'Jay Jagannath'

Vector Algebra

Scalars and vector : A scalar is a quantity that is specified by magnitude only. It is represented by a real number ~~only~~ along with suitable units.

Ex : Length, mass, volume, density, temperature etc.

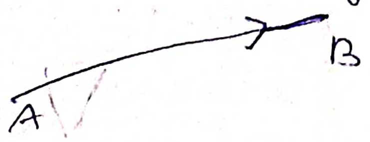
A vector is a quantity that is specified by both magnitude and direction. It is represented by a directed line segment.

Ex. : Force, displacement, velocity, accn etc.

Generally bold type letters as **A**, **B**, - - - etc are used to denote vectors and ordinary type letters like a, b, - - - etc are used to denote scalars. While writing vectors, arrows are put over letters ~~etc~~ i.e $\vec{a}, \vec{b}$ - - - etc.

A directed line segment is written in bold type as **AB** or $\overrightarrow{AB}$ where A is called initial point and B is called terminal point.

The length or magnitude of $\overrightarrow{AB}$ is written as $|\overrightarrow{AB}|$ or AB. The magnitude of $\vec{a}$ is denoted by $|\vec{a}|$ or a.



Zero vector

A vector whose magnitude is zero is called a null or zero vector. It is denoted by $\vec{O}$. The magnitude of $\vec{O}$ is constant but direction is ~~arbitarary~~ arbitrary. So it is not unique.

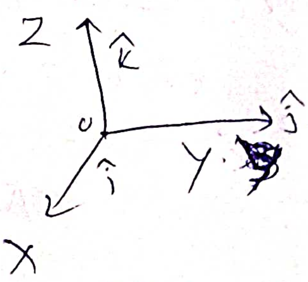
Unit vector :- A vector whose magnitude is unity is called unit vector. Here we put Cap ($\wedge$) over letters to denoted a unit vector.

Ex :- $\hat{a}, \hat{b}, \hat{c}$ - - - - etc.

The magnitude of a unit vector is constant but direction is arbitrary. So it is not unique.

Notes:-

① We denote $\hat{i}, \hat{j}, \hat{k}$ as unit vectors along X, Y, Z axes respectively. Thus $\hat{i}, \hat{j}, \hat{k}$ are constant vectors.



② Unit vector along a vector denotes the direction of the vector.

③ Suppose $\vec{a}$ is a vector, then $\vec{a} = |\vec{a}|\,\hat{a}$ where $\hat{a}$ is

Unit vector along $\vec{a}$.

$\Rightarrow$ $\boxed{\hat{a} = \dfrac{\vec{a}}{|\vec{a}|}}$

(4) Any vector $\vec{a}$ in space has 3 components along X, Y, Z axes and in plane has 2 components along X and Y axis.

Suppose X – component of $\vec{a} = a_1$

Y – " " " $= a_2$

Z – " " " $= a_3$

$\therefore \vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

or we write $\vec{a} = (a_1, a_2, a_3)$

In plane $\vec{a} = a_1\hat{i} + a_2\hat{j} = (a_1, a_2)$

(5) If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ then

$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

(6) Component of $\hat{i}$ are 1, 0, 0

" " $\hat{j}$ " 0, 1, 0

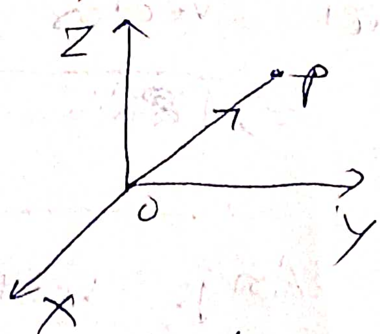
" " $\hat{k}$ " 0, 0, 1

(7) Suppose O is origin and P (x, y, z) is any point.

$\therefore$ The position vector P $= \vec{OP} = x\hat{i} + y\hat{j} + z\hat{k}$ – we denote the position vector as $\vec{r}$ i.e $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

8. $\vec{O} = 0\hat{i} + 0\hat{j} + 0\hat{k} = (0, 0, 0)$

Fig for note-7



Equality of two vectors

Two vectors $\vec{a}$ and $\vec{b}$ are said to be equal when they have the same magnitude. i.e $|\vec{a}| = |\vec{b}|$ and the same direction. we write $\vec{a} = \vec{b}$

Collinear vectors

Two or more vectors are called Collinear vectors vector when they have same line of action or they are parallel to the same line. If $\vec{a}$ and $\vec{b}$ are collinear then we write

$\vec{a} = K\vec{b}$ where K is scalar.

Coplaner vectors :-

Two or more than two vectors are said to be coplaner if they lie in the same plane or are parallel to the same plane. If $\vec{a}, \vec{b}, \vec{c}$ are co-planer then we write

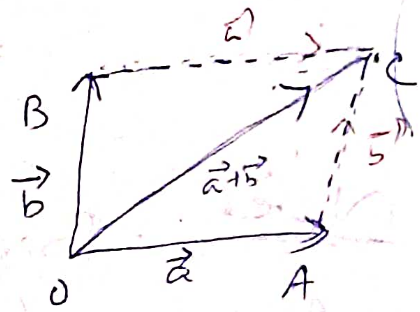
$$\vec{a} = \lambda\vec{b} + \mu\vec{c}$$

where λ and μ are scalars.

Addition of vectors

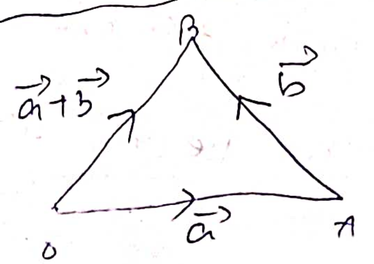
Parallelogram ~~or vectors~~ law for vector addition

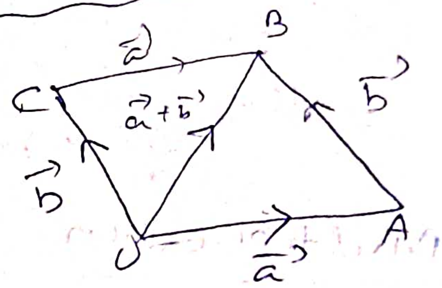
If $\vec{a}$ and $\vec{b}$ are represented as two adjacent side of a parallelogram $\vec{OA}$ and $\vec{OB}$ respectively then the diagonal $\vec{OC}$ represents the vector $\vec{a}+\vec{b}$



i.e $\vec{OA} + \vec{OB} = \vec{OC}$

Triangle law of addition





If $\vec{OA} = \vec{a}$, $\vec{AB} = \vec{b}$ then

$\vec{OB} = \vec{a}+\vec{b}$

i.e $\vec{OB} = \vec{OA} + \vec{AB}$

It is equivalant to parallelogram law because if we construct the ▱ OABC, then $\vec{OC} = \vec{AB} = \vec{b}$

and $\vec{OA} = \vec{a}$ $\therefore \vec{OB} = \vec{a}+\vec{b}$

Definition of addition :

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$\vec{a}+\vec{b} = (a_1+b_1)\hat{i} + (a_2+b_2)\hat{j} + (a_3+b_3)\hat{k}$

Also $\vec{a}+\vec{b} = \vec{b}+\vec{a}$ (Commutative law for addition ~~vectors~~)

Proof ÷ $\vec{a} + \vec{b} = (a_1+b_1)\hat{i} + (a_2+b_2)\hat{j} + (a_3+b_3)\hat{k}$

$= (b_1+a_1)\hat{i} + (b_2+a_2)\hat{j} + (b_3+a_3)\hat{k}$

Let $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ $\quad \vec{b} + \vec{a}$ (Proved)

Note: Prove the associative law for addition.

$\vec{a} + (\vec{b} + \vec{c}) = \vec{a} + (b_1+c_1)\hat{i} + (b_2+c_2)\hat{j} + (b_3+c_3)\hat{k}$

$= a_1\hat{i} + a_2\hat{j} + a_3\hat{k} + (b_1+c_1)\hat{i} + (b_2+c_2)\hat{j} + (b_3+c_3)\hat{k}$

$= (a_1\hat{i} + b_1\hat{i}) + (a_2\hat{j} + b_2\hat{j}) + (a_3\hat{k} + b_3\hat{k}) + c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

$= (a_1+b_1)\hat{i} + (a_2+b_2)\hat{j} + (a_3+b_3)\hat{k} + c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

$= (\vec{a} + \vec{b}) + \vec{c}$

Multiplication of a vector with a scalar →

Let $\vec{a}$ be a vector and

$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ then

$K\vec{a} = (Ka_1)\hat{i} + (Ka_2)\hat{j} + Ka_3\hat{k}$

Linear Combination (L.C)

Linearly ~~combination~~ Independent (L.I)

Linearly Dependent (L.D)

A vector $\vec{d}$ is ~~called~~ said to be a L.C of the vectors $\vec{a}, \vec{b}, \vec{c}$ if $\exists$ scalars l, m, n

Structure

$\vec{d} = l\vec{a} + m\vec{b} + n\vec{c}$

A system of vectors $\vec{a_1}, \vec{a_2}, \dots \vec{a_n}$

is said to be L.D if ∃ scalars $m_1, m_2, --- m_n$ not all zero such that

$$m_1 \vec{a_1} + m_2 \vec{a_2} + ------ + m_n \vec{a_n} = \vec{0}$$

A system of vectors $\vec{a_1}, \vec{a_2}, --- --- \vec{a_n}$ is said to be L.I if

$$m_1 \vec{a_1} + m_2 \vec{a_2} + ------ + m_n \vec{a_n} = \vec{0}$$

$$\Rightarrow m_1 = m_2 = ------ m_n = 0$$

Method :- $\boxed{\text{Topics mathematics } \boxed{\text{problem}}}$

(1) To prove that any 3 points A, B, C are collinear find $\vec{AB}$ and $\vec{BC}$ and show that $\vec{AB} = K\vec{BC}$ where K is scalar.

(2) To prove that any 3 vectors are coplaner, express $\vec{a}, \vec{b}, \vec{c}$ are say $\vec{a}$ as L.C of one of them other two. i.e $\vec{a} = K_1\vec{b} + K_2\vec{c}$ where K_1, K_2 are scalars.

(3) To prove that 4 points are coplaner. First A, B, C, D and then find $\vec{AB}, \vec{BC}, \vec{CD}$ and show that $\vec{AB} = K_1\vec{BC} + K_2\vec{CD}$ where K_1, K_2 are scalars.

(4) Suppose position vectors of A be $\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and position vector of B be $\vec{b} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$

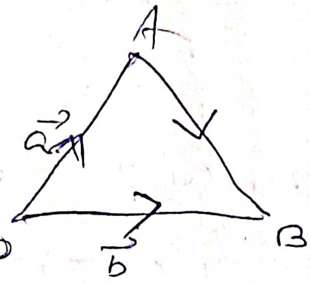
then $\vec{AB}$ = position vector B - position vector A

$= (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$

$= \vec{b} - \vec{a}$

<u>Proof</u> Now $\vec{OA} = \vec{a}$

$\vec{OB} = \vec{b}$



By Δ law of addition

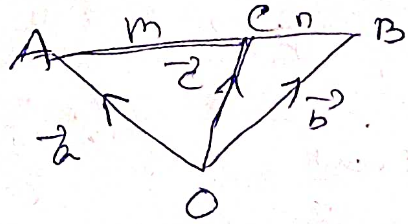
$\vec{OA} + \vec{AB} = \vec{OB}$

$\Rightarrow \vec{AB} = \vec{OB} - \vec{OA} = \vec{b} - \vec{a} \equiv$ position vector of B - position vector of A.

Section Formula

Let position vector of A be $\vec{a}$ and position vector of B be $\vec{b}$. Let C be a point which divides the line ~~externally~~ internally in the segment AB ratio $m:n$, then position vectors of C is $\dfrac{m\vec{b} + n\vec{a}}{m+n}$

<u>Proof</u> :-



Given that position vector of A is $\vec{a}$ and " " " B is $\vec{b}$, Let O be origin.

$\because \vec{OA} = \vec{a}$, $\vec{OB} = \vec{b}$ positions

Now C is the point with the vector $\vec{c}$ which divides AB internally in the ratio m : n i.e $\frac{AC}{CB} = \frac{m}{n}$

$\Rightarrow nAC = mCB$

$\Rightarrow n\vec{AC} = m\vec{CB}$ ($\because$ Direction of $\vec{AC}$ and $\vec{CB}$ are same)

$\Rightarrow n(\vec{c} - \vec{a}) = m(\vec{b} - \vec{c})$

($\because \vec{AC}$ = position vector of C − position vector of A
$= \vec{c} - \vec{a}$
$\vec{CB} = \vec{b} - \vec{c}$

$\Rightarrow \vec{c}(m+n) = m\vec{b} + n\vec{a}$

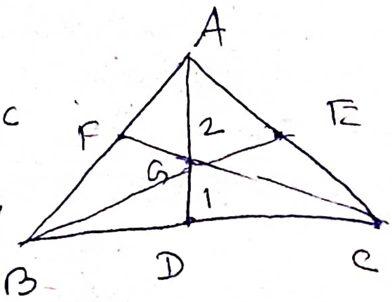
$= \dfrac{m\vec{b} + n\vec{a}}{m+n}$

$\Rightarrow \vec{c}$

$\therefore$ position vector of C $= \dfrac{m\vec{b} + n\vec{a}}{m+n}$.

Notes :- ① The mid point of AB has the position vector $\frac{\vec{a}+\vec{b}}{2}$ ($\because m = n = 1$)

② Let the position vector of A, B, C be $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively



position vector of $D = \dfrac{\vec{b} + \vec{c}}{2}$

But the centroid G divides AD in 2:1

$\therefore$ Position of $G = \dfrac{2\left(\frac{\vec{b}+\vec{c}}{2}\right)+1.\vec{a}}{2+1}$

$= \dfrac{\vec{a}+\vec{b}+\vec{c}}{3}$

Dot Product or Scalar product or inner product

Def^n : The dot product of two vectors $\vec{a}$ and $\vec{b}$ is denoted $\vec{a}.\vec{b}$ and is defined by

$\vec{a}.\vec{b} = |\vec{a}|.|\vec{b}| \cos\theta = ab\cos\theta$

where θ is the angle between the vectors $\vec{a}$ and $\vec{b}$

Component form :

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

then $\vec{a}.\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

Notes (1) $\cos\theta = \dfrac{\vec{a}.\vec{b}}{|\vec{a}|.|\vec{b}|}$

(2) If the two vectors $\vec{a}$ and $\vec{b}$ are $\perp$, then $\vec{a}.\vec{b} = 0$ $(\because \theta = \frac{\pi}{2} \Rightarrow \cos\theta = 0)$

(3) If the two vectors are parallel $\theta = 0 \Rightarrow \cos\theta = 1$

$\therefore \vec{a}.\vec{b} = |\vec{a}|.|\vec{b}|$

If $\vec{a} = \vec{b}$ then ~~$\vec{a}.\vec{a} = |\vec{a}|.|\vec{a}|$~~

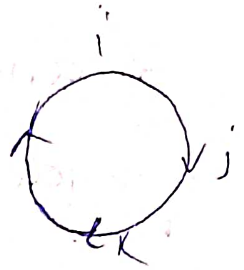
$$\vec{a}.\vec{a} = |\vec{a}|.|\vec{a}|$$

$$\boxed{\begin{aligned} \Rightarrow \vec{a}.\vec{a} &= (|\vec{a}|)^2 \\ \text{or } (\vec{a})^2 &= |\vec{a}|^2 \end{aligned}}$$

4. If $\hat{i}.\hat{i} = |i|^2 = 1$

$\hat{j}.\hat{j} = 1$

$\hat{k}.\hat{k} = 1$



$\hat{i}.\hat{j} = 0$, $\hat{j}.\hat{k} = 0$, $\hat{k}.\hat{i} = 0$

($\because$ They are mutually perpendicular)

(5) $\vec{a}.\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

$= b_1a_1 + b_2a_2 + b_3a_3$

$= \vec{b}.\vec{a}$

$\therefore$ Commutative law holds for dot product.

6. Null vector is $\perp$ to every other vector. (Because dot product is zero)

7. The dot product of two vector is scalar.

8. The dot product is distributive with respect to vector addition.

Prove: To prove that $\vec{a}.(\vec{b}+\vec{c}) = \vec{a}.\vec{b} + \vec{a}.\vec{c}$

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

$$\vec{b}+\vec{c} = (b_1+c_1)\hat{i} + (b_2+c_2)\hat{j} + (b_3+c_3)\hat{k}$$

$$\vec{a}\cdot(\vec{b}+\vec{c})$$

$$= a_1(b_1+c_1) + a_2(b_2+c_2) + a_3(b_3+c_3)$$

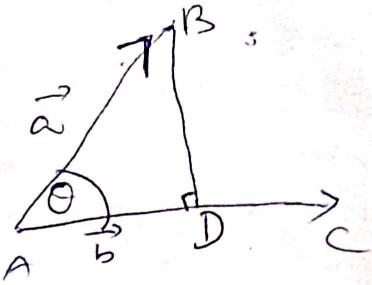
$$= a_1b_1 + a_1c_1 + a_2b_2 + a_2c_2 + a_3b_3 + a_3c_3$$

$$= (a_1b_1 + a_2b_2 + a_3b_3) + (a_1c_1 + a_2c_2 + a_3c_3)$$

$$= \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c} \quad \text{(proved)}$$

9) Scalar projection of a vector $\vec{a}$ on another vector $\vec{b}$ $= \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

<u>Proof</u>: Scalar projection of $\vec{a}$ on $\vec{b}$



$$= AD = AB\cos\theta$$

$$= |\vec{a}|\cos\theta$$

$$= |\vec{a}|\,\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|}$$

$$= \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$$

10) Vector projection of $\vec{a}$ on $\vec{b}$

$$= \vec{AD} = |\vec{AD}|\,\hat{b}$$

$$= AD\cdot\hat{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\cdot\hat{b}$$

Also component of $\vec{a}$ along $\vec{b}$ $= \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\hat{b} = \left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\right)\vec{b}$

where $\hat{b}$ is unit vector along $\vec{b}$

11) The workdone by a force $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ where

its point of application under goes a displacement.

$$\vec{d} = d_1\hat{i} + d_2\hat{j} + d_3\hat{k}$$ is given by

$$\vec{F}\cdot\vec{d} = F_1d_1 + F_2d_2 + F_3d_3$$

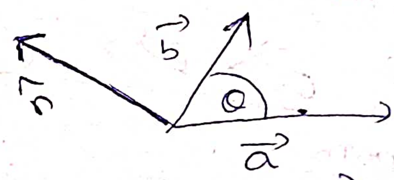
12) $K.E = \frac{1}{2}mv^2$

Cross product or vector product

The cross product of two vectors $\vec{a}$ & $\vec{b}$ written as $\vec{a}\times\vec{b}$ and is defined as a vector $\vec{a}\times\vec{b}$ ~~$=(\vec{a}\cdot\vec{b})$~~

$$= |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$$

where θ is the angle between $\vec{a}$ and $\vec{b}$ and $\hat{n}$ is the unit vector $\perp$ to the plane containing both $\vec{a}$ and $\vec{b}$, such that the vectors $\vec{a}, \vec{b}, n$ forms a right-handed system.



