}{3}x + \frac{5}{3}y - \frac{z}{3} - \frac{19}{3} = 0$

$\Rightarrow 8x + 5y - z - 19 = 0$

24. $x+2 = 2y-1$

$\Rightarrow x - 2y + 3 = 0$

$3z = 2y - 1 \Rightarrow 2y - 3z - 1 = 0$

The plane containing the line is
$(x-2y+3) + \lambda(2y-3z-1) = 0$

$\Rightarrow x + (2\lambda-2)y - 3\lambda z + (3-\lambda) = 0$

Drs. of its normal are $1, 2\lambda-2, -3\lambda$

Plane is // to line $x = 1-5y = 2z-7$

$$\Rightarrow \frac{x-1}{5} = \frac{y}{-1} = \frac{z-4}{5/2}$$

D. r's of this line are $5, -1, \frac{5}{2}$

Since they are $\perp$

$\therefore 5 - 2\lambda + 2 - \frac{15}{2}\lambda = 0 \Rightarrow \lambda = \frac{14}{19}$

$\therefore$ Eq^n of plane is

$$x + \left(2 \times \frac{14}{19} - 2\right)y - 3 \times \frac{14}{19} z + \left(3 - \frac{14}{19}\right) = 0$$

$$\Rightarrow 19x + 10y - 42z + 43 = 0$$

To find the S.D. express the both lines in symmetrical form

as $\frac{x+2}{1} = \frac{y-\frac{1}{2}}{\frac{1}{2}} = \frac{z}{\frac{1}{3}}$

& $\frac{x-1}{5} = \frac{y}{-1} = \frac{z-4}{\frac{5}{2}}$ and

proceed.

26. Let the eq^n of line L be

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$$

If is $\perp$ to $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-6}{7}$

$\therefore 3l + 4m + 7n = 0$ ——— (1)

Since the plane contains the line L

$\therefore l_1 - 2m + 4n = 0$ ——— (2)

& $x_1 - 2y_1 + 4z_1 - 51 = 0$ ——— (3)

From (1) and (2) $\frac{l}{30} = \frac{m}{-5} = \frac{n}{-10}$

$\Rightarrow \frac{l}{-6} = \frac{m}{1} = \frac{n}{2}$

$\boxed{\text{Take } x_1 = 5,\ y_1 = 3 \quad \therefore z_1 = 13}$? See Krita

$\therefore$ Eqn is $\frac{x-5}{-6} = \frac{y-3}{1} = \frac{z-13}{2}$

—o—

Sphere

4. (i)
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The feet of the $\perp$ drawn from the pt (2,3,6) on XY-plane, YZ-plane & ZX plane are (2,3,0), (0,3,6) (2,0,6) respectively.

Let the eqn of sphere be

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ —(1)

Since it passes through $(2,3,6)$ $(2,3,0)$, $(0,3,6)$ and $(2,0,6)$, we have

$4+9+36+4u+6v+12w+d=0$ — ②

$4+9+4u+6v+d=0$ — ③

$9+36+6v+12w+d=0$ — ④

$4+36+4u+12w+d=0$ — ⑤

From ② and ③

$36+12w=0 \Rightarrow w=-3$

From ② and ④ $4+4u=0 \Rightarrow u=-1$

From ② and ⑤ $9+6v=0 \Rightarrow v=-\frac{3}{2}$

From ③, $13-4-9+d=0$
$\Rightarrow d=0$

$\therefore$ The eqn of sphere is

$x^2+y^2+z^2-2x-3y-6z=0$

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No 14. Let the eqn of plane be $Ax+By+Cz+D=0$

It contains the parallel lines

$\frac{x-4}{1} = \frac{y-3}{-4} = \frac{z-2}{5}$ and

$\frac{x-3}{1} = \frac{y-3}{-4} = \frac{z-0}{5}$

$\therefore$ The plane contains the pt. s

$(4,3,2)$ and $(3,2,0)$ & its

normal is $\perp$ to both lines.

$\therefore$ $4A + 3B + 2C + D = 0$ ——— (1)

$3A + 3B + D = 0$ ——— (2)

The d.r.s of normal are A, B, C

$\therefore$ $A \times 1 + B(-4) + C \times 5 = 0$

$\Rightarrow A - 4B + 5C = 0$ ——— (3)

From (1) and (2)

$A + 2C = 0$

Now $A - 4B + 5C = 0$

$A + 0B + 2C = 0$.