Notes →

(1) The cross product of $\vec{a}$ and $\vec{b}$ is a vector quantity.

(2) The cross product is a three dimensional concept and hence it is not available for vectors in R^2.

(3) Component form.

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

then $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

(4) $\vec{a}\times\vec{b} = -(\vec{b}\times\vec{a})$

i.e. Cross product obeys the anti-commutative law.

Commutative law does not hold in this case.

<u>Proof</u> $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

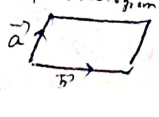
$= - \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = -(\vec{b}\times\vec{a})$

(5) $\vec{a}\times\vec{b}$ is a vector $\perp$ to $\vec{a}$ & $\vec{b}$ both.

(6) The unit vector $\perp$ to both $\vec{a}$ & $\vec{b}$ $= \dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$

(7) $|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta \quad (\because |\hat{n}| = 1)$

$\Rightarrow \sin\theta = \dfrac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$

(8) Area of 

having $\vec{a}$ and $\vec{b}$

as its adjacent sides $= |\vec{a}\times\vec{b}|$

In vector area $= \vec{a}\times\vec{b}$

The area of the Δ determined by the vectors $= \frac{1}{2}|\vec{a}\times\vec{b}|$, vector area of Δ $\frac{1}{2}\,\vec{a}\times\vec{b}$

9. $\hat{i}\times\hat{i}=\vec{0}$, $\hat{j}\times\hat{j}=\vec{0}$, $\hat{k}\times\hat{k}=\vec{0}$

10. If $\vec{a}$ & $\vec{b}$ are $\parallel$ $\theta=0$

$\therefore\ \vec{a}\times\vec{b}=\vec{0}$

Hence $\vec{a}\times\vec{a}=\vec{0}$,

11. If $\vec{a}$ and $\vec{b}$ are $\perp$ then $\theta=90^\circ$

then $\sin\theta=1$

$\therefore\ \vec{a}\times\vec{b}=|\vec{a}||\vec{b}|\,\hat{n}$

In particular $\hat{i}\times\hat{j}=|\hat{i}||\hat{j}|\hat{k}$ $= \hat{k}$ → ($\because$ $\hat{k}$ is $\perp$ to both $\hat{i}$ and $\hat{j}$)

Similarly $\hat{j}\times\hat{k}=\hat{i}$, $\hat{k}\times\hat{i}=\hat{j}$

Hence $\hat{i},\hat{j},\hat{k}$ obey cyclic order.

Then $\hat{j}\times\hat{i}=-\hat{k}$

$\hat{k}\times\hat{j}=-\hat{i}$

$\hat{i}\times\hat{k}=-\hat{j}$

{These results can be verified by using determinant form}

12) If $\vec{v}$ denotes the linear velocity, $\vec{r}$ is the position vector of a point P on the body. Angular velocity be $\vec{\omega}$, then

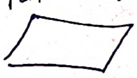
$$\vec{v} = \vec{\omega} \times \vec{r}$$

(13) $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

$(\vec{b} + \vec{c}) \times \vec{a} = \vec{b} \times \vec{a} + \vec{c} \times \vec{a}$

(It can be proved easily by taking components & using determinant form)

$$\hat{i} \times \hat{j} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \hat{k}$$

(14). If the diagonals of parallelogram  are $\vec{d_1}$ and $\vec{d_2}$ then area is given by $\frac{1}{2} |\vec{d_1} \times \vec{d_2}|$ & vector area $= \frac{1}{2} \vec{d_1} \times \vec{d_2}$

(15) If the diagonals of a quadrilateral are given by $\vec{d_1}$ and $\vec{d_2}$ then its

its vector area $= \frac{1}{2} \vec{d_1} \times \vec{d_2}$

and area $= \frac{1}{2} |\vec{d_1} \times \vec{d_2}|$

(16). If three points A, B, C are collinear then $\vec{AB} \times \vec{BC} = \vec{0}$

If two vectors are collinear then their cross product is zero.

Sun - 6.30 PM

Scalar Triple product

The scalar triple product of 3 vectors $\vec{a}$, $\vec{b}$ & $\vec{c}$ is denoted by $[\vec{a}\ \vec{b}\ \vec{c}]$ and is defined by

$$[\vec{a}\ \vec{b}\ \vec{c}] = (\vec{a}\times\vec{b}).\vec{c}$$

Notes - ① Suppose θ is angle between $\vec{a}$ & $\vec{b}$ and ϕ is angle between $\vec{a}\times\vec{b}$ & $\vec{c}$

Then $[\vec{a}\ \vec{b}\ \vec{c}] = (\vec{a}\times\vec{b}).\vec{c}$

$= |\vec{a}||\vec{b}| \sin\theta\ \hat{n}.\vec{c}$

$= |\vec{a}||\vec{b}| \sin\theta\ |\hat{n}|\ |\vec{c}| \cos\phi$

$= |\vec{a}||\vec{b}||\vec{c}| \sin\theta \cos\phi$

2. $[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

where $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

③ Scalar Triple product is a scalar quantity.

④ The dot & cross can be interchanged in scalar Triple product i.e

$[\vec{a}\ \vec{b}\ \vec{c}] = \vec{a}\cdot(\vec{b}\times\vec{c})$ or $(\vec{a}\times\vec{b})\cdot\vec{c}$

(5) $[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}]$

i.e we maintain cyclic order. It can be proved by taking determinant form.

(6) $[\vec{a}\ \vec{b}\ \vec{c}] = -[\vec{a}\ \vec{c}\ \vec{b}]$

i.e with every change in the cyclic order there will be a change in sign. It can be proved by taking determinant ~~form~~ form.

(7) Scalar ~~triple~~ triple product is zero if at least two vectors are equal i.e $[\vec{a}\ \vec{b}\ \vec{c}] = 0$

It can be proved by taking determinant form.

8. Volume of the parallelepiped formed by the 3 vectors $\vec{a}, \vec{b}, \vec{c}$ originated from a common point

$= [\vec{a}\ \vec{b}\ \vec{c}]$

(9.) Volume of the tetrahedron formed by 3 vectors $\vec{a}, \vec{b}, \vec{c}$ originated from a common point $= \frac{1}{6}[\vec{a}\,\vec{b}\,\vec{c}]$

(10) If Volume of the parallelopiped $=0$, then the 3 vectors $\vec{a}, \vec{b}\,\vec{c}$, are Coplanner then $[\vec{a}\,\vec{b}\,\vec{c}]=0$

~~Vectoriple~~

Vector Triple Product

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$$

$$= \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b})$$

bac − cab

$$\boxed{\begin{aligned}&(\vec{a}\times\vec{b})\times\vec{c}\\ &= -\vec{c}\times(\vec{a}\times\vec{b})\\ &= -(\vec{c}\cdot\vec{b}\,\vec{a} - \vec{c}\cdot\vec{a}\,\vec{b})\\ &= (\vec{c}\cdot\vec{a})\vec{b} - (\vec{c}\cdot\vec{b})\vec{a}\end{aligned}}$$

Elements problems page ~~22~~ 3 Eg Ex-15 (a)

5. (i) A is $(4,5,5)$

Position vector of A is $(4\hat{i}+5\hat{j}+5\hat{k})$

B is $(3,3,3)$

Position vector of B is $3\hat{i}+3\hat{j}+3\hat{k}$

$\therefore \vec{AB}$ = Position vector of B − Position vector of A

$= (3\hat{i}+3\hat{j}+3\hat{k}) - (4\hat{i}+5\hat{j}+5\hat{k})$

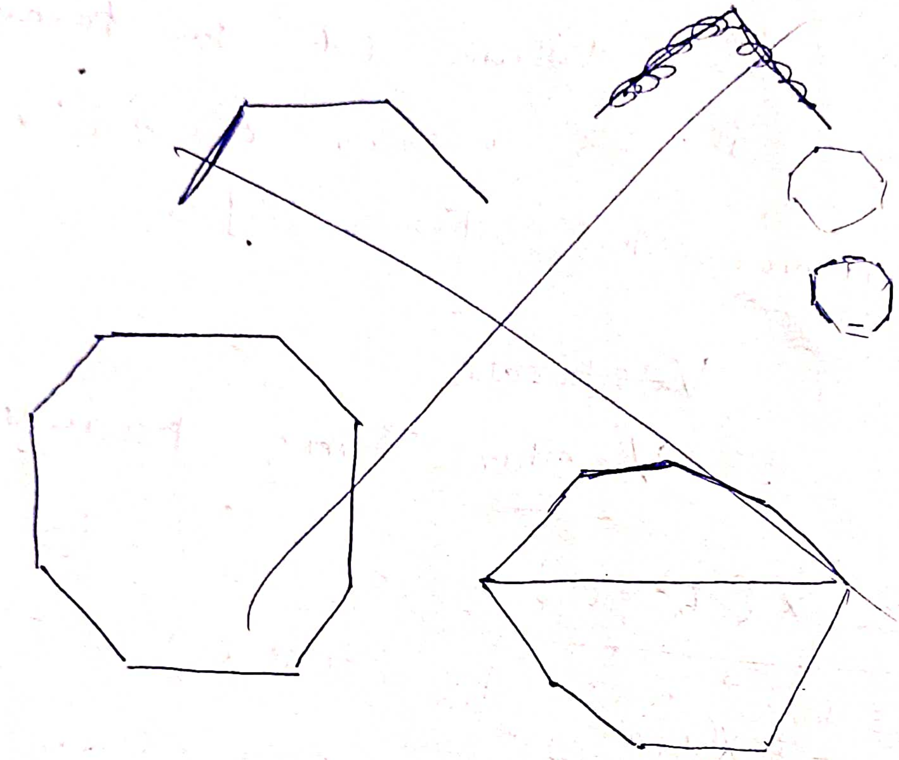
$= -\hat{i} - 2\hat{j} - 2\hat{k}$

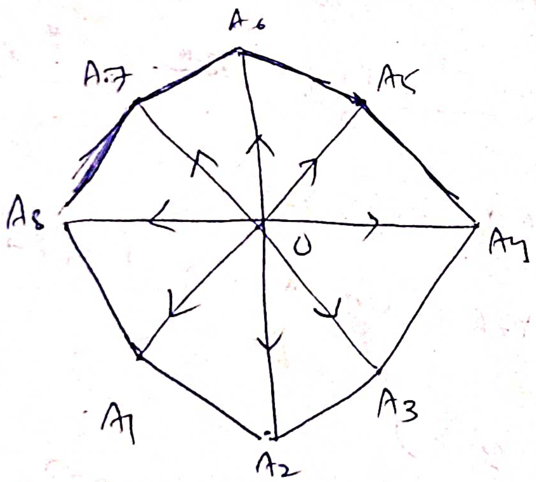
$$|\vec{AB}| = \sqrt{(-1)^2 + (-2)^2 + (-2)^2}$$

$$= \sqrt{1+4+4}$$

$$= 3.$$

9.





$\vec{OA_1} = -\vec{OA_5}$, $\vec{OA_2} = -\vec{OA_6}$

$\vec{OA_3} = -\vec{OA_7}$, $\vec{OA_4} = -\vec{OA_8}$

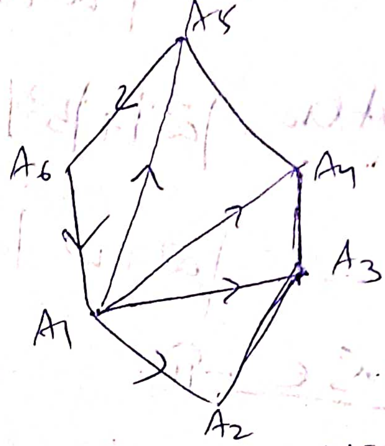
$\therefore$ $\vec{OA_1} + \vec{OA_2} + \vec{OA_3} + \vec{OA_4} + \vec{OA_5}$

$+ \vec{OA_6} + \vec{OA_7} + \vec{OA_8} = -\vec{OA_5} = \vec{OA_8}$

$-\overrightarrow{OA_7} - \overrightarrow{OA_8} + \overrightarrow{OA_5} + \overrightarrow{OA_6} + \overrightarrow{OA_7} + \overrightarrow{OA_8}$

$= \vec{O}$

10) Consider a closed hexagon.



$(\overrightarrow{A_1A_2} + \overrightarrow{A_2A_3}) + \overrightarrow{A_3A_4}$

$+ \overrightarrow{A_4A_5} + \overrightarrow{A_5A_6} + \overrightarrow{A_6A_1}$

$= (\overrightarrow{A_1A_3} + \overrightarrow{A_3A_4}) + \overrightarrow{A_4A_5}$

$+ \overrightarrow{A_5A_6} + \overrightarrow{A_6A_1}$ (By $\triangle$ law of addition)

$= (\overrightarrow{A_1A_4} + \overrightarrow{A_4A_5}) + \overrightarrow{A_5A_6} + \overrightarrow{A_6A_1}$

$= (\overrightarrow{A_1A_5} + \overrightarrow{A_5A_6}) + \overrightarrow{A_6A_1} = \overrightarrow{A_1A_6} + \overrightarrow{A_6A_1}$

$= \overrightarrow{A_1A_6} - \overrightarrow{A_1A_6} = \vec{0}$

Now we will generalise it.

Let $A_1 A_2$ - - ——— A_n be a closed polygon.

Now $(\overrightarrow{A_1A_2} + \overrightarrow{A_2A_3}) + (\overrightarrow{A_3A_4} + \overrightarrow{A_4A_5})$ - - - - - - -

$+ \overrightarrow{A_{n-1}A_n} + \overrightarrow{A_nA_1}$

$= \overrightarrow{A_1A_3} + \overrightarrow{A_3A_4} + - - - - \overrightarrow{A_{n-1}A_n} + \overrightarrow{A_nA_1}$

- - - $+ \overrightarrow{A_{n-1}A_n} + \overrightarrow{A_nA_1}$

$= \overrightarrow{A_1A_4} +$ - - -

$= \overrightarrow{A_1A_n} + \overrightarrow{A_nA_1}$ (Repeating the $\triangle$ law of addition n times)

$= \overrightarrow{A_1A_n} - \overrightarrow{A_1A_n} = \vec{0}$ (Proved)

11.(a)(i)

Proof → Case-I   Suppose $\vec{a} = \vec{0}$

Then $|\vec{a}+\vec{b}| = |\vec{0}+\vec{b}| = |\vec{b}|$

Also $|\vec{a}|+|\vec{b}| = |\vec{0}|+|\vec{b}| = |\vec{b}|$

∴ $|\vec{a}+\vec{b}| = |\vec{a}|+|\vec{b}|$

Case-II   Suppose $\vec{a} = \vec{0}, \vec{b} = \vec{0}$

Then obviously $|\vec{a}+\vec{b}| = 0 = |\vec{a}|+|\vec{b}|$

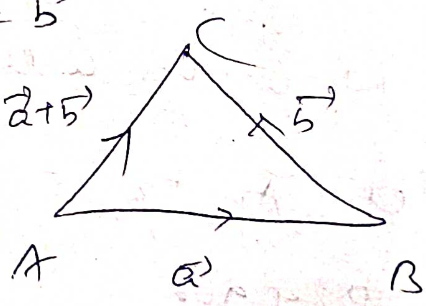
Case-III   or $\vec{a} \neq \vec{0}$ & $\vec{b} \neq \vec{0}$

Let these vectors be represented by $\vec{AB}$ & $\vec{BC}$ respectively.

i.e $\vec{AB} = \vec{a}$ & $\vec{BC} = \vec{b}$

By △ law addition

$\vec{AC} = \vec{a} + \vec{b}$



∴ $AB = |\vec{a}|$, $BC = |\vec{b}|$

$AC = |\vec{a}+\vec{b}|$

But if A, B, C are not collinear

then, $AC < \cancel{BC}\ AB + BC$

$\Rightarrow |\vec{a}+\vec{b}| < |\vec{a}|+|\vec{b}|$

Further when A, B, C are collinear then $AC = \cancel{BC\,AB}\ AB + BC$

$\Rightarrow |\vec{a}+\vec{b}| = |\vec{a}| + |\vec{b}|$

$\therefore$ In general $|\vec{a}+\vec{b}| \leq \vec{a}+\vec{b}$

Equality arises

(1) When either $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$

or both are zero vectors.

(2) When $\vec{a}$ & $\vec{b}$ are collinear

i.e $\vec{a} = k\vec{b}$ where k is scalar

(ii) Case-I If $\vec{a}=\vec{0}$, $\vec{b} \neq 0$

then $|\vec{a}-\vec{b}| = |\vec{b}|$

$|\vec{a}| - |\vec{b}| = -|\vec{b}|$

$\therefore |\vec{a}-\vec{b}| > |\vec{a}| - |\vec{b}|$

Case-II If $\vec{a} \neq 0$, $\vec{b} = \vec{0}$ then

$|\vec{a}-\vec{b}| = \vec{a}$

$|\vec{a}| - |\vec{b}| = |\vec{a}|$

$\therefore |\vec{a}-\vec{b}| = |\vec{a}| - |\vec{b}|$

Case-III If $\vec{a}=\vec{0}$ & $\vec{b}=\vec{0}$

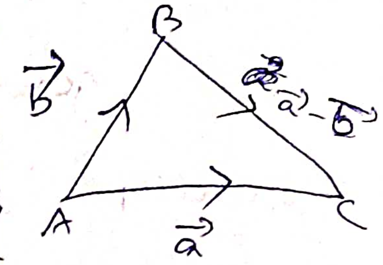
then $|\vec{a}-\vec{b}| = |\vec{0}| = 0$

$|\vec{a}| - |\vec{b}| = 0-0=0$

$\Rightarrow |\vec{a}-\vec{b}| = |\vec{a}| - |\vec{b}|$

Case - IV   Suppose $\vec{a} \neq \vec{0}, \vec{b} \neq \vec{0}$

Let ~~suppose~~ there vectors be represented by $\vec{AC}$ and $\vec{AB}$ respectively.



i.e $\vec{AC} = \vec{a}$, $\vec{AB} = \vec{b}$

Also by $\triangle$ law of addition.

$\Rightarrow \vec{AB} + \vec{BC} = \vec{AC}$

$\Rightarrow \vec{BC} = \vec{AC} - \vec{AB} = \vec{a} - \vec{b}$

$\therefore |\vec{AC}| = |\vec{a}|$, $|\vec{AB}| = |\vec{b}|$, $|BC| = |\vec{a}-\vec{b}|$

But if A, B, C are not ~~DA~~ collinear, then

$AB + BC > AC$

$\Rightarrow |\vec{b}| + |\vec{a}-\vec{b}| > |\vec{a}|$

$\Rightarrow |\vec{a}-\vec{b}| > |\vec{a}| - |\vec{b}|$

If A, B, C are collinear then

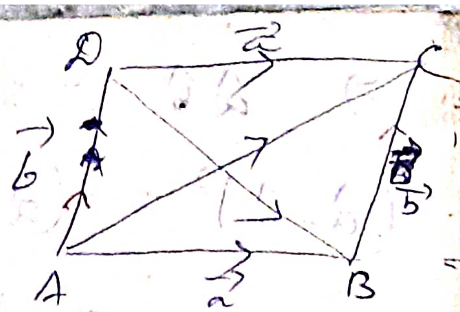
$AB + BC = AC \Rightarrow |\vec{b}| + |\vec{a}-\vec{b}| = |\vec{a}|$

$\Rightarrow |\vec{a}-\vec{b}| = |\vec{a}| - |\vec{b}|$

In general $|\vec{a}-\vec{b}| \geq |\vec{a}| - |\vec{b}|$

11. (b) Complete the ▱gm

ABCD with $\vec{a}$ & $\vec{b}$ as adjacent sides.



Join AC & BD.

$\therefore |\vec{AC}| = |\vec{AB} + \vec{BC}| = |\vec{a} + \vec{b}|$

by Δ law of addition.

$\vec{AD} + \vec{DB} = \vec{AB}$

$\Rightarrow \vec{DB} = \vec{AB} - \vec{AD} = \vec{a} - \vec{b}$

$\therefore |\vec{DB}| = |\vec{a} - \vec{b}|$

But $|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$

$\Rightarrow |\vec{AC}| = |\vec{DB}|$

$\Rightarrow AC = DB$

i.e Two diagonals of ▱ are equal.

$\Rightarrow$ The parallelogram is a rectangle

i.e $\vec{a} \perp \vec{b}$

18. Given that $\hat{a} + \hat{b} = \hat{c}$

$\Rightarrow (\hat{a} + \hat{b})^2 = (\hat{c})^2$

$\Rightarrow (\hat{a} + \hat{b}).(\hat{a} + \hat{b}) = |\hat{c}|^2 = 1$

$\Rightarrow \hat{a}.\hat{a} + \hat{a}.\hat{b} + \hat{b}.\hat{a} + \hat{b}.\hat{b} = 1$

$\Rightarrow (\hat{a})^2 + 2\hat{a}.\hat{b} + (\hat{b})^2 = 1$

$\Rightarrow 1 + 2\hat{a}.\hat{b} + \cancel{1} = \cancel{1}$

$\Rightarrow \hat{a}\,\hat{b} = \frac{-1}{2}$

$(\hat{a}-\hat{b})^2 = (\hat{a})^2 - 2\hat{a}.\hat{b} + (\hat{b})^2$

$= 1 - 2.(-\frac{1}{2}) + 1$

$= 1+1+1 = 3$

$\therefore |\hat{a}-\hat{b}|^2 = 3$

$\Rightarrow |\hat{a}-\hat{b}| = \sqrt{3}$

i.e the magnitude of their ~~distance~~ difference is $\sqrt{3}$ (Proved)

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7. Given that

$(\vec{a}+\vec{b}).(\vec{a}-\vec{b}) = 0$

$\Rightarrow \vec{a}.\vec{a} - \vec{a}.\vec{b} + \vec{b}.\vec{a} - \vec{b}.\vec{b} = 0$

$\Rightarrow |\vec{a}|^2 \cancel{- \vec{a}.\vec{b} + \vec{a}.\vec{b}} - |\vec{b}|^2 = 0$

$(\because \vec{a}.\vec{b} = \vec{b}.\vec{a})$

$\Rightarrow |\vec{a}|^2 = |\vec{b}|^2$

$\Rightarrow |\vec{a}| = |\vec{b}|$

8. (1) Since $\vec{a} \perp \vec{b}$

$\therefore \vec{a}.\vec{b} = 0$

Now $(\vec{a}+\vec{b})^2 = (\vec{a})^2 + 2\vec{a}.\vec{b} + (\vec{b})^2$

$= (\vec{a})^2 + (\vec{b})^2$

Also $(\vec{a}-\vec{b})^2 = (\vec{a})^2 - 2\vec{a}\cdot\vec{b} + (\vec{b})^2$

$= (\vec{a})^2 + (\vec{b})^2$

$\therefore (\vec{a}+\vec{b})^2 = (\vec{a}-\vec{b})^2$

(ii) Suppose $\vec{a} \perp \vec{b}$

then $\vec{a}\cdot\vec{b} = 0$

$\therefore |\vec{a}+\vec{b}|^2 = (\vec{a}+\vec{b})^2$

$= (\vec{a})^2 + 2\vec{a}\cdot\vec{b} + (\vec{b})^2$

$= (\vec{a})^2 + (\vec{b})^2$

$= |\vec{a}|^2 + |\vec{b}|^2$

Conversely let

$|\vec{a}+\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2$

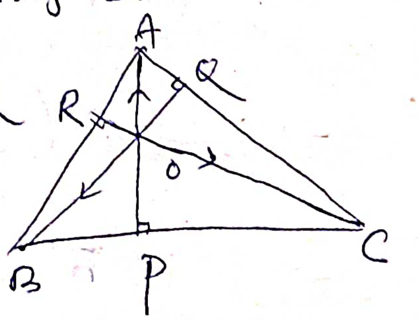
$\Rightarrow (\vec{a}+\vec{b})^2 = (\vec{a})^2 + (\vec{b})^2$

$\Rightarrow (\vec{a})^2 + (\vec{b})^2 + 2\vec{a}\cdot\vec{b} = \cancel{(\vec{a})^2} + \cancel{(\vec{b})^2}$

$\Rightarrow \vec{a}\cdot\vec{b} = 0 \Rightarrow \vec{a} \perp \vec{b}$ (proved)

10.(i) Let ABC be any Δ.

Let the $\perp$ CR and BQ meet at O.



Take O as origin.

Join AO & extend it to meet BC at P.

To prove $AP \perp BC$

Let position vector of A, B, C be $\vec{a}$, $\vec{b}$ & $\vec{c}$ respectively with respect to O as origin.

$$\overrightarrow{BQ} = -\mu\vec{b}, \quad \overrightarrow{CR} = -\gamma\vec{c}, \quad \overrightarrow{AP} = -\lambda\vec{a}$$

$$\overrightarrow{AC} = \vec{c} - \vec{a}, \quad \overrightarrow{AB} = \vec{b} - \vec{a}, \quad \overrightarrow{BC} = \vec{c} - \vec{b}$$

Now $BQ \perp AC$

$\therefore \overrightarrow{BQ} \perp \overrightarrow{AC}$

$\therefore \overrightarrow{BQ} \cdot \overrightarrow{AC} = 0$

$\Rightarrow (-\mu\vec{b})(c - \vec{a}) = 0$

$\Rightarrow -\mu(\vec{b}\cdot\vec{c} - \vec{b}\cdot\vec{a}) = 0$

$\Rightarrow \vec{b}\cdot\vec{c} = \vec{b}\cdot\vec{a} = \vec{a}\cdot\vec{b}$

Also $CR \perp AB$

$\Rightarrow \overrightarrow{CR} \perp \overrightarrow{AB}$

$\therefore \overrightarrow{CR} \cdot \overrightarrow{AB} = 0$

$\Rightarrow (-\gamma\vec{c}) \cdot (\vec{b} - \vec{a}) = 0$

$\Rightarrow -\gamma(\vec{c}\cdot\vec{b} - \vec{c}\cdot\vec{a}) = 0$

$\Rightarrow \vec{c}\cdot\vec{b} - \vec{c}\cdot\vec{a} = 0$

$\Rightarrow \vec{c}\cdot\vec{b} = \vec{c}\cdot\vec{a}$

Now consider $\overrightarrow{AP} \cdot \overrightarrow{BC}$

$= (-\lambda \vec{a}) \cdot (\vec{c} - \vec{b}) = -\lambda \{ \vec{a}\cdot\vec{c} - \vec{a}\cdot\vec{b} \}$

$= -\lambda \{ \vec{c}\,\vec{b} - \vec{b}\cdot\vec{c} \} = -\lambda \{ \vec{b}\cdot\vec{c} - \vec{b}\cdot\vec{c} \}$

$= -\lambda \cdot 0 = 0$

$\therefore \vec{AP} \perp \vec{BC}$

i.e $AP \perp BC$

i.e The three altitudes AP, BQ and CR are concurrent.

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7. (elements) $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta \, \hat{n}$

where $\hat{n}$ is the unit vector $\perp$ to the plane containing $\vec{a}$ & $\vec{b}$ and θ is the angle between $\vec{a}$ & $\vec{b}$

$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta \quad (\because |\hat{n}| = 1)$

$= ab \sin\theta$

$\Rightarrow |\vec{a} \times \vec{b}|^2 = a^2 b^2 \sin^2\theta$

$\Rightarrow (\vec{a} \times \vec{b})^2 = a^2 b^2 \sin^2\theta = a^2 b^2 (1 - \cos^2\theta)$

$= a^2 b^2 - a^2 b^2 \cos^2\theta$

$= a^2 b^2 - (ab\cos\theta)^2$

$= a^2 b^2 - (\vec{a}\cdot\vec{b})^2$

(proved)

Q-) If $\vec{a}+\vec{b}+\vec{c}=0$, then prove that

$$\vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}$$

Proof Given that $\vec{a}+\vec{b}+\vec{c}=0$

$$\Rightarrow \vec{a} = -(\vec{b}+\vec{c})$$

$$\Rightarrow \vec{a}\times\vec{b} = -(\vec{b}+\vec{c})\times\vec{b}$$

$$\Rightarrow \vec{a}\times\vec{b} = -\vec{b}\times\vec{b} - (\vec{c}\times\vec{b})$$

$$\Rightarrow \vec{a}\times\vec{b} = 0 + \vec{b}\times\vec{c}$$

$$\Rightarrow \vec{a}\times\vec{b} = \vec{b}\times\vec{c}$$

Again $\vec{a}+\vec{b}+\vec{c}=0$

$$\Rightarrow \vec{b} = -(\vec{a}+\vec{c})$$

$$\Rightarrow \vec{b}\times\vec{c} = -(\vec{a}+\vec{c})\times\vec{c}$$

$$= -(\vec{a}\times\vec{c}) - \vec{c}\times\vec{c}$$

$$= \vec{c}\times\vec{a} - 0$$

$$= \vec{c}\times\vec{a}$$

$$\therefore \vec{a}\times\vec{b} = \vec{b}\times\vec{c} = \vec{c}\times\vec{a}$$

8. $\vec{a}\times\vec{b} = \vec{b}\times\vec{c} = -\vec{c}\times\vec{b}$

$$\Rightarrow \vec{a}\times\vec{b} + \vec{c}\times\vec{b} = \vec{0}$$

$$\Rightarrow (\vec{a}+\vec{c})\times\vec{b} = \vec{0}$$

$\Rightarrow \quad \vec{a}+\vec{c}=0 \quad$ or $\quad \vec{b}=0 \quad$ or $\quad \vec{a}+\vec{c} \parallel \vec{b}$

But $\quad \vec{a}+\vec{c} \neq 0 \;\&\; \vec{b} = \vec{0}$

because they are arbitrary vectors

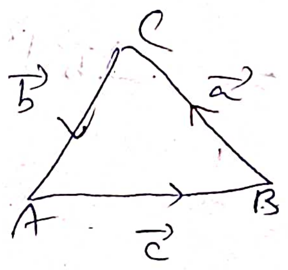
& $\vec{a}\times\vec{b} \neq \vec{0}$ and $\vec{b}\times\vec{c} \neq \vec{0}$

$\therefore \quad \vec{a}+\vec{c} \parallel \vec{b}$

$\Rightarrow \quad \vec{a}+\vec{c} = m\vec{b}$ where m is scalar

(proved)

<u>Ex 3</u>: Given that



$\vec{a}, \vec{b}, \vec{c}$ represents $\vec{BC}, \vec{AC}$, & $\vec{AB}$ of $\triangle ABC$

$\vec{AB}+\vec{BC} = \vec{AC}$ (By $\triangle$ law of addition)

$\Rightarrow \quad \vec{c}+\vec{a} = -\vec{b}$

$\Rightarrow \quad \vec{a}+\vec{b}+\vec{c} = \vec{0}$

$\therefore \quad \vec{a}\times\vec{b} = \vec{b}\times\vec{c} = \vec{c}\times\vec{a}$ (proved before)

$\Rightarrow |\vec{a}\times\vec{b}| = |\vec{b}\times\vec{c}| = |\vec{c}\times\vec{a}|$

$\Rightarrow \quad |\vec{a}||\vec{b}| \sin C = |\vec{b}||\vec{c}| \sin A$

$= |\vec{c}||\vec{a}| \sin B$

$\Rightarrow \quad ab \sin C = bc \sin A = ca \sin B$

$\Rightarrow \quad \dfrac{\sin C}{c} = \dfrac{\sin A}{a} = \dfrac{\sin B}{b}$

$\Rightarrow \quad \dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

(Proved)

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6. $\left(\vec{a}\cdot(\vec{b}\times\vec{c})\right)^2 = \left[\vec{a}\cdot|\vec{b}|\,|\vec{c}|\sin\frac{\pi}{2}\,\hat{n}\right]^2$

where $\hat{n}$ is $\perp$ to $\vec{b}$ and $\vec{c}$ i.e
$\hat{n}$ is $\parallel$ to $\vec{a}$ ($\because \vec{a},\vec{b},\vec{c}$ are mutually $\perp$)

$= |\vec{b}|^2\cdot|\vec{c}|^2\cdot 1\,(\vec{a}\cdot\hat{n})^2$

$= b^2c^2\left(|\vec{a}|^2\,|\hat{n}|^2\right)$ ($\because \hat{n}\parallel\vec{a}$ & $\cos^2\theta=1$)

$= b^2c^2a^2\cdot 1 = a^2b^2c^2$ (proved)

7: Elements p.g.239 $\left[\vec{a}+\vec{b}\quad \vec{b}+\vec{c}\quad \vec{c}+\vec{a}\right]$

$= (\vec{a}+\vec{b})\cdot\left((\vec{b}+\vec{c})\times(\vec{c}+\vec{a})\right)$

$= (\vec{a}+\vec{b})\left[\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{c}+\vec{c}\times\vec{a}\right]$

$= (\vec{a}+\vec{b})\left[\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{a}\right]$

$= \vec{a}\cdot\vec{b}\times\vec{c}+\vec{a}\cdot\vec{b}\times\vec{a}+\vec{a}\cdot\vec{c}\times\vec{a}+\vec{b}\cdot\vec{b}\times\vec{c}+\vec{b}\cdot\vec{b}\times\vec{a}+\vec{b}\cdot\vec{c}\times\vec{a}$

$= [\vec{a}\,\vec{b}\,\vec{c}]+[\vec{a}\,\vec{b}\,\vec{a}]+[\vec{a}\,\vec{c}\,\vec{a}]+[\vec{b}\,\vec{b}\,\vec{c}]+[\vec{b}\,\vec{b}\,\vec{a}]+[\vec{b}\,\vec{c}\,\vec{a}]$

$= [\vec{a}\,\vec{b}\,\vec{c}] + \cancel{[\vec{a}\,\vec{b}\,\vec{a}]} + \cancel{[\vec{a}\,\vec{c}\,\vec{a}]} + \cancel{[\vec{b}\,\vec{b}\,\vec{c}]} + \cancel{[\vec{b}\,\vec{b}\,\vec{a}]} + [\vec{b}\,\vec{c}\,\vec{a}]$

$= [\vec{a}\,\vec{b}\,\vec{c}] + [\vec{b}\,\vec{c}\,\vec{a}] = [\vec{a}\,\vec{b}\,\vec{c}] + [\vec{b}\,\vec{c}\,\vec{a}]$

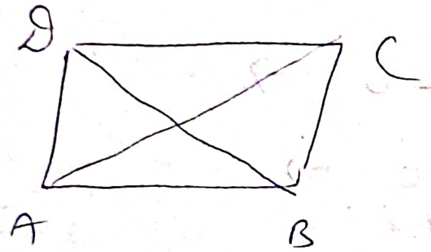
(∵ A Scalar triple product is zero if at least two vectors are equal)

$= 2\,[\vec{a}\cdot\vec{b}\cdot\vec{c}]$ (proved)

Topics P-g - 337

3. (xxvii), (xiii), (xxx), 18, 22, 23, 27.

High- secondary. P.g -428, (8)



Topics

27. P.g -341

Since $\vec{a}, \vec{b}, \vec{c}$ are coplaner.

$\therefore \vec{c} = \lambda\vec{a} + \mu\vec{b}$ —— (1)

$\Rightarrow \vec{c}\cdot\vec{a} = \lambda\,\vec{a}\cdot\vec{a} + \mu\,\vec{b}\cdot\vec{a}$ —— (2)

& $\vec{c}\cdot\vec{b} = \lambda\,\vec{a}\cdot\vec{b} + \mu\,\vec{b}\cdot\vec{b}$ —— (3)

Eliminating λ, μ from (1), (2) and (3)

$$\begin{vmatrix} \vec{c} & \vec{a} & \vec{b} \\ \vec{c}\cdot\vec{a} & \vec{a}\cdot\vec{a} & \vec{b}\cdot\vec{a} \\ \vec{c}\cdot\vec{b} & \vec{a}\cdot\vec{b} & \vec{b}\cdot\vec{b} \end{vmatrix} = 0$$

$$\Rightarrow \vec{c}\begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{b}\cdot\vec{a} \\ \vec{a}\cdot\vec{b} & \vec{b}\cdot\vec{a} \end{vmatrix} - \vec{a}\begin{vmatrix} \vec{c}\cdot\vec{a} & \vec{b}\cdot\vec{a} \\ \vec{c}\cdot\vec{b} & \vec{b}\cdot\vec{b} \end{vmatrix} + \vec{b}$$

$$\begin{vmatrix} \vec{c}\cdot\vec{a} & \vec{a}\cdot\vec{a} \\ \vec{c}\cdot\vec{b} & \vec{a}\cdot\vec{b} \end{vmatrix} = 0 \quad \text{(expanding)}$$

~~Eliminating λ, μ from (1), (2) & (3)~~

$$\cancel{\begin{vmatrix} \vec{c} & \vec{a} & \vec{b} \\ \vec{c}\cdot\vec{a} & \vec{a}\cdot\vec{a} & \vec{b}\cdot\vec{a} \\ \vec{c}\cdot\vec{b} & \vec{a}\cdot\vec{b} & \vec{b}\cdot\vec{b} \end{vmatrix}}$$

23. Since $\vec{a}\cdot\vec{b} = 0 \Rightarrow \vec{a} \perp \vec{b}$

$\vec{a}\cdot\vec{c} = 0 \Rightarrow \vec{a} \perp \vec{c}$

but $\vec{b}\times\vec{c} \perp \vec{b}$ & $\vec{c}$

i.e $\vec{a} \perp \vec{b}$ & $\vec{c}$

is along $\therefore \vec{a} \parallel \vec{b}\times\vec{c}$

But $\vec{a}$ is unit vector.

$\therefore \vec{a}$ is the unit vector $\parallel$ to $\vec{b}\times\vec{c}$

$\therefore \vec{a} = K\, \vec{b}\times\vec{c}$

$|\vec{a}| = |K|\,|\vec{b}\times\vec{c}|$

$\Rightarrow 1 = |K|\,|\vec{b}|\,|\vec{c}| \sin\frac{\pi}{6} = |K|.1.1.\frac{1}{2}$

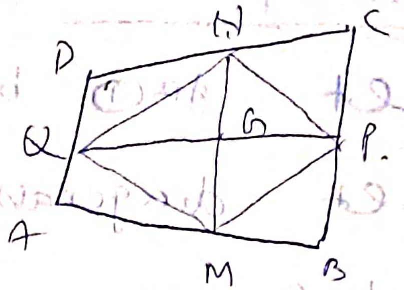
$\Rightarrow |K| = 2 \Rightarrow K = \pm 2$

$\therefore \vec{a} = \pm 2(\vec{b}\times\vec{c})$

22. Here M, N, P, Q are mid points.
NQMP is ▭gm. Given mid

Of MN and of PQ.