Solving $\frac{A}{-8} = \frac{B}{5-2} = \frac{C}{4}$

$\Rightarrow \frac{A}{-8} = \frac{B}{3} = \frac{C}{4} = K$

$\therefore A = -8K \;, B = 3K \;, \quad C = 4K$

From eqⁿ $\quad -24K + 9K + D = 0$

$\Rightarrow D = 15K$

$\therefore$ The eqⁿ of plane is

$-8Kx + 3Ky + 4KZ + 15K = 0$

$\Rightarrow \quad 8x - 3y - 4Z - 15 = 0$

12. The given line is

$$\frac{x-1}{2} = \frac{y+3}{1} = \frac{Z}{1}$$

Taking 1st and last $\frac{x-1}{2} = \frac{Z}{1}$

$\Rightarrow x - 1 - 2Z = 0$ — ①

Taking last two

$$\frac{y+3}{1} = \frac{Z}{1}$$

$\Rightarrow y + 3 - Z = 0$

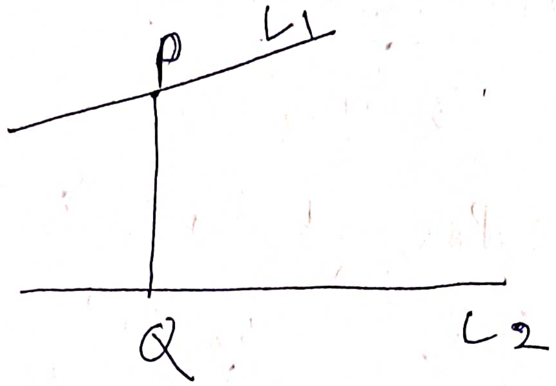
$\Rightarrow 2y + 6 - 2Z = 0$ ——— ②

Now general eqⁿ of plane is

$(x - 1 - 2Z) + \lambda(2y + 6 - 2Z) = 0$

$\Rightarrow x + 2\lambda y - 2Z(\lambda + 1) + 6\lambda - 1 = 0$

27. St. line



The eq^n of L_1 is $\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1} = r_1$

$\Rightarrow x = 3r_1 + 3, \quad y = 8 - r_1, \quad z = 3 + r_1$

Eq^n of line L_2 is

$$\frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4} = r_2$$

$$\Rightarrow x = -3r_2 - 3$$
$$y = -7 + 2r_2$$
$$z = 6 + 4r_2$$

Let $P(3r_1+3,\ 8-r_1,\ 3+r_1)$ and

$Q(-3r_2-3,\ -7+2r_2,\ 6+4r_2)$

be the end pts of the S.D. PQ.

Drs of PQ are

$-3r_2 - 3 - 3r_1 - 3,\ -7 + 2r_2 - 8 + r_1,$

$6 + 4r_2 - 3 - r_1$

i.e

$-3r_1 - 3r_2 - 6, \quad -15 + 2r_2 + r_1,$

$3 + 4r_2 - r_1$

Since PQ $\perp$ L_1 & d.r.s of

L_1 are $3, -1, 1$

$\therefore \quad 3(-3r_1 - 3r_2 - 6) - 1(-15 + 2r_2 + r_1)$

$+ 1(3 + 4r_2 - r_1) = 0$

$\Rightarrow \quad -11r_1 - 7r_2 = 0$

$\Rightarrow \quad 11r_1 + 7r_2 = 0 \quad \text{——} (1)$

Since d.r.s of L_2 are $(-3, 2, 4)$

& L_2 is $\perp$ PQ

$\therefore \quad -3(-3r_1 - 3r_2 - 6) + 2(-15 + 2r_2 + r_1)$

$+ 4(3 + 4r_2 - r_1) = 0$

$\Rightarrow \quad 7r_1 + 29r_2 = 0$

$\Rightarrow \quad r_1 = -\frac{29}{7} r_2$

Putting it in (1) we get

$11 \times \left(-\frac{29}{7}\right) r_2 + 7r_2 = 0$

$\Rightarrow r_2 = 0 \quad \Rightarrow r_1 = 0$

∴ P is the pt (3, 8, 3)

and Q is " " (−3, −7, 6)

$$|PQ| = \sqrt{(6)^2 + (15)^2 + (3)^2}$$

$$= \sqrt{270}$$

$$= 3\sqrt{30}$$

Eqⁿ of PQ is

$$\frac{x-3}{-3-3} = \frac{y-8}{-7-8} = \frac{z-3}{6-3}$$

$$\Rightarrow \frac{x-3}{-6} = \frac{y-8}{-15} = \frac{z-3}{3}$$

$$\Rightarrow \frac{x-3}{2} = \frac{y-8}{5} = \frac{z-3}{-1}$$

— 0 —