Take O as origin.



$\vec{OA} + \vec{OB} = 2\vec{OM}$

$\vec{OC} + \vec{OD} = 2\vec{ON}$

$\therefore \vec{OA} + \vec{OB} + \vec{OC} + \vec{OD} = 2(\vec{OM} + \vec{ON})$

$= 2(2\vec{OG})$

$= 4\vec{OG}$ (proved)

18. $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = \vec{m} \times (\vec{c} \times \vec{d})$

where $\vec{m} = \vec{a} \times \vec{b}$

$= (\vec{m} \cdot \vec{d})\vec{c} - (\vec{m} \cdot \vec{c})\vec{d}$

$= (\vec{a} \times \vec{b} \cdot \vec{d})\vec{c} - (\vec{a} \times \vec{b} \cdot \vec{c})\vec{d}$

$= \lambda \vec{c} + \mu \vec{d}$

where $\lambda = \vec{a} \times \vec{b} \cdot \vec{d}$, $\mu = -(\vec{a} \times \vec{b} \cdot \vec{c})$

12. P.g - 235 elements

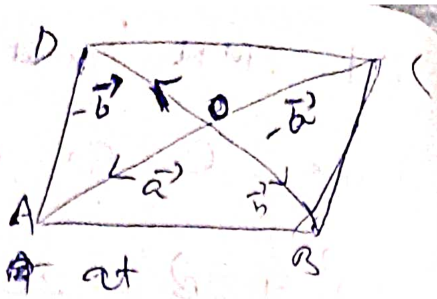
$(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b})$

$= \vec{a} \times \vec{a} + \vec{a} \times \vec{b} - \vec{b} \times \vec{a} - \vec{b} \times \vec{b}$

$= 0 + \vec{a} \times \vec{b} + \vec{a} \times \vec{b} - \vec{0}$

$= 2(\vec{a} \times \vec{b})$

Interpretation



Let ABCD be ▱

Let diagonals meet at at O. Take O be origin

position vector of A $= \vec{OA} = \vec{a}$,

position vector of B $= \vec{OB} = \vec{b}$

$\therefore \vec{BA} = \vec{a} - \vec{b}$ $\quad \therefore \vec{OC} = -\vec{a}$

$\therefore \vec{CB} = \vec{b} + \vec{a} = \vec{a} + \vec{b}$

$\therefore (\vec{a} - \vec{b}) \times (\vec{a} + \vec{b})$ represents vector area of ▱ ABCD. and it is $2\,\vec{a} \times \vec{b}$ i.e twice the vector area of the ▱gm. whose adjacent sides are semi diagonals of the 1st ▱.

or four times the vector area of the △ whose adjacent sides are semi-diagonals of 1st ▱ i.e

$4 \times$ Area of △OAB.

Topic Pg- 339

8. $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 3 \\ 3 & 2 & -1 \end{vmatrix}$ Find $\hat{n} = \dfrac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$

Required vector $= \pm 4\hat{n}$

3. (vii) $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 4 & 0 \end{vmatrix} = 0$

(x) $\alpha \vec{u} + \beta \vec{v} + \gamma \vec{w} = \hat{i}(\alpha+\beta+\gamma) + \hat{j}(\alpha+\beta) + \hat{k}(\alpha)$

$\therefore a = \alpha+\beta+\gamma$, $b = \alpha+\beta$, $c = \alpha$

$\beta = b-c$, $\gamma = a-c-b+c = a-b$

(xii) Unit vector $= \cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k}$

$= l\hat{i} + m\hat{j} + n\hat{k}$

$l = \cos 45^\circ = \frac{1}{\sqrt{2}}$, $m = \cos 60^\circ = \frac{1}{2}$

$l^2+m^2+n^2 = 1 \Rightarrow \frac{1}{2} + \frac{1}{4} + n^2 = 1 \Rightarrow n^2 = \frac{1}{4}$

$\Rightarrow n = \pm\frac{1}{2}$

$\therefore$ Unit vector $= \frac{\hat{i}}{\sqrt{2}} + \frac{\hat{j}}{2} \pm \frac{\hat{k}}{2}$

(xiii) $\vec{d}$ is $\perp \vec{a}$

Let $\vec{d} = d_1\hat{i} + d_2\hat{j} + d_3\hat{k}$

$\vec{d}\cdot\vec{a} = 0 \Rightarrow d_1 + d_2 + 0\cdot d_3 = 0$

$\Rightarrow d_1 + d_2 = 0 \Rightarrow d_1 = -d_2$

$\therefore \vec{d} = d_2\hat{i} + d_2\hat{j} + d_3\hat{k}$

Let $\vec{c}$ be the vector $\parallel$ to $\vec{a}$.

$\therefore \vec{c} = K_1 \vec{a}$

Now $\vec{b} = 3\hat{j} + 4\hat{k} = \vec{c} + \vec{d}$

$= K_1 \vec{a} - d_2 \hat{i} + d_2 \hat{j} + d_3 \hat{k}$

$= K_1 \hat{i} + K_1 \hat{j} - d_2 \hat{i} + d_2 \hat{j} + d_3 \hat{k}$

$= (K_1 - d_2)\hat{i} + (K_1 + d_2)\hat{j} + d_3 \hat{k}$

Comparing $d_3 = 4$, $K_1 + d_2 = 3$, $K_1 - d_2 = 0$

$\Rightarrow 2K_1 = 3 \Rightarrow K_1 = \frac{3}{2} \therefore d_2 = \frac{3}{2}$

$\therefore \vec{b} = \frac{3}{2}\vec{a} - \frac{3}{2}\hat{i} + \frac{3}{2}\hat{j} + 4\hat{k}$

$= \frac{3}{2}(\hat{i} + \hat{j}) + \left[-\frac{3}{2}\hat{i} + \frac{3}{2}\hat{j} + 4\hat{k}\right]$

xvii $\vec{a} \cdot \vec{b} = 0 \Rightarrow \vec{a} \perp \vec{b} \Rightarrow \theta = \frac{\pi}{2}$

$\vec{a} \times \vec{b}$ is $\perp \vec{b}$ also $\vec{b} \cdot \vec{c} = 0$

$\Rightarrow \vec{b} \perp \vec{c}$

$\therefore \vec{a} \times \vec{b} \parallel \vec{c}$ $\therefore$ $\alpha = 0$

$\therefore \vec{c}$ is $\perp$ to $\vec{a}$ and $\vec{b}$ both

$\therefore \vec{c} \cdot \vec{a} = 0$

$(\vec{a} \times \vec{b}) \cdot \vec{c} = |\vec{a}||\vec{b}||\vec{c}| \sin\theta \cos\alpha$

$= 2 \times 2 |\hat{i}| \cdot 1 \cdot 1$

$\Rightarrow |\vec{c}| = 2$ $= 8$

$\vec{b} \times \vec{c}$ is $\perp$ $\vec{b}$ & $\vec{a} \perp \vec{b}$

$\therefore$ $\vec{b} \times \vec{c} \parallel \vec{a}$ $\therefore$ $\vec{b} \times \vec{c} = m\vec{a}$

Now $\vec{a} \times \vec{b} \cdot \vec{c} = 8 \Rightarrow \vec{a} \cdot \vec{b} \times \vec{c} = 8$

$\Rightarrow \vec{a} \cdot m\vec{a} = 8$

$\Rightarrow m\vec{a} \cdot \vec{a} = 8$

$\Rightarrow m4 = 8$ $\Rightarrow m = 2$

$\therefore$ $\vec{b} \times \vec{c} = 2\vec{a}$

(xviii) $(\hat{i} \cdot \hat{i})\vec{r} - (\hat{i} \cdot \vec{r})\hat{i} + (\hat{j} \cdot \hat{j}) \cdot \vec{r}$

$- (\hat{j} \cdot \vec{r}) \cdot \hat{j} + (\hat{k} \cdot \hat{k})\vec{r} - (\hat{k} \cdot \vec{r})\hat{k}$

$= (\vec{r} + \vec{r} + \vec{r}) - \left\{ \sum (\hat{i} \cdot \vec{r})\hat{i} \right\} = 3\vec{r} - \vec{r} = 2\vec{r}$

(xxx) $\vec{r} \times \vec{b} = \vec{c} \times \vec{b}$

$\Rightarrow \vec{a} \times (\vec{r} \times \vec{b}) = \vec{a} \times (\vec{c} \times \vec{b})$

$\Rightarrow (\vec{a} \cdot \vec{b}) \cdot \vec{r} - (\vec{a} \cdot \vec{r})\vec{b} = (\vec{a} \cdot \vec{b})\vec{c} - (\vec{a} \cdot \vec{c})\vec{b}$

$\Rightarrow \vec{r} = \vec{c} - \dfrac{\vec{a} \cdot \vec{c}}{\vec{a} \cdot \vec{b}} \cdot \vec{b}$

$(\because \vec{a} \cdot \vec{b} \neq 0)$

(xx) Applying determinant methug

(xxv) $\pm (\hat{i} + \hat{j}) \times (\hat{j} + \hat{k}) \to$ So 2 vectors

(xxxiii) Let $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$\vec{a} \cdot \vec{b} = 3 \Rightarrow b_1 + b_2 + b_3 = 3$ — (1)

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & K \\ 1 & 1 & 1 \\ b_1 & b_2 & b_3 \end{vmatrix} = \hat{i}(b_3 - b_2) + \hat{j}(b_1 - b_3) + \hat{k}(b_2 - b_1)$$

$$= \hat{j} - \hat{k}$$

Comparing $b_3 - b_2 = 0 \Rightarrow b_3 = b_2$

$b_1 - b_3 = 1$, $b_2 - b_1 = -1$

$\Rightarrow b_1 = b_2 + 1$

$\therefore b_2 + 1 + b_2 + b_2 = 3 \Rightarrow b_2 = \frac{2}{3} = b_3$

$b_1 = \frac{2}{3} + 1 = \frac{5}{3}$

$\therefore \vec{b} = \frac{5}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}$

19) Pg-341

$(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = (\vec{a} \times \vec{b}) \cdot \vec{x}$ where $\vec{x} = \vec{c} \times \vec{d}$

$$= \vec{a} \cdot (\vec{b} \times \vec{x})$$

$$= \vec{a} \left(\vec{b} \times (\vec{c} \times \vec{d})\right)$$

$$= \vec{a} \left((\vec{b} \cdot \vec{d})\vec{c} - (\vec{b} \cdot \vec{c}) \cdot \vec{d}\right)$$

$$= (\vec{b} \cdot \vec{d})(\vec{a} \cdot \vec{c}) - (\vec{b} \cdot \vec{c})(\vec{a} \cdot \vec{d})$$

$$= \begin{vmatrix} \vec{a}.\vec{c} & \vec{b}.\vec{c} \\ \vec{a}.\vec{d} & \vec{b}.\vec{d} \end{vmatrix}.$$

30) The given relation can be written as

$$\left((1-\lambda)x + 3y - 4z\right)\hat{i} + \left(x - (3+\lambda)y + 5z\right)\hat{j}$$

$$+ (3x + y - \lambda z)\hat{k} = \vec{0}$$

$$\therefore \left.\begin{aligned} (1-\lambda)x + 3y - 4z &= 0 \\ x - (3+\lambda)y + 5z &= 0 \\ 3x + y - \lambda z &= 0 \end{aligned}\right\} \longleftarrow (1)$$

Given that $(x, y, z) \neq (0, 0, 0)$

ie The system (1) has non-trival solution

$$\therefore \begin{vmatrix} 1-\lambda & 3 & -4 \\ 1 & -(3+\lambda) & 5 \\ 3 & 1 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^3 + 2\lambda^2 + \lambda = 0 \Rightarrow \lambda(\lambda+1)^2 = 0$$

$$\therefore \lambda = 0, -1,$$

28. $|\vec{a}| = \sqrt{51}$. Let $\vec{a}$ makes θ with each of the given

Vectors $\vec{b}, \vec{c}, \vec{d}$.

where $\vec{b} = \frac{1}{3}(\hat{i} - 2\hat{j} + 2\hat{k})$,

$\vec{c} = \frac{1}{5}(-4\hat{i} - 3\hat{k})$, $\vec{d} = \hat{j}$

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\vec{a}.\vec{b} = |\vec{a}||\vec{b}| \cos\theta$

$\Rightarrow \frac{1}{3}(a_1 - 2a_2 + 2a_3) = \sqrt{51} .1. \cos\theta$ — ①

$\vec{a}.\vec{c} = |\vec{a}|.|\vec{c}| \cos\theta$

$\Rightarrow \frac{1}{5}(-4a_1 - 3a_3) = \sqrt{51} .1. \cos\theta$ — ②

$\vec{a}.\vec{d} = |\vec{a}||\vec{d}| \cos\theta$ — ③

From ① and ③ $\frac{1}{3}(a_1 - 2a_2 + 2a_3) = a_2$

$\Rightarrow a_1 - 5a_2 + 2a_3 = 0$ — ④

From ② and ③

$4a_1 + 5a_2 + 3a_3 = 0$ — ⑤

From ④ and ⑤

$\frac{a_1}{-25} = \frac{a_2}{5} = \frac{a_3}{25}$

$\Rightarrow \frac{a_1}{5} = \frac{a_2}{-1} = \frac{a_3}{-5}$ $\therefore \vec{a} = 5\hat{i} - \hat{j} - 5\hat{k}$

17) Let P be (x, y, z)

$\therefore \quad \vec{OP} = x\hat{i} + y\hat{j} + z\hat{k} = \vec{r}$ (say)

where $|\vec{OP}| = 1$, $\Rightarrow x^2+y^2+z^2 = 1$

Now P lies on the plane of $\vec{r_1}$ and $\vec{r_2}$.

$\vec{r_1} = (1, -1, 1)$

$\vec{r_2} = (2, 3, -1)$

$\vec{r}$ lies on the plane of $\vec{r_1}$ and $\vec{r_2}$, $\vec{r_1} \times \vec{r_2} = \begin{vmatrix} i & j & k \\ 1 & -1 & 1 \\ 2 & 3 & -1 \end{vmatrix}$

i.e $\vec{r}$ is $\perp$ $\vec{r_1} \times \vec{r_2}$

$= -2\hat{i} - 3\hat{j} + 5\hat{k} = \vec{b}$ (say)

$\therefore \quad \vec{r} \cdot \vec{b} = 0 \quad \Rightarrow -2x - 3y + 5z = 0$ —— (1)

$\vec{r}$ is $\perp$ to $\vec{r_3} \Rightarrow \vec{r} \cdot \vec{r_3} = 0$

$\Rightarrow -x + 2y + 2z = 0$ —— (2)

Solving (1) and (2) $\Rightarrow \frac{x}{16} = \frac{y}{1} = \frac{z}{7}$

$\frac{x}{16} = \frac{y}{1} = \frac{z}{7}$

$= \frac{\sqrt{x^2+y^2+z^2}}{\pm\sqrt{306}}$

$= \frac{1}{\pm\sqrt{306}}$

$\therefore \quad x = \frac{16}{\pm\sqrt{306}}, \quad y = \frac{1}{\pm\sqrt{306}}, \quad z = \frac{7}{\pm\sqrt{306}}$

(See problem no-)

EX – 15(a)

8) $\left[\vec{a}\times\vec{b} \quad \vec{b}\times\vec{c} \quad \vec{c}\times\vec{a}\right]$

$= (\vec{a}\times\vec{b}) \cdot \{(\vec{b}\times\vec{c}) \times (\vec{c}\times\vec{a})\}$

$= (\vec{a}\times\vec{b}) \cdot \{((\vec{b}\times\vec{c}).\vec{a})\vec{c} - ((\vec{b}\times\vec{c}).\vec{c})\vec{a}\}$

$= (\vec{a}\times\vec{b}) \cdot \{[\vec{b}\ \vec{c}\ \vec{a}].\vec{c} - [\vec{b}\ \vec{c}\ \vec{c}]\vec{a}\}$

$= (\vec{a}\times\vec{b}) \cdot \{[\vec{b}\ \vec{c}\ \vec{a}].\vec{c} - \vec{0}\}$

$= (\vec{a}\times\vec{b}) \cdot \{[\vec{b}\ \vec{c}\ \vec{a}]\vec{c}\}$

$= (\vec{a}\times\vec{b}.\vec{c})\,[\vec{b}\ \vec{c}\ \vec{a}]$

$= [\vec{a}\ \vec{b}\ \vec{c}]\ [\vec{a}\ \vec{b}\ \vec{c}] = [\vec{a}\ \vec{b}\ \vec{c}]^2$

(Proved)

10) $\vec{a}\times(\vec{b}\times\vec{c}) + \vec{b}\times(\vec{c}\times\vec{a}) + \vec{c}(\vec{a}\times\vec{b})$

$= (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b}).\vec{c} + (\vec{b}.\vec{a})\vec{c} - (\vec{b}.\vec{c}).\vec{a}$

$+ (\vec{c}.\vec{b})\vec{a} - (\vec{c}.\vec{a}).\vec{b}$

$= (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c} + (\vec{a}.\vec{b}).\vec{c}$

$- (\vec{b}.\vec{c})\vec{a} + (\vec{b}.\vec{c})\vec{a} - (\vec{a}.\vec{c})\vec{b}$

$= \vec{0}$

Let $\vec{x} = \vec{a}\times(\vec{b}\times\vec{c})$

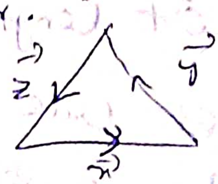
$\vec{y} = \vec{b}\times(\vec{c}\times\vec{a})$

$\vec{z} = \vec{c}\times(\vec{a}\times\vec{b})$

$\therefore \vec{x} + \vec{y} + \vec{z} = \vec{0}$

$\Rightarrow \vec{x}, \vec{y}, \vec{z}$ are represented along the sides of $\triangle$ taken in order.

$\Rightarrow \vec{x}, \vec{y}, \vec{z}$ are Co planner.



11) Given that

$$\vec{a} \times (\hat{b} \times \hat{c}) = \frac{1}{2}\hat{b}$$

$\Rightarrow (\vec{a} \cdot \hat{c})\hat{b} - (\hat{a} \cdot \hat{b})\hat{c} = \frac{1}{2}\hat{b}$

$\Rightarrow (\hat{a}\,\hat{c} - \frac{1}{2})\hat{b} - (\hat{a} \cdot \hat{b})\vec{c} = \vec{0}$

$\Rightarrow l\hat{b} + m\hat{c} = \vec{0}$

where $l = \hat{a}\hat{c} - \frac{1}{2}$

$m = -(\hat{a} \cdot \hat{b})$

Also given that $\hat{b}$ & $\vec{c}$ are not //.

$\therefore l = 0$ & $m = 0$ ($\because$ $l\vec{x} + m\vec{y} = \vec{0}$ then $\vec{x} \parallel \vec{y}$ or co-efficient are zero)

$\Rightarrow \hat{a} \cdot \hat{c} - \frac{1}{2} = 0$ & $-\hat{a} \cdot \hat{b} = 0$

$\hat{a}\,\hat{b} = 0$

$\Rightarrow \hat{a}\,\hat{c} = \frac{1}{2}$ & $\hat{a} \perp \hat{b}$

$\Rightarrow |\hat{a}| \cdot |\hat{c}| \cos\theta = \frac{1}{2}$ & $\hat{a} \perp \hat{b}$.

$\Rightarrow \cos\theta = \frac{1}{2}$ & angle between $\hat{a}$ & $\hat{b}$

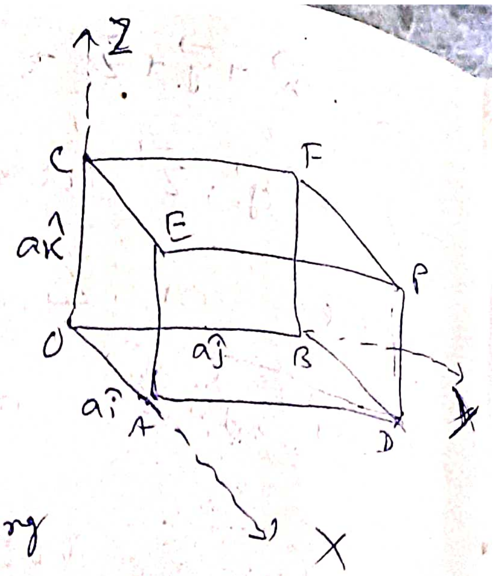
$\Rightarrow \theta = \frac{\pi}{3}$ & is $\frac{\pi}{2}$

$\therefore$ Angle between $\hat{a}$ and $\hat{c}$ is $\frac{\pi}{3}$ & angle between $\hat{a}$ & $\hat{b}$ is $\frac{\pi}{2}$

Elements 10 · Pg - 230

(viii) Let each side of length "a"

Let one corner be origin.



OA, OB, OC are along X, Y, Z axes respectively.

$\therefore \overrightarrow{OA} = a\hat{i}$, $\overrightarrow{OB} = a\hat{j}$, $\overrightarrow{OC} = a\hat{k}$

Consider the square OADB

$\therefore \overrightarrow{OD} = a\hat{i} + a\hat{j}$ ($\square$gm law of addition)

Now $\overrightarrow{DP} = \overrightarrow{OC} = a\hat{k}$

$\therefore \overrightarrow{OP} = \overrightarrow{OD} + \overrightarrow{DP} = a\hat{i} + a\hat{j} + a\hat{k}$

$= a(\hat{i}+\hat{j}+\hat{k})$

$\overrightarrow{CD} = \overrightarrow{CO} + \overrightarrow{OD}$

$= -\overrightarrow{OC} + \overrightarrow{OD} = -a\hat{k} + a\hat{i} + a\hat{j}$

$= a(\hat{i}+\hat{j}-\hat{k})$

$\overrightarrow{OP} \cdot \overrightarrow{CD} = a(\hat{i}+\hat{j}+\hat{k}) \cdot a(\hat{i}+\hat{j}-\hat{k})$

$= a^2\{\hat{i}\cdot\hat{i} + 0 - 0 + 0 + \hat{j}\cdot\hat{j} - 0 + 0 + 0 - \hat{k}\cdot\hat{k}\}$

$= a^2\{1+1-1\}$

$= a^2$

But $\overrightarrow{OP} \cdot \overrightarrow{CD} = |\overrightarrow{OP}|\,|\overrightarrow{CD}| \cos\theta$ where θ is angle between OP & CD

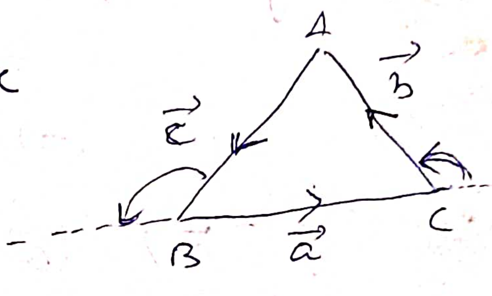
$= \sqrt{a^2+a^2+a^2} \cdot \sqrt{a^2+a^2+a^2} \cdot \cos\theta$

$= 3a^2 \cos\theta$

$\Rightarrow 3a^2 \cos\theta = a^2 \Rightarrow 3\cos\theta = 1 \Rightarrow \cos\theta = \frac{1}{3}$

$\Rightarrow \theta = \cos^{-1}\left(\frac{1}{3}\right)$

(vi) Let $\vec{a}, \vec{b}, \vec{c}$ be represented along BC, CA, AB of Δ ABC respectively.



$\therefore \overrightarrow{BC} = \vec{a}, \ \overrightarrow{CA} = \vec{b}, \ \overrightarrow{AB} = \vec{c}$

$\therefore \vec{a} + \vec{b} + \vec{c} = 0$

$\Rightarrow \overrightarrow{BC} + \overrightarrow{CA} + \overrightarrow{AB} = 0$

$\Rightarrow \overrightarrow{BC} = -(\overrightarrow{CA} + \overrightarrow{AB})$

$\Rightarrow \overrightarrow{BC} \cdot \overrightarrow{BC} = -(\overrightarrow{CA} + \overrightarrow{AB}) \cdot \overrightarrow{BC}$

$\Rightarrow (\overrightarrow{BC})^2 = -\overrightarrow{CA} \cdot \overrightarrow{BC} - \overrightarrow{AB} \cdot \overrightarrow{BC}$

$\Rightarrow a^2 = -(CA)(BC)\cos(\pi - C) - (AB)(BC)\cos(\pi - B)$

$= ba \cos C + ca \cos B$

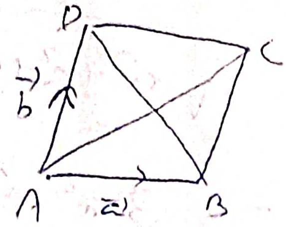
$\Rightarrow a = b\cos C + c\cos B$

(proved)

(iii) Let $\overrightarrow{AB} = \vec{a}, \ \overrightarrow{AD} = \vec{b}$

$\therefore \overrightarrow{AC} = \vec{a} + \vec{b}$

$\vec{BD} = \vec{BA} + \vec{AD}$



$= -\vec{AB} + \vec{AD}$

$= -\vec{a} + \vec{b}$

$= \vec{b} - \vec{a}$

But given that $AC = BD$

$\Rightarrow AC^2 = BD^2$

$\Rightarrow (\vec{AC})^2 = (\vec{BD})^2$

$\Rightarrow (\vec{a} + \vec{b})^2 = (\vec{b} - \vec{a})^2$

$\Rightarrow \cancel{(\vec{a})^2} + \cancel{(\vec{b})^2} + \cancel{2}\,\vec{a}\cdot\vec{b} = \cancel{(\vec{b})^2} + \cancel{(\vec{a})^2} - \cancel{2}\,\vec{b}\cdot\vec{a}$

$\Rightarrow \vec{a}\cdot\vec{b} = -\vec{b}\cdot\vec{a} = -\vec{a}\cdot\vec{b}$

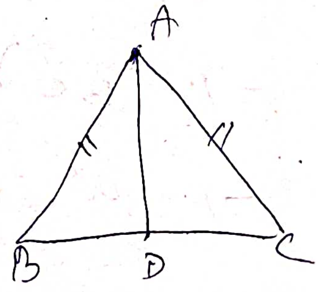
$\Rightarrow 2\,\vec{a}\cdot\vec{b} = 0 \Rightarrow \vec{a}\cdot\vec{b} = 0$

$\Rightarrow \vec{a} \perp \vec{b} \Rightarrow \vec{AB} \perp \vec{AD}$.

$\therefore AB \perp AD$

i.e $ABCD$ is a rectangle.

(ii) Let A be origin.



Let position vector of $B = \vec{b} = \vec{AB}$

" " $C = \vec{c} = \vec{AC}$

" " " $D = \dfrac{\vec{b} + \vec{c}}{2}$

$\therefore \vec{AD} = \dfrac{\vec{b} + \vec{c}}{2}$

$\vec{BC} = \vec{c} - \vec{b}$

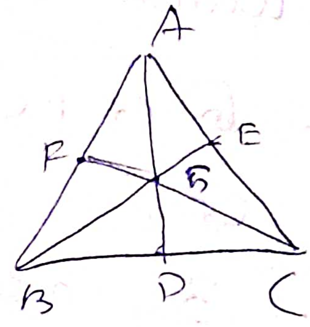
Also $AB = AC \Rightarrow AB^2 = AC^2 \Rightarrow (\vec{AB})^2 = (\vec{AC})^2$

$\Rightarrow (\vec{b})^2 = (\vec{c})^2$.

Now $\vec{AD} \cdot \vec{BC} = \left(\dfrac{\vec{b}+\vec{c}}{2}\right) \cdot (\vec{c}-\vec{b})$

$= \dfrac{(\vec{c})^2 - (\vec{b})^2}{2} = \dfrac{0}{2} = 0$

$\therefore \vec{AD} \perp \vec{BC} \Rightarrow AD \perp BC$ (Proved)



22(a)

Let position vector of A be $\vec{a}$

" " " B be $\vec{b}$

" " " C be $\vec{c}$

" " "

Now position vector of $D = \dfrac{\vec{b}+\vec{c}}{2}$

Let G be the point which divides AD in the ratio 2:1

$\therefore$ Position vector of G is $\dfrac{2 \times \left(\dfrac{\vec{b}+\vec{c}}{2}\right) + 1 \cdot \vec{a}}{2+1}$

$= \dfrac{\vec{a}+\vec{b}+\vec{c}}{3}$

$\therefore$ Position vector of $E = \dfrac{\vec{a}+\vec{c}}{2}$

The point which divides BE in the ratio 2:1 has position vector

$$\dfrac{2 \times \dfrac{\vec{a}+\vec{c}}{2} + 1 \cdot \vec{b}}{2+1} = \dfrac{\vec{a}+\vec{b}+\vec{c}}{3}$$

Which is position vector of G.

Position vector of $F = \dfrac{\vec{a}+\vec{b}}{2}$

The point which divides CF in the ratio 2 : 1 has position vector

$$\frac{2 \cdot \frac{\vec{a}+\vec{b}}{2} + 1 \cdot \vec{c}}{2+1} = \frac{\vec{a}+\vec{b}+\vec{c}}{3}$$

which is the position vector of G

∴ G is the common point of AD, BE, CF.

∴ 3 medians meet at G.

i.e they are concurrent.

(Proved)

Exercise-15(a)

1. (i) $\vec{a} = \hat{i} + 2\hat{j} + \hat{k}$

$\vec{b} = 2\hat{i} - 2\hat{j} + 2\hat{k}$

$\vec{c} = -\hat{i} + 2\hat{j} + \hat{k}$

(d) No pair of vectors have same direction.

(ii) $\vec{a} = 2\hat{i} + 3\hat{j} - 6\hat{k}$ are parallel.

$\vec{b} = \alpha\hat{i} - \hat{j} + 2\hat{k}$

Since $\vec{a} \parallel \vec{b}$

$\vec{a} = k\vec{b}$

$3 = -k$, $-6 = 2k$

$2 = k\alpha$ $\Rightarrow k = -3$

$\therefore \alpha = \frac{2}{k} = \frac{2}{-3}$

(iii) $\vec{A} = 3\hat{i} + a\hat{j} + 1\hat{k}$

$\vec{B} = 2\hat{i} + \hat{j} - \hat{k}$

$\vec{BA} = \vec{A} - \vec{B} = \hat{i} - \hat{j} + 2\hat{k}$

(iv) $\cancel{|k\vec{a}| = 1}$

$\cancel{k \cdot |\vec{a}| = 1}$

$\cancel{\Rightarrow |\vec{a}| = \frac{1}{k}}$

or $\cancel{k|\vec{a}| = 1}$

$\cancel{\Rightarrow k = \frac{1}{|\vec{a}|}}$

or

$\cancel{k\vec{a} = \pm 1}$

$\cancel{\Rightarrow k = \pm\frac{1}{\vec{a}}}$

Ans

$|k\vec{a}| = 1$

$\Rightarrow |k||\vec{a}| = 1$

$\Rightarrow |k| = \frac{1}{|\vec{a}|}$

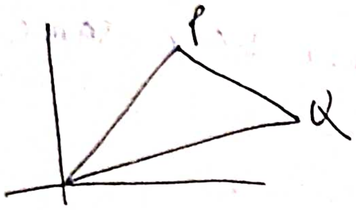
$\Rightarrow k = \pm\frac{1}{|\vec{a}|}$

(v) dcs of $\overrightarrow{PQ}$? where $\overrightarrow{OP} = \hat{i} - 2\hat{k}$

$\overrightarrow{OQ} = 3\hat{i} - 2\hat{j}$

$\therefore$ P is the point ~~(1, -2)~~ (1, 0, -2)

Q is the point (3, -2, 0)



D r.s of PQ

(2, -2, 2)

D C.s

$= \left\langle \frac{2}{2\sqrt{3}}, \frac{-2}{2\sqrt{3}}, \frac{2}{2\sqrt{3}} \right\rangle$

$= \left\langle \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$

2. (i) $\vec{a} - \vec{a} = \vec{0}$

(ii) The vector $\vec{0}$ has indefinite direction.

(iii) All unit vectors are equal in magnitude & ~~but~~ but different in direction.

(iv) $\vec{a} = \vec{b}$ $\Rightarrow |\vec{a}| = |\vec{b}|$

(v) Subtraction of vector is not commutative

3. $\vec{a} = (2, 1)$, $\vec{b} = (-1, 0)$

$\therefore \vec{a} = 2\hat{i} + \hat{j}$

$\vec{b} = -\hat{i}$

$3\vec{a} + 2\vec{b} = 6\hat{i} + 3\hat{j} - 2\hat{i}$

$= 4\hat{i} + 3\hat{j}$

$= (4, 3)$

(ii) $\vec{a} = (1,1,1)$, $\vec{b} = (-1,3,0)$

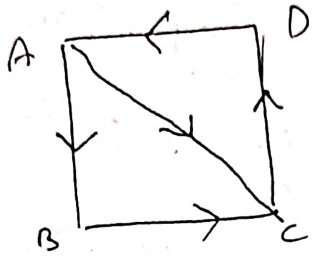
$\vec{c} = (2,0,2)$

$\vec{a} + 2\vec{b} - \frac{1}{2}\vec{c}$

$(1,1,1) + (-2,6,0) - (1,0,1)$

$= (-2,7,0)$

(iii) 4.



$\vec{AB} + \vec{BC} = \vec{AC}$ (Δ law of addition)

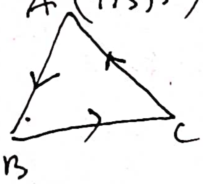
$\vec{CD} + \vec{DA} = \vec{CA} = -\vec{AC}$

$\therefore \vec{AB} + \vec{BC} + \vec{CD} + \vec{DA}$

$= \vec{AC} - \vec{AC}$

$= \vec{0}$

5. (i) A(4,5,5)



B (3,3,3) C (1,2,5)

$\vec{AB} = (-1,-2,-2)$

$\vec{BC} = (-2,-1,2)$

$\vec{CA} = (3,3,0)$

Length of AB $= \sqrt{1+4+4} = 3$

" " BC $= \sqrt{4+1+4} = 3$

" " CA $= \sqrt{9+9+0} = 3\sqrt{2}$

(ii) A (8,6,1)

(2,0,1) B C (-4,0,-5)

$\vec{AB} = -6\hat{i} - 6\hat{j}$

$\vec{BC} = -6\hat{i} - 6\hat{k}$

$\vec{CA} = 12\hat{i} + 6\hat{k} + 6\hat{k}$

$|\vec{AB}| = 6\sqrt{2}$, $|\vec{BC}| = 6\sqrt{2}$

$CA = 6\sqrt{6}$ ($\because \sqrt{216} = 6\sqrt{6}$)

6. P_1 is $(4,3)$, P_2 is $(8,-5)$

$\overrightarrow{P_1P_2}$ is $4\hat{i} - 8\hat{j}$

mid point of $\overrightarrow{P_1P_2}$ $\cancel{\frac{4\hat{i} - 8\hat{j}}{2}}$ $(2,-4)$

i.e. $\cancel{2\hat{i} - 4\hat{j}}$

Position vector of mid point from origin $(0,0)$ is $2\hat{i} - 4\hat{j}$

6. P_1 is the point $(4,3)$ & P_2 is the point $(8,-5)$

Position vector of $\overrightarrow{P_1}$ is $4\hat{i} + 3\hat{j}$

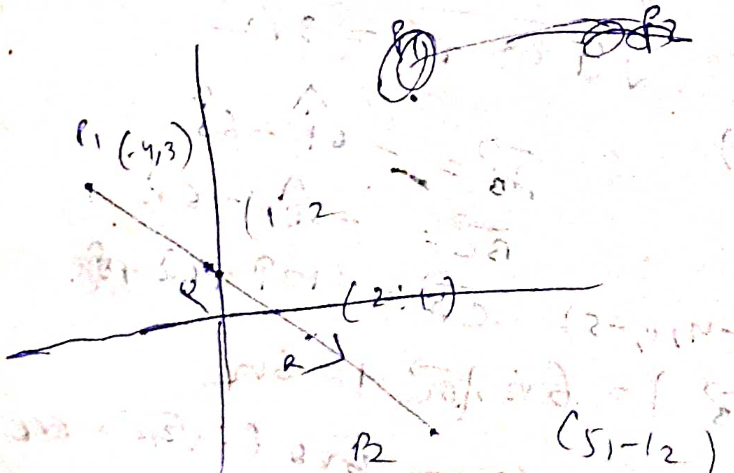
$\overrightarrow{P_2}$ is $8\hat{i} - 5\hat{j}$

$\overrightarrow{P_1P_2} = 4\hat{i} - 8\hat{j}$

mid point of P_1P_2 is $(6, -1)$

Position vector of ~~PBH~~ mid point from origin $6\hat{i} - \hat{j}$

7.



Let Q & R be the points of trisection.

Q is point $\left(\frac{5 \cdot 2 - 8}{3}, \frac{-12 + 2 \cdot 3}{3}\right)$

$= (-1, -2)$

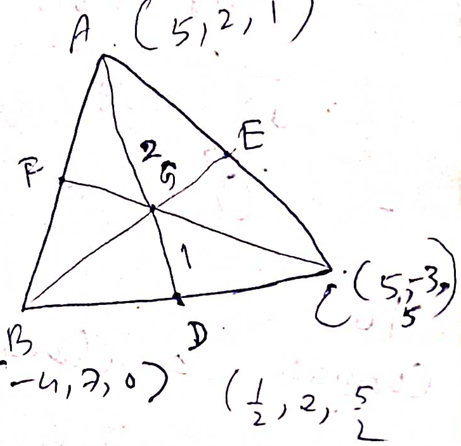
R is the point $\left(\frac{2 \cdot (5) + -4}{3}, \frac{2(-12) + 3}{3}\right)$

$= (2, -7)$

vector from the origin to the Q $-\hat{i} - 2\hat{j}$ & $2\hat{i} - 7\hat{j}$

8.

Let D be the mid point of BC



D has the co-ordinate $\left(\frac{1}{2}, 2, \frac{5}{2}\right)$

$\left(\frac{1}{2}, 2, \frac{5}{2}\right)$

G divides AD in the ~~ratio~~ ratio 2 : 1

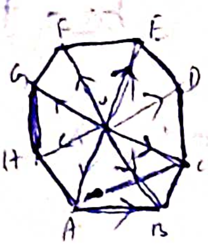
$\therefore$ Co-ordinate of G

$= \left(\frac{2 \cdot \frac{1}{2} + 1 \cdot 5}{3}, \frac{2 \cdot (2) + 1 \cdot 2}{3}, \frac{2 \cdot \frac{5}{2} + 1 \cdot 1}{3}\right)$

$= (2, 2, 2)$

$\therefore$ Position vector of G $2\hat{i} + 2\hat{j} + 2\hat{k}$

9.



Let the centre of octagon be O.

The vector drawn from the point O to the vertices A, B, C, D, E, F, G, H are

$\vec{OA}, \vec{OB}, \vec{OC}, \vec{OD}, \vec{OE}, \vec{OF}, \vec{OG}, \vec{OH}$

But $\vec{OA} = -\vec{OE}$, $\vec{OB} = -\vec{OF}$, $\vec{OC} = -\vec{OG}$

$\vec{OD} = -\vec{OH}$

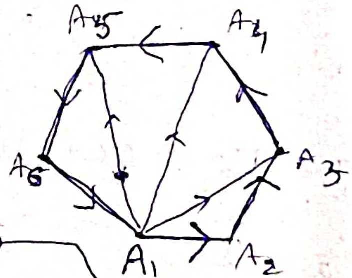
$\therefore$ Sum of the vectors

$= \vec{OA} + \vec{OB} + \vec{OC} + \vec{OD} + \vec{OE} + \vec{OF} + \vec{OG} + \vec{OH}$

$= \cancel{\vec{OA}} + \cancel{\vec{OB}} + \cancel{\vec{OC}} + \cancel{\vec{OD}} + \cancel{-\vec{OA}} \cancel{-\vec{OB}} \cancel{-\vec{OC}} \cancel{-\vec{OD}}$

$= \vec{O}$ = Null vector.

10.



Consider a closed hexagon.

$(\vec{A_1A_2} + \vec{A_2A_3}) + \vec{A_3A_4} + \vec{A_4A_5} + \vec{A_5A_6} + \vec{A_6A_1}$

$= (\vec{OA_2} + \vec{A_2A_3}) + \vec{A_3A_4} + \vec{A_4A_5} + \vec{A_5O}$

$= (\vec{OA_3} + \vec{A_3A_4}) + \vec{A_4A_5} + \vec{A_5O}$

$= (\vec{OA_4} + \vec{A_4A_5}) + \vec{A_5O}$

$= \vec{OA_5} + \vec{A_5O}$

$= \vec{O}$

Now we will generalise it.

Let A_1, A_2 ---------- A_n be a closed polygon of n sides.

$A_1 + A_2$ --- $+ A$ -- -- ---------- $+ A_n$

$(\overrightarrow{A_1A_2} + \overrightarrow{A_2A_3}) + \overrightarrow{A_3A_4} + \overrightarrow{A_4A_5}$ ---------- $\overrightarrow{A_{n-2}A_{n-1}} + \overrightarrow{A_{n-1}A_n} + \overrightarrow{A_nA_1}$

$= (\overrightarrow{A_1A_3}) + (\overrightarrow{A_3A_4}) + \overrightarrow{A_4A_5}$ ---- $+ \overrightarrow{A_{n-1}A_n} + \overrightarrow{A_nA_1}$

= ----------------------

= ----------------------

$= \overrightarrow{A_1A_n} + \overrightarrow{A_nA_1}$ (Repeating n times)

$= \overrightarrow{A_1A_n} - \overrightarrow{A_1A_n}$

$= \vec{0}$ (proved)

$(\overrightarrow{A_1A_3} + \overrightarrow{A_3A_4}) + \overrightarrow{A_4A_5} + \overrightarrow{A_5A_6} + \overrightarrow{A_6A_1}$

$= (\overrightarrow{A_1A_4} + \overrightarrow{A_4A_5}) + \overrightarrow{A_5A_6} + \overrightarrow{A_6A_1}$

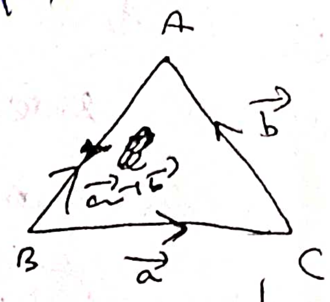
$= (\overrightarrow{A_1A_5} + \overrightarrow{A_5A_6}) + \overrightarrow{A_6A_1}$

$= \overrightarrow{A_1A_6} - \overrightarrow{A_1A_6}$

$= \vec{0}$

11. (i) To prove

Case-I $\quad |\vec{a}+\vec{b}| \leq |\vec{a}|+|\vec{b}|$



Suppose $\vec{a} = \vec{0}$

Then

$|\vec{a}+\vec{b}| = |\vec{0}+\vec{b}| = |\vec{b}|$

Also $|\vec{a}|+|\vec{b}| = \vec{0}+|\vec{b}| = |\vec{b}|$

$\therefore \quad |\vec{a}+\vec{b}| = |\vec{a}|+|\vec{b}|$

Case-II $\quad$ Suppose $\vec{b} = 0$

Then $|\vec{a}+\vec{b}| = |\vec{a}+\vec{0}| = |\vec{a}|$

$|\vec{a}|+|\vec{b}| = |\vec{a}|+|\vec{0}| = |\vec{a}|$

$\therefore \quad |\vec{a}+\vec{b}| = |\vec{a}+\vec{b}|$

Case-III $\quad$ Suppose $\vec{a} = \vec{0}, \vec{b} = 0$

$|\vec{a}+\vec{b}| = |\vec{0}+\vec{0}| = |\vec{0}| = 0$

$|\vec{a}|+|\vec{b}| = |\vec{0}|+|\vec{0}| = 0+0 = 0$

$\therefore \quad |\vec{a}+\vec{b}| = |\vec{a}+\vec{b}|$

Case-IV

Suppose $\vec{a} \neq 0, \vec{b} \neq 0$

These two vector can be shown as the sides of triangle ABC. i.e ~~$|\vec{BC}| = |\vec{a}|$~~ i.e $\vec{BC} = \vec{a}$, $\vec{CA} = \vec{b}$

$\therefore |\vec{BC}| = |\vec{a}|, \quad |\vec{CA}| = \vec{b}$

$BC = |\vec{a}| \quad CA = |\vec{b}|$

By Δ law of addition $\vec{BA} = \vec{a}+\vec{b}$

$AB = |\vec{a}+\vec{b}|$ $\quad |\vec{BA}| = |\vec{a}+\vec{b}|$

But if A, B, C are not collinear

then $AB < BC + AC$

$\Rightarrow |\vec{a}+\vec{b}| < |\vec{a}| + |\vec{b}|$

$\therefore$ Further when A, B, C are collinear

~~$AC = AB + BC$~~

~~$\Rightarrow |\vec{a}+\vec{b}| = |\vec{a}| + |\vec{b}|$~~

$AB = BC + AC$

$\Rightarrow |\vec{a}+\vec{b}| = |\vec{a}| + |\vec{b}|$

$\therefore$ In general $|\vec{a}+\vec{b}| \leq |\vec{a}| + |\vec{b}|$

(A) (i) Equality arises when $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or both of them are zero vector.

(ii) when $\vec{a}$ & $\vec{b}$ are collinear.

B (ii) To prove $|\vec{a}-\vec{b}| \geq |\vec{a}| - |\vec{b}|$

Case 1 when $\vec{a} = \vec{0}$ & $\vec{b} \neq \vec{0}$

Then $|\vec{a}-\vec{b}| = |\vec{0}-\vec{b}| = |\vec{b}|$

$|\vec{a}| - |\vec{b}| = |\vec{0}| - |\vec{b}| = -|\vec{b}|$

$\therefore |\vec{a}-\vec{b}| > |\vec{a}| - |\vec{b}|$

Case 2

when $\vec{b} = \vec{0}$ & $\vec{a} \neq \vec{0}$

then $|\vec{a}-\vec{b}| = |\vec{a}-\vec{0}| = |\vec{a}|$

$|\vec{a}|-|\vec{b}| = |\vec{a}| - |\vec{0}| = |\vec{a}|$

$\therefore |\vec{a}-\vec{b}| = |\vec{a}|-|\vec{b}|$

Case-III when $\vec{a}=\vec{0}$ & $\vec{b}=\vec{0}$

$|\vec{a}-\vec{b}| = |\vec{0}-\vec{0}| = |\vec{0}| = 0$

$|\vec{a}|-|\vec{b}| = |\vec{0}|-|\vec{0}| = 0-0 = 0$

$\therefore |\vec{a}-\vec{b}| = |\vec{a}|-|\vec{b}|$

Case-IV $\vec{a} \neq 0$ & $\vec{b} \neq \vec{0}$

Let there are vectors be represented by $\vec{CB}$ & $\vec{CA}$

i.e $\vec{CA}=\vec{b}$ & $\vec{CB}=\vec{a}$

Also by △ law of addition

$\vec{CA}+\vec{AB} = \vec{CB}$

$\Rightarrow \vec{b}+\vec{AB} = \vec{a}$

$\Rightarrow \vec{AB} = \vec{a}-\vec{b}$.

But A, B, C are not collinear.

$\therefore$ $\cancel{|\vec{CB}| + |\vec{CA}| > |\vec{AB}|}$

$\Rightarrow \cancel{|\vec{c}| +}$

$|\vec{CA}|+|\vec{AB}| > |\vec{CB}|$

$\Rightarrow |\vec{b}|+|\vec{a}-\vec{b}| > |\vec{a}|$

$\Rightarrow |\vec{a}-\vec{b}| > |\vec{a}|-|\vec{b}|$

Or A, B, C are colliner.

$|\vec{CA}| + |\vec{AB}| = |\vec{CB}|$

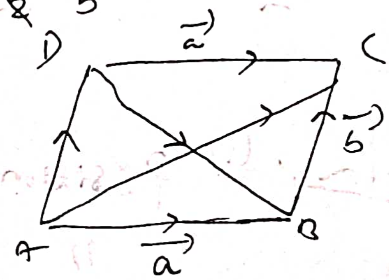
$\Rightarrow |\vec{b}| + |\vec{a}-\vec{b}| = |\vec{a}|$

$\Rightarrow |\vec{a}-\vec{b}| = |\vec{a}|-|\vec{b}|$

$\therefore$ In general

$|\vec{a}-\vec{b}| \geq |\vec{a}|-|\vec{b}|$

(b) Complete the $\square$ of parallelogram ABCD with $\vec{a}$ & $\vec{b}$ as adjacent sides.



Join AC & BD

$|\vec{AC}| = |\vec{AB} + \vec{BC}| = |\vec{a}+\vec{b}|$ (by Δ law of addition)

$\vec{AD} + \vec{DB} = \vec{AB}$

$\Rightarrow \vec{DB} = \vec{AB} - \vec{AD} = \vec{a} - \vec{b}$

$|\vec{DB}| = |\vec{a}-\vec{b}|$

But given that $|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}|$

$\Rightarrow |\vec{AC}| = \vec{BD}$

$AC = BD$

$\therefore$ Two diagonals of $\square$ are equal

$\therefore$ The $\square$ is a rectangle $\therefore \vec{a} \perp \vec{b}$.

Q11 $|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}|$

Squaring both the sides.

$\Rightarrow \cancel{(\vec{a})^2} + \cancel{(\vec{b})^2} + 2(\vec{a}).(\vec{b}) = \cancel{(\vec{a})^2} + \cancel{(\vec{b})^2} - 2(\vec{a})(\vec{b})$

$\Rightarrow 2\,\vec{a}.\vec{b} = 0$

$\Rightarrow \vec{a}.\vec{b} = 0$

$\Rightarrow |\vec{a}||\vec{b}|.\cos\theta = 0$

$\therefore \theta = \pi/2$

$\therefore \vec{a} \perp \vec{b}$.

not Geometrical interpretation

12. (i) Position vector of P $= -\hat{i} + 3\hat{j}$

" " Q $= \hat{i} + 2\hat{j}$

$\vec{PQ} = 2\hat{i} - \hat{j}$

$|\vec{PQ}| = \sqrt{2^2+1} = \sqrt{5}$

~~Scalar Components ~~ ~~2, -1~~ ~~$2\hat{i} - \hat{j}$~~

~~vector Component along direction~~

$\widehat{PQ}$

~~Thus unit vector $\widehat{PQ}$ $\frac{2}{\sqrt{5}}\hat{i} - \frac{1}{\sqrt{5}}\hat{j}$~~

~~Therefore Scalar Components are~~

Scalar Components are 2 & -1

~~Vector Components~~

Componant vectors along Co-ordinate axes $= 2\hat{i} - \hat{j}$

(ii) Position vector of P $-\hat{i} - 2\hat{j}$

" " Q $-5\hat{i} - 6\hat{j}$

$\overrightarrow{PQ} = -4\hat{i} - 4\hat{j}$

$|\overrightarrow{PQ}| = \sqrt{4^2 + 4^2} = \sqrt{32} = 4\sqrt{2}$

Scalar Componants -4 & -4

Componet vectors $-4\hat{i} - 4\hat{j}$.

(iii) Position vector of P $\hat{i} + 4\hat{j} - 3\hat{k}$

" " Q $2\hat{i} - 2\hat{j} - \hat{k}$

$\overrightarrow{PQ} = \hat{i} - 6\hat{j} + 2\hat{k}$

$|\overrightarrow{PQ}| = \sqrt{1 + 36 + 4} = \sqrt{41}$

Scalar Componants $1, -6, 2$

Compont vector $\hat{i} - 6\hat{j} + 2\hat{k}$

13. $P = (2, -1, -1)$ Position vector of P $= 2\hat{i} - \hat{j} - \hat{k}$

$Q = (-1, -3, 2)$ Position vector of Q $= -\hat{i} - 3\hat{j} + 2\hat{k}$

$\overrightarrow{PQ} = -3\hat{i} - 2\hat{j} + 3\hat{k}$

$|\overrightarrow{PQ}| = \sqrt{9 + 4 + 9} = \sqrt{22}$

Drs of PQ $= (-3, -2, 3)$

DCs of PQ $= \left(\frac{-3}{\pm\sqrt{22}}, \frac{-2}{\pm\sqrt{22}}, \frac{3}{\sqrt{22}}\right)$

(ii) P has the position vector $3\hat{i} - \hat{j} + 7\hat{k}$

Q has " " $4\hat{i} - 3\hat{j} - \hat{k}$

$\overrightarrow{PQ}$ is $\hat{i} - 2\hat{j} - 8\hat{k}$

$|\overrightarrow{PQ}| = \sqrt{1+4+64} = \sqrt{69}$

DR's of PQ $= (1, -2, -8)$

DC's of PQ $= \left(\frac{1}{\sqrt{69}}, \frac{-2}{\sqrt{69}}, \frac{-8}{\sqrt{69}}\right)$

14. $\vec{a} = 2\hat{i} - 2\hat{j} + \hat{k}$

$\vec{b} = 2\hat{i} + 3\hat{j} + 6\hat{k}$

$\vec{c} = -\hat{i} + 2\hat{k}$

$\vec{a} - \vec{b} + 2\vec{c}$

$= \cancel{2\hat{i}} - 2\hat{j} + \hat{k} - \cancel{2\hat{i}} - 3\hat{j} - 6\hat{k} - 2\hat{i} + 4\hat{k}$

$= -2\hat{i} - 5\hat{j} - \hat{k}$

magnitude $= \sqrt{4+25+1} = \sqrt{30}$

DCs are $\left(\frac{-2}{\sqrt{30}}, \frac{-5}{\sqrt{30}}, \frac{-1}{\sqrt{30}}\right)$

15. (i) The given vector is $5\hat{i} - 12\hat{j}$

Its magnitude is $\sqrt{25+144} = 13$

Unit vector $= \frac{5}{13}\hat{i} - \frac{12}{13}\hat{j}$

(ii) Let $\vec{A} = 2\hat{i} + \hat{j}$

$|\vec{A}| = \sqrt{2^2+1^2} = \sqrt{5}$

$\hat{A} = \frac{\vec{A}}{|\vec{A}|} = \frac{2}{\sqrt{5}}\hat{i} + \frac{1}{\sqrt{5}}\hat{j}$

(iii) Let $\vec{A} = 3\hat{i} + 6\hat{j} - \hat{k}$

$|\vec{A}| = 9 + 36 + 1 = \sqrt{46}$

$\hat{A} = \frac{3}{\sqrt{46}}\hat{i} + \frac{6}{\sqrt{46}}\hat{j} - \frac{1}{\sqrt{46}}\hat{k}$

(iv) Let $\vec{A} = 3\hat{i} + \hat{j} - 2\hat{k}$

$|\vec{A}| = \sqrt{9+1+4} = \sqrt{14}$

$\hat{A} = \frac{3}{\sqrt{14}}\hat{i} + \frac{1}{\sqrt{14}}\hat{j} - \frac{2}{\sqrt{14}}\hat{k}$

16. The given vector is

$\vec{A} = \vec{r_1} - \vec{r_2}$

$= \hat{i} + 2\hat{j} + \hat{k} - 3\hat{i} - \hat{j} + 5\hat{k}$

$= -2\hat{i} + \hat{j} + 6\hat{k}$

$|\vec{A}| = \sqrt{4+1+36} = \sqrt{41}$

Unit vector $= \frac{-2}{\sqrt{41}}\hat{i} + \frac{1}{\sqrt{41}}\hat{j} + \frac{6}{\sqrt{41}}\hat{k}$

17: The vector is sum of the two vector.

Let $\vec{A} = 2\hat{i} + \hat{i} + 4\hat{j} + 2\hat{j} - 5\hat{k} + 3\hat{k}$

$= 3\hat{i} + 6\hat{j} - 2\hat{k}$

$\therefore |\vec{A}| = \sqrt{9+36+4y} = 7$

Unit vector $= \frac{3}{7}\hat{i} + \frac{6}{7}\hat{j} + \frac{-2}{7}\hat{k}$

D.C.S or unit vector $\left\langle \frac{3}{7}, \frac{6}{7}, \frac{-2}{7} \right\rangle$

18. Given that

$$\hat{a}+\hat{b} = \hat{c}$$

$$\Rightarrow (\hat{a}+\hat{b})^2 = (\hat{c})^2$$

$$\Rightarrow (\hat{a}+\hat{b})(\hat{a}+\hat{b}) = |\hat{c}|^2 = 1$$

$$\Rightarrow \hat{a}.\hat{a} + \hat{a}.\hat{b} + \hat{b}.\hat{a} + \hat{b}.\hat{b} = 1$$

$$\Rightarrow (\hat{a})^2 + 2(\hat{a}.\hat{b}) + (\hat{b})^2 = 1$$

$$\Rightarrow 1 + 2(\hat{a}.\hat{b}). + \cancel{1} = \cancel{1}$$

$$\Rightarrow \hat{a}.\hat{b} = -\frac{1}{2}$$

$$(\hat{a}-\hat{b})^2 = (\hat{a})^2 - 2\hat{a}.\hat{b} + (\hat{b})^2$$

$$= 1 - \cancel{2}.\left(\frac{-1}{\cancel{2}}\right) + 1$$

$$= 1+1+1 = 3$$

$$\Rightarrow (\hat{a}-\hat{b}) = \sqrt{3}$$

i.e The magnitude of their difference is $\sqrt{3}$.

19. Position vector of A $= 4\hat{i} + 3\hat{j} - \hat{k}$

" " " B $= 5\hat{i} + 2\hat{j} + 2\hat{k}$

" " " C $= 2\hat{i} - 2\hat{j} - 3\hat{k}$

" " " D $= 4\hat{i} - 4\hat{j} + 3\hat{k}$

$$\overrightarrow{AB} = \hat{i} - \hat{j} + 3\hat{k}$$

$$\overrightarrow{CD} = 2\hat{i} - 2\hat{j} + 6\hat{k}$$

$$= 2(\hat{i} - \hat{j} + 3\hat{k})$$

$$= K(\quad)$$

$\therefore$ $\overrightarrow{CD}$ is a scalar multiple of $\overrightarrow{AB}$ $\therefore$ $\overrightarrow{AB} \parallel \overrightarrow{CD}$.

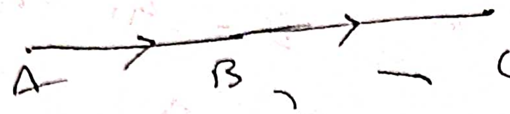
20. (i) A is the point with co-ordinate $(2,6,3)$

B " " " " " $(1,2,7)$

C " " " " " $(3,10,-1)$

$\vec{AC} = 1, 4, -4$

$\hat{i}+4\hat{j}-4\hat{k}$



$$\vec{AB} = -\hat{i}-4\hat{j}+4\hat{k}$$

$$\vec{BC} = 2\hat{i}+8\hat{j}-8\hat{k}$$

$$= -2(-\hat{i}-4\hat{i}+4\hat{k})$$

$\therefore$ $\cancel{\vec{a} = k\vec{b}}$, ~~and $\vec{AB} = K\vec{BC}$~~, $\vec{BC} = K\vec{AB}$ & B is the common point.

So A, B, C are ~~B~~colliner.

(ii) $\vec{PQ} = \hat{i}-4\hat{j}-2\hat{k}$

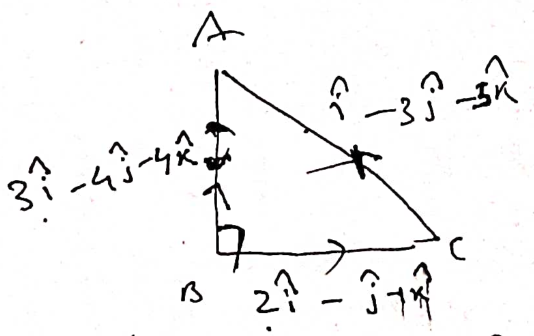
$\vec{QR} = -4\hat{i}+16\hat{j}+8\hat{k}$

$= -4(\hat{i}-4\hat{j}-2\hat{k})$

$\cancel{\vec{a} = k\vec{b}}$ $\vec{QR} = k\vec{PQ}$ & Q is the common point.

$\therefore$ P, Q, R are ~~[illegible]~~ Collinear.

21.



~~6+4-4~~

$2+3-5$

Let $\vec{a} = 2\hat{i}-\hat{j}+\hat{k}$, $\vec{c} = 3\hat{i}-4\hat{j}-4\hat{k}$

$\vec{b} = \hat{i}-3\hat{j}-5\hat{k}$,

Then $\vec{a}+\vec{b} = 3\hat{i}-4\hat{j}-4\hat{k} = \vec{c}$

shows $\vec{a}, \vec{b}, \vec{c}$ satisfy the $\triangle$ law of addition & hence AB, BC, CA are the sides of a triangle respectively.

$|\vec{a}| = \sqrt{4+1+1} = \sqrt{6}$ $\therefore |\vec{a}|^2 = 6$

$|\vec{b}| = \sqrt{1+9+25} = \sqrt{35}$ $|\vec{b}|^2 = 35$

$|\vec{a}|^2 + |\vec{b}|^2 = 41$

$|\vec{c}| = \sqrt{9+16+16}$. $|\vec{c}|^2 = 41$.

$\therefore m\angle ABC = 90°$.

OR $\cos\theta = \cancel{2(-1)\ldots}$

$\cancel{6 - 3 + 3 - 8 + 4 - 4}$

$\cancel{-8 + 8 - 4}$

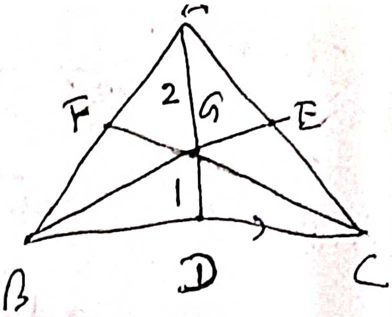
$= \cancel{6 - 3 + 3 - 8 + 4 - 4 - 8 + 4 - 4}$

$\cancel{\theta \; \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|} = \frac{2 - 6 - 10 + 1 + 3 + 5 + 1 - 3 - 1}{|\vec{a}|\cdot|\vec{b}|}}$

$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|} = \frac{2+3-5}{|\vec{a}||\vec{b}|} = 0$

$\theta = \pi/2$ $\therefore$ ABC is $\triangle$.

22.(a)



Let in $\triangle ABC$ AB, BC, CA be the sides.

Let D be the mid point of BC

Position vector of $A = \vec{a}$

" " of $B = \vec{b}$

" " " $C = \vec{c}$

Now position vector of $D = \frac{\vec{b}+\vec{c}}{2}$

Let G be the point which divides AD in the ratio 2 : 1

∴ Position vector of G

$$\frac{2\left(\frac{\vec{b}+\vec{c}}{2}\right)+1.\vec{a}}{3} = \frac{\vec{a}+\vec{b}+\vec{c}}{3}.$$

E be the mid point of AC

∴ Position vector of $E = \frac{\vec{a}+\vec{c}}{2}$

The point which divides BE in the ratio 2 : 1 has the position vector

$$\frac{2.\left(\frac{\vec{a}+\vec{c}}{2}\right)+1.\vec{b}}{2+1} = \frac{\vec{a}+\vec{b}+\vec{c}}{3}$$

which is position vector of G.

F be the midpoint of AB

position vector of $\vec{F} = \frac{\vec{a}+\vec{b}}{2}$

The point which divides CF in the ratio 2 : 1 has the position vector

$$\frac{2\cdot\left(\frac{\vec{a}+\vec{b}}{2}\right)+1\cdot\vec{c}}{2+1} = \frac{\vec{a}+\vec{b}+\vec{c}}{3}$$

which is the position vector of G

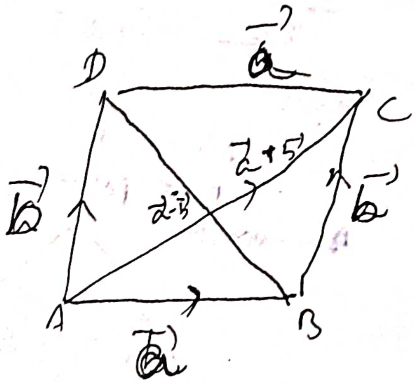
$\therefore$ G is the common point of

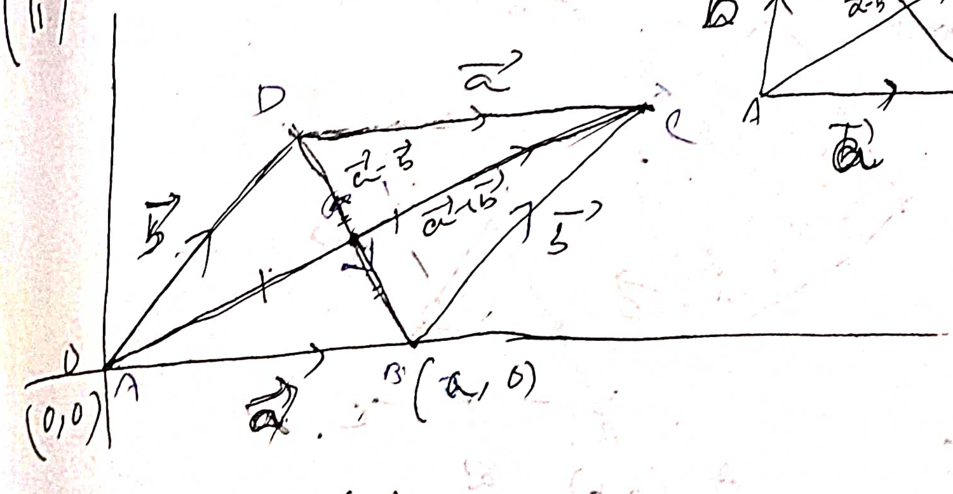
AD, BE & CF.

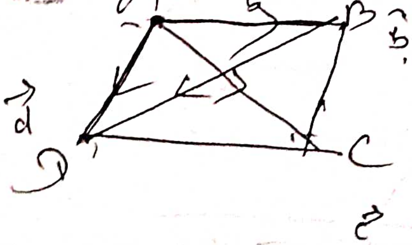
$\therefore$ 3 medians meet at G.

$\therefore$ " " are concurrent.

(ii)



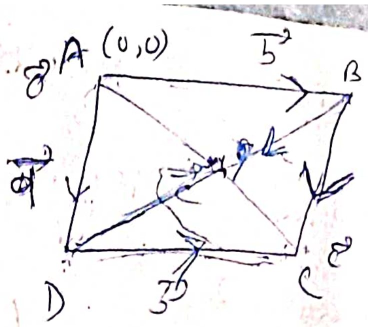




$\vec{d} = \vec{c} - \vec{b}$

$\frac{\vec{d}+\vec{b}}{2} = \frac{\vec{c}}{2}$

Let ABCD be a Parallelogram.



Let A be the origin.

Let position vector of B, C and D be $\vec{b}$, $\vec{c}$ and $\vec{d}$.

For a 

$\vec{AB} = \vec{BC}$

$\Rightarrow \vec{b} - \vec{d} = \vec{c} - \vec{b}$

$\Rightarrow \frac{\vec{d} + \vec{b}}{2} = \frac{\vec{c}}{2}$

$\vec{AB} = \frac{\vec{b} - \vec{a}}{2}$

~~The two diagonals bisects each other.~~

Mid point of $\vec{AC} = \frac{\vec{c}}{2}$.

Mid point point of

But $\vec{d} + \vec{b} = \vec{c} \Rightarrow \frac{\vec{c}}{2} = \frac{\vec{b} + \vec{d}}{2}$

$\vec{b} - \vec{a} = \vec{c} - \vec{b}$

$\vec{a} + \vec{c} = \frac{\vec{b} + \vec{d}}{2}$

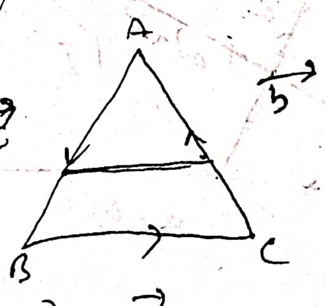
$\frac{\vec{a} + \vec{c}}{2} = \frac{\vec{b} + \vec{d}}{2}$

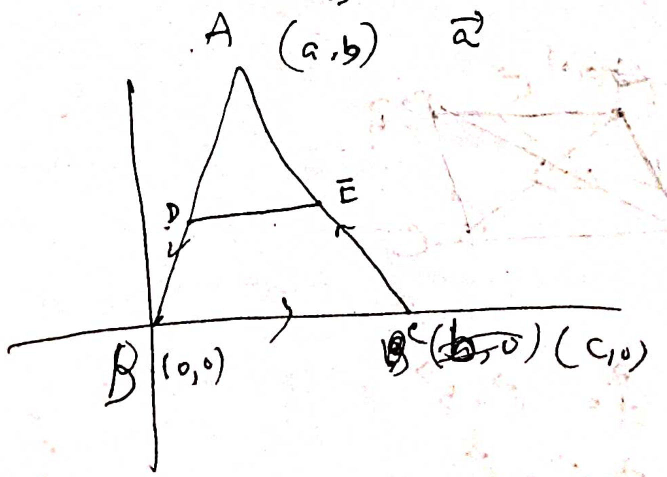
$\Rightarrow$ P.V.e of mid point = P. vector mid point of BD

B (C)

$\vec{a}, \vec{b}, \vec{c}$

$\frac{\vec{a} + \vec{b}}{2}, \frac{\vec{a} + \vec{c}}{2}$





~~Position~~ vector

Let ABC be a $\triangle$. Consider B as origin. and BC along x-axis.

B is the point (0,0), C is the point (c,0), A is the point (a,b).

Position vector of $\vec{A}$ $a\hat{i}+b\hat{j}$

" " $\vec{C}$ $c\hat{i}$

" " $\vec{B}$ $=\vec{0}$

"

Position vector of D = $\frac{a\hat{i}+b\hat{j}}{2}$.

Position " " $\vec{E}$ = $\frac{a\hat{i}+b\hat{j}+c\hat{i}}{2}$

$= \frac{(a+c)\hat{i}+b\hat{j}}{2}$,

$$\vec{DE} = \frac{(a+c)\hat{i}+b\hat{j} - a\hat{i}-b\hat{j}}{2}$$

$$= \frac{\cancel{a\hat{i}}+c\hat{i}+\cancel{b\hat{j}}-\cancel{a\hat{i}}-\cancel{b\hat{j}}}{2} = \frac{c\hat{i}}{2}$$

$$\vec{BC} = (c-0)\hat{i} + (0-0)\hat{j}$$

$$= c\hat{i}$$

$$|\vec{DE}| = \sqrt{\frac{c^2}{4}} = \frac{c}{2}$$

$$|\vec{BC}| = \sqrt{c^2} = c$$

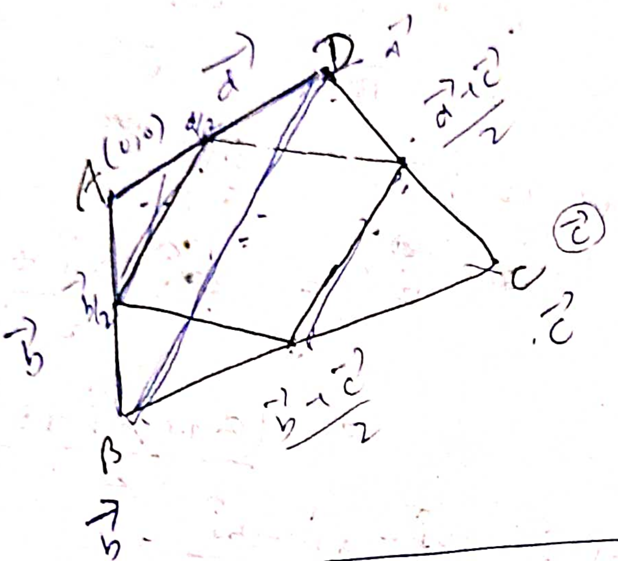
$\therefore BC = 2 \cdot DE$

Since $\vec{BC} = 2 \cdot \vec{DE}$

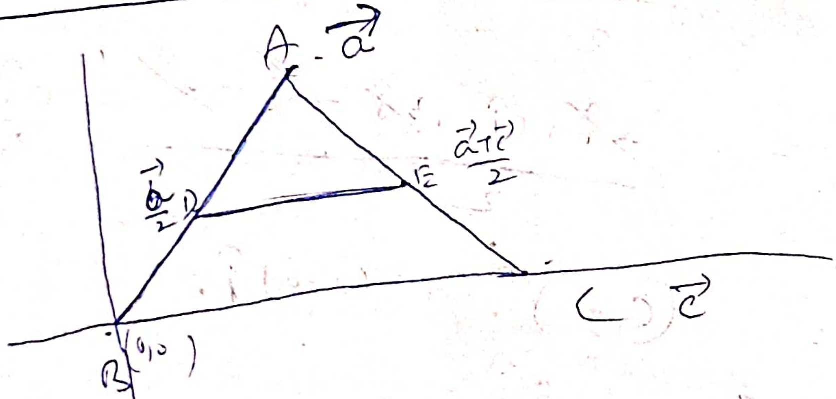
$\vec a, \vec b, \vec c, \vec d$

$\frac{\vec a + \vec d}{2}, \frac{\vec b + \vec c}{2}$

(d)



(c)



Let ABC be a triangle. Let B be the origin.

Position vector of A be $\vec a$,

" " C be $\vec c$

D, E be the mid points of AB & AC respectively.

D has the position vector $\frac{\vec{Ba}}{2}$

& E has the position vector $\frac{\vec{a}+\vec{c}}{2}$,

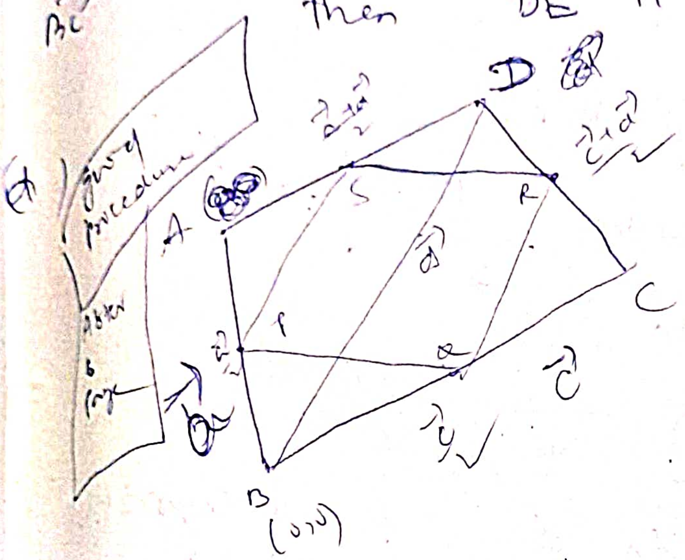
$\vec{BC} = \vec{c}$.

$\vec{DE} = \frac{\vec{a}+\vec{c}}{2} - \frac{\vec{a}}{2} = \frac{\vec{c}}{2}$

$\vec{DE} = \frac{1}{2}\vec{BC}$ (Proved)

Since $\vec{DE}$ is a scalar multiple of $\vec{BC}$ then $\vec{DE} \parallel \vec{BC}$ (Proved)

Ex)



Let ABCD be a quadrilateral.

Let P, Q, R, S be the mid points of ~~....~~, & AB, BC, CD, AD respectively.

Let B point be the ~~The poss~~ origin. The position vectors of points are shown in figure.

In $\triangle ABD$, $\vec{BD} = \vec{d}$

$\vec{PS} = \frac{\vec{a}+\vec{d}}{2} - \frac{\vec{a}}{2} = \frac{\vec{d}}{2}$

Since $\vec{PS} = \frac{1}{2}\vec{BD}$ i.e a scalar multiple of $\vec{BD}$ $\therefore \vec{PS} \parallel \vec{BD}$

In ΔBCD, $\overrightarrow{BD} = \vec{d}$

$$\overrightarrow{QR} = \frac{\vec{c}+\vec{d}}{2} - \frac{\vec{c}}{2}$$

$$= \frac{\vec{d}}{2}$$

Since $\overrightarrow{BD}$ is scalar multiple of $\overrightarrow{QR}$ then $\overrightarrow{QR} \parallel \overrightarrow{BD}$

But, before it is proved that $\overrightarrow{PS} \parallel \overrightarrow{BD}$.

$\therefore \overrightarrow{QR} \parallel \overrightarrow{PS}$.

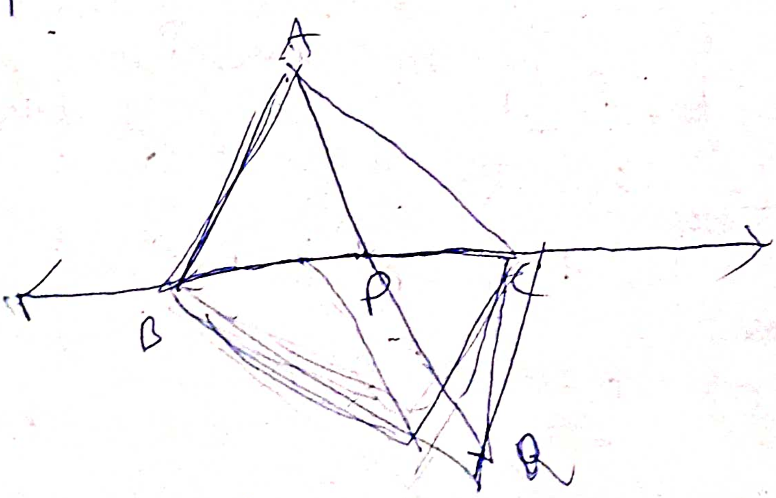
Similarly it can be proved that.

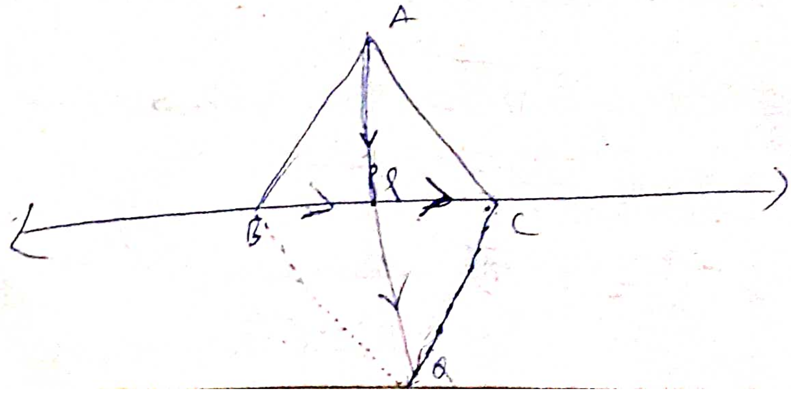
$\overrightarrow{PQ} \parallel \overrightarrow{AC} \parallel \overrightarrow{SR}$

$\therefore \overrightarrow{PQ} \parallel \overrightarrow{SR}$.

$\therefore$ PQRS is a parallelogram

(e)





$$\vec{PQ} = \vec{AP} + \vec{PB} + \vec{PC} \quad (\text{Given})$$

$$= \vec{AB} + \vec{PC}$$

$$\Rightarrow \vec{AB} = \vec{PQ} - \vec{PC}$$

But $\vec{CQ} = \vec{PQ} - \vec{PC}$ $\quad (\because \vec{PC} + \vec{CQ} = \vec{PQ})$

$$\therefore \vec{AB} = \vec{CQ}$$

The lines of action of $\vec{AB}$ & $\vec{CQ}$ are in same direction and $AB = CQ$.

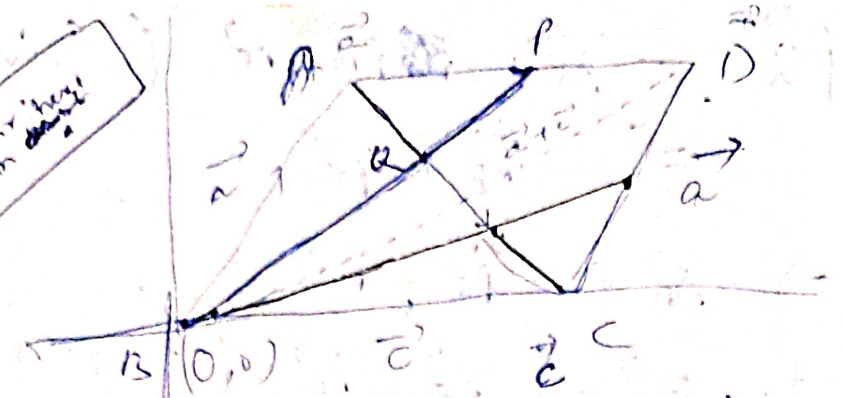
i.e $AB \parallel CQ$.

$\therefore$ and $AB = CQ$

$\therefore$ ABQC is a parallelogram.

(f)

Another similar procedure in next page



Let ABCD be a parallelogram.

Position vector of A, D, C are $\vec a$, $\vec d$ & $\vec c$ respectively.

Let the line joining the midpoint of AD and origin cuts the diagonal at Q.

Mid point of AD, 'P' has the position vector $\frac{\vec a + \vec d}{2}$.

$$\overrightarrow{BP} = \frac{\vec a + \vec d}{2} - \vec 0 = \frac{\vec a + \vec d}{2}$$

$$= \frac{\vec a + \vec a + \vec c}{2} \quad (\because \vec d = \vec a + \vec c \text{ by } \triangle \text{ law of addition})$$

$$= \frac{2\vec a + \vec c}{2}$$

The ~~point~~ vector which divides AC in the ratio 1:2

$$\frac{1.\vec c + 2\vec a}{3} = \frac{\vec c + 2\vec a}{3}$$

$$= \frac{3}{2} \times \frac{\vec c + 2\vec a}{3} = \frac{3}{2} \times \overrightarrow{BP}$$

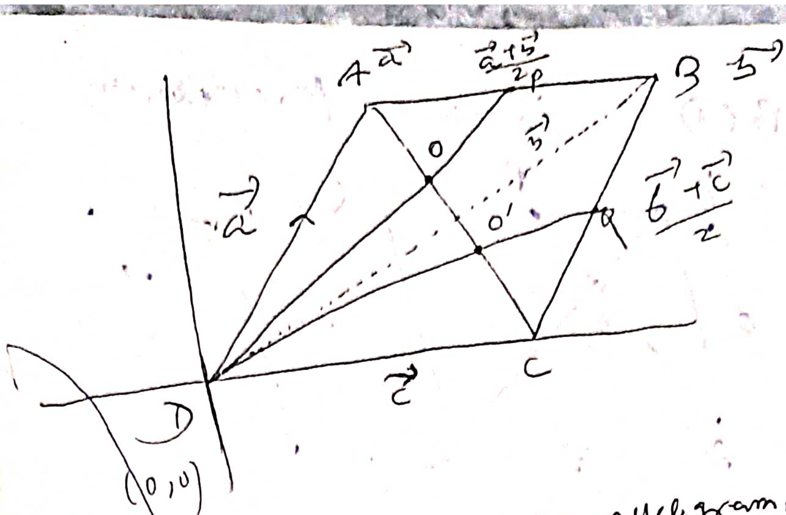
Since this vector is scalar multiple of $\overrightarrow{BP}$, ∴ $\overrightarrow{BP}$ & that vector are coincident.

∴ $\overrightarrow{BP}$ divides the diagonal in the ratio 1:3

~~∴ Q is the point in the given ratio~~ $\boxed{1:2}$ ..

The line joining a vertex to the midpoint of opposite side trisects the other diagonal (Proved)

Again,



Let ABCD be a parallelogram.
Let D be the origin.
position vector of A, B, & C are $\vec{a}$, $\vec{b}$ & $\vec{c}$.
P be the mid point of AB.
Q be the mid point of BC.
position vector of P = $\frac{\vec{a}+\vec{b}}{2}$.
" " Q = $\frac{\vec{b}+\vec{c}}{2}$.
Let DP & DQ intersects the diagonal AC at O' & O'.
Let O divides PD in the ratio K : 1

Position vector of O = $\frac{1.\left(\frac{\vec{a}+\vec{b}}{2}\right) + K.\vec{0}}{K+1} = \frac{(\vec{a}+\vec{b})}{2(K+1)}$

Let O' divides QD in the ratio $\lambda : 1$

" " $O' = \frac{1\left(\frac{\vec{b}+\vec{c}}{2}\right) + \lambda.\vec{0}}{\lambda+1} = \frac{(\vec{b}+\vec{c})}{2(\lambda+1)}$

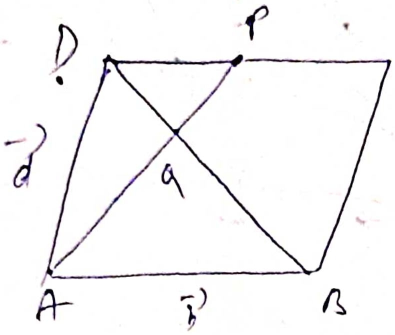
AO =

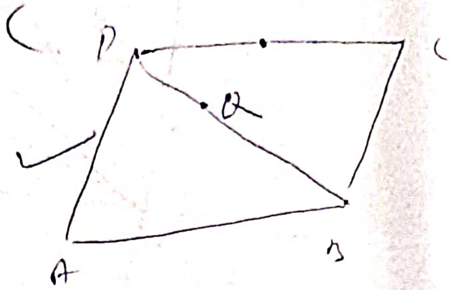
OO' =

O'C =

So the diagonal is trisected.

(f) ABCD in a parallelogram





Let A be the origin.

Position vector of B $= \vec{b}$

P. V of D $= \vec{d}$

$\vec{AC} = \vec{b} + \vec{d}$ (Parallelogram of addition)

Let Q be the point of trisection of $\vec{DB}$ which divides it in the ratio 1:2

$\therefore$ Position vector of Q $= \dfrac{2\vec{d} + 1.\vec{b}}{3}$

$= \dfrac{2\vec{d} + \vec{b}}{3}$

$\vec{AQ} = \dfrac{2\vec{d} + \vec{b}}{3}$.

Position vector of mid point P of DC $=$

$\dfrac{\vec{AC} + \vec{AD}}{2}$

$= \dfrac{\vec{b} + \vec{d} + \vec{d}}{2} = \dfrac{2\vec{d} + \vec{b}}{2}$

$\therefore$ $\vec{AP}$ = Position Vector of P $= \dfrac{2\vec{d} + \vec{b}}{2}$

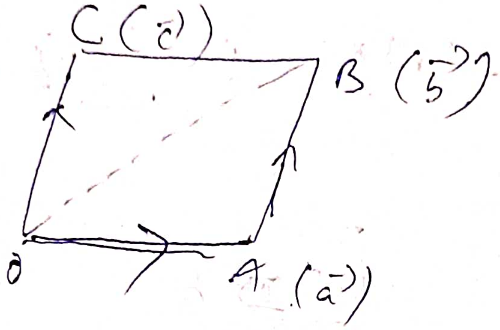
$= \frac{3}{2} \times \frac{2\vec{a}+\vec{b}}{3}$

$= \frac{3}{2} \times \vec{AQ}$

$\therefore$ $\vec{AP}$ and $\vec{AQ}$ are collinear.

i.e AP passes through the point of trisection Q.

(b)



$\vec{OB} = \vec{b}$, $OA = \vec{a}$

Now $\vec{OC} = \vec{c}$

By ▱gm law

$\vec{OA} + \vec{OC} = \vec{OB}$

$\Rightarrow \vec{a} + \vec{c} = \vec{b}$

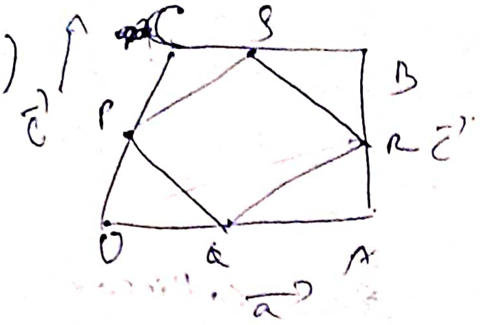
The mid point of $\vec{OB}$ has the

$p.v = \frac{\vec{b}}{2} = \frac{\vec{a}+\vec{c}}{2}$

The mid point of $\vec{CA}$ has

$p.v = \frac{\vec{a}+\vec{c}}{2}$

$\therefore$ The mid pt of $\vec{OB}$ & mid pt of $\vec{CA}$ are same. i.e OB & CA bisect each other.

(d) 

P.V of P $= \frac{\vec{c}}{2}$,

P.V Q $= \frac{\vec{a}}{2}$, (position of $\vec{B}$)

P.V $\vec{R} = \frac{\vec{a}+\vec{b}}{2}$ (position vector of $\vec{B}$)

P.V $\vec{S} = \frac{\vec{b}+\vec{c}}{2}$

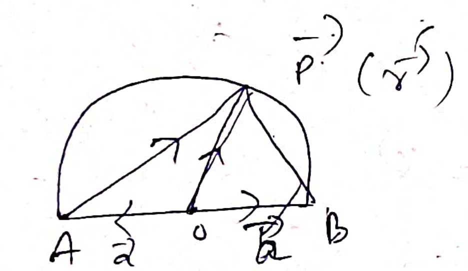
$\vec{PQ} = \frac{\vec{a}-\vec{c}}{2}$, $\vec{SR} = \frac{\vec{a}-\vec{c}}{2}$

$\vec{PQ} = \vec{SR}$. $\Rightarrow$ PQ = SR & PQ ∥ SR

∴ PQRS is ▱ Parallelogram.

15 - b

(V)



Let P.V of B $= \vec{a}$

" of A $= -\vec{a}$

Let P.V P $= \vec{r}$

∴ $|\vec{r}| = |\vec{a}| =$ radius.

Now $\vec{AP} = \vec{r} + \vec{a}$

$\vec{BP} = \vec{r} - \vec{a}$

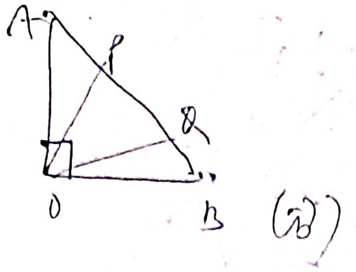
∴ $\vec{AP} \cdot \vec{BP} = (\vec{r}+\vec{a}) \cdot (\vec{r}-\vec{a})$

$= (\vec{r})^2 - (\vec{a})^2$

$= |\vec{r}|^2 - |\vec{a}|^2 = |\vec{a}|^2 - |\vec{a}|^2 = 0$

∴ AP ⊥ BP ∴ ∠APB = 90° (∵ $|\vec{r}| = \vec{a}$ = radius)

(viii)



Let O be origin.

Let P.V of B be $\vec{b}$, P.V of A $= \vec{a}$

$\vec{AB} = \vec{b} - \vec{a}$

$AB^2 = (\vec{AB})^2 = \vec{AB} \cdot \vec{AB}$

$= (\vec{b} - \vec{a}) \cdot (\vec{b} - \vec{a})$

$= (\vec{b} - \vec{a})^2$

$= \vec{b}^2 + \vec{a}^2 - 2\vec{b} \cdot \vec{a}$

$= \vec{a}^2 + \vec{b}^2 - 2\vec{a} \cdot \vec{b}$

P is the pt which divides AB in the ratio 1 : 2.

$\therefore$ PV of P $= \dfrac{2\vec{a} + \vec{b}}{3}$.

Similarly Q divides AB in the ratio 2 : 1

P.V of Q $= \dfrac{2\vec{b} + \vec{a}}{3}$

Also given $\angle AOB = 90°$.

$\therefore \vec{OA} \perp \vec{OB}$.

$\therefore \vec{a} \perp \vec{b} \Rightarrow \vec{a} \cdot \vec{b} = 0$

$\therefore AB^2 = \vec{a}^2 + \vec{b}^2$

$OP^2 = \vec{OP} \cdot \vec{OP} = \left(\dfrac{2\vec{a} + \vec{b}}{3}\right) \cdot \left(\dfrac{2\vec{a} + \vec{b}}{3}\right)$

$= \left(\dfrac{2\vec{a} + \vec{b}}{3}\right)^2 = \dfrac{4\vec{a}^2 + \vec{b}^2 + 4\vec{a} \cdot \vec{b}}{9}$

$= \dfrac{4\vec{a}^2 + \vec{b}^2}{9}$

$OQ^2 = (\overrightarrow{OQ})^2$

$= \left(\frac{2\vec{b}+\vec{a}}{3}\right)^2$

$= \frac{4\vec{b}^2 + \vec{a}^2 + 4\vec{b}\cdot\vec{a}}{9} = \frac{(4\vec{b}+\vec{a})^2}{9}$

$\therefore OP^2 + OQ^2 = \cancel{\frac{5}{9}(\vec{a}^2+\vec{b}^2)} = \frac{5}{9} AB^2$.

$= \frac{4\vec{a}^2 + \vec{b}^2}{9} + \frac{4\vec{b}^2 + \vec{a}^2}{9}$

$= \frac{1}{9}\left(5\vec{a}^2 + 5\vec{b}^2\right)$

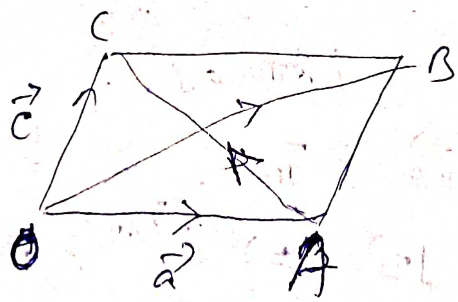
$= \frac{5}{9}\left(\vec{a}^2 + \vec{b}^2\right)$

$= \frac{5}{9}(\overrightarrow{AB})^2$

$= \frac{5}{9} AB^2$ (Proved)

10

iii



Let O be the origin.

P.V of $A = \vec{a}$

P.V of $\vec{C} = \vec{c}$

$\overrightarrow{AC} = \vec{c} - \vec{a}$

$\left(\because \vec{a} + \overrightarrow{AC} = \vec{c} \Rightarrow \overrightarrow{AC} = \vec{c} - \vec{a}\right)$

$\vec{OB} = \vec{a} + \vec{c}$ (By $\square$ law of addition)

Given that

$AC = OB$

$\Rightarrow AC^2 = OB^2$

$\Rightarrow (\vec{AC})^2 = (\vec{OB})^2$

$\Rightarrow (\vec{c} - \vec{a})^2 = (\vec{a} + \vec{c})^2$

$\Rightarrow \cancel{(\vec{c})^2} + \cancel{(\vec{a})^2} - 2 \cdot \vec{c} \cdot \vec{a} = \cancel{(\vec{a})^2} + \cancel{(\vec{c})^2} + 2 \vec{a} \cdot \vec{c}$

$\Rightarrow - \vec{a} \cdot \vec{c} = \vec{a} \cdot \vec{c}$

$\Rightarrow 2 \vec{a} \cdot \vec{c} = 0$

$\Rightarrow \vec{a} \cdot \vec{c} = 0$

$\Rightarrow \vec{a} \perp \vec{c}$

$\Rightarrow \vec{OA} \perp \vec{OC}$

$\therefore$ OABC is a rectangle.