Module-2 : (Chapter-5) Time-Varying Fields and Maxwell's Equations

Henceforth, we shall examine situations in which electric and magnetic fields are dynamic or time varying. In static EM fields, electric and magnetic fields are independent of each other, whereas in dynamic EM fields, the two fields are interdependent.

In otherwords, a time-varying electric field necessarily involves a corresponding time-varying magnetic field. Second, time-varying EM fields, represented by $E(x,y,z,t)$ and $H(x,y,z,t)$, are of more practical value than static EM fields.

Third, electrostatic fields are usually produced by stationary / static electric charges, where as magnetostatic fields are due to motion of electric charges with uniform velocity (direct current) or static magnetic charges (magnetic poles); time-varying fields or waves are usually due to accelerated charges or time-varying currents such as shown in fig 5.1. Any pulsating current will produce radiation (time-varying fields).

In summary:

Stationary Charges → Electrostatic fields

Steady currents → Magnetostatic fields

time-varying currents → Electromagnetic fields (or Waves)

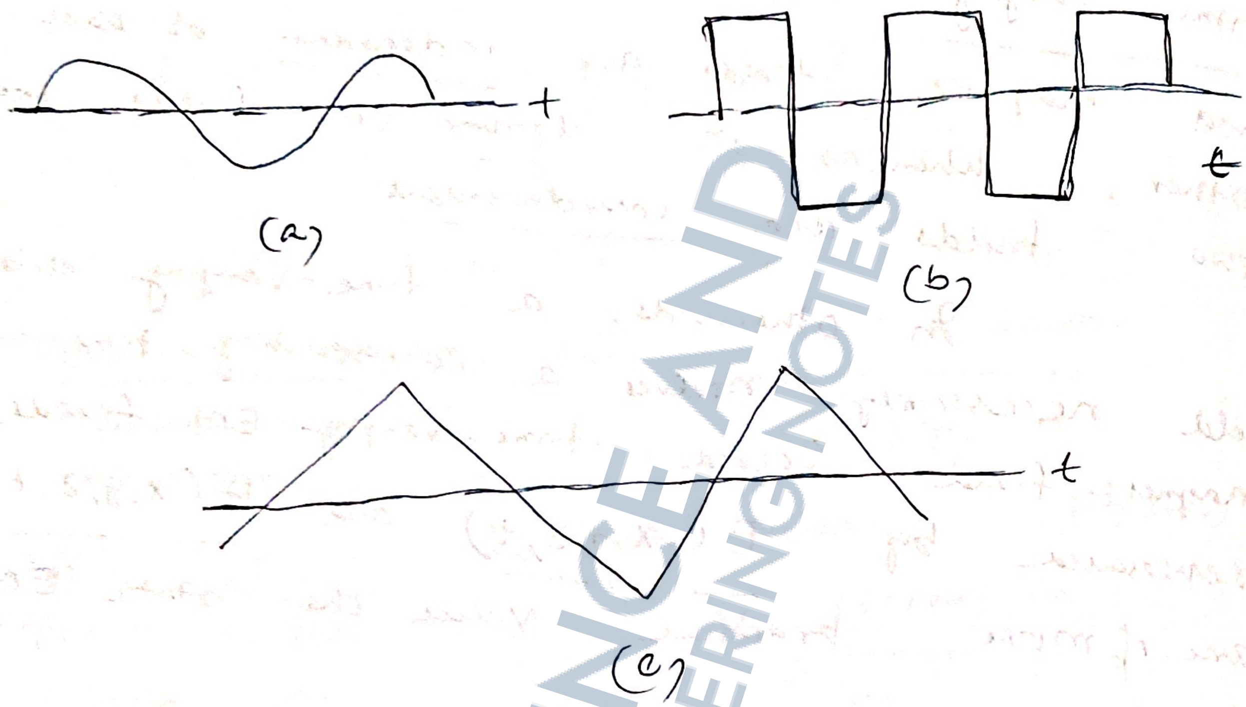


Fig 5.1 Example of time-varying current:
(a) Sinusoidal (b) Rectangular (c) Triangular

Faraday's law of Electromagnetic Induction :-

According to Faraday's experiments, a static magnetic field produces no current flow, but a time-varying field produces an induced voltage (Called electromotive force or simply emf) in a Closed circuit, which causes a flow of current.

Faraday discovered that the induced emf, Vemf (in volts), in any closed circuit is equal to the time rate of change of the magnetic flux linkage by the circuit.

This is called Faraday's law, and it can be expressed as

$$V_{emf} = -\frac{d\lambda}{dt} = -N\frac{d\Psi}{dt} \qquad (5.1)$$

where $\lambda = N\Psi$ is the flux linkage, N is the number of turns in the circuit, and Ψ is the flux through each turn.

The $-ve$ sign shows that the induced voltage act in such a way as to oppose the flux producing it. This is known as Lenz's law, and it emphasizes that the direction of current flow in the circuit is such that the induced magnetic field produced by the induced current will oppose the change in the original magnetic field.

Transformer and Motional Electromotive Forces

Having considered the connection between emf and electric field, we may examine how Faraday's law links electric and magnetic fields. For a circuit with a single turn $(N=1)$, eq^n (5.1) becomes,

$$V_{emf} = -\frac{d\psi}{dt} \qquad (5.4)$$

In terms of E and B, eq^n (5.4) can be written as

$$V_{emf} = \oint_L E \cdot dl = -\frac{d}{dt}\int_S B \cdot ds \qquad (5.5)$$

Where ψ has been replaced by $\int_S B \cdot ds$ and S is the surface area of the circuit bounded by the closed path L. It is clear from eqn (5.5) that in a time-varying situation, both electric and magnetic fields are present and interrelated. Note that dl and ds in eqn (5.5) are in accordance with Stoke's theorem.

This should be observed in Fig 5.3. The variation of flux with time as in eqn (5.1) may be caused in three ways:

1. By ~~keeping~~ having a stationary loop in a time-varying B field
2. By having a time-varying loop area in a static B field
3. By having a time-varying loop area in a time-varying B field

A. Stationary loop in time-varying B field (Transformer Emf)

In figure (5.3) a stationary conducting loop is in a time-varying magnetic 'B' field. Equation (5.5) becomes

$$V_{emf} = \oint_L E.dl = -\int_S \frac{\partial B}{\partial t}.ds \quad \text{—(5.6)}$$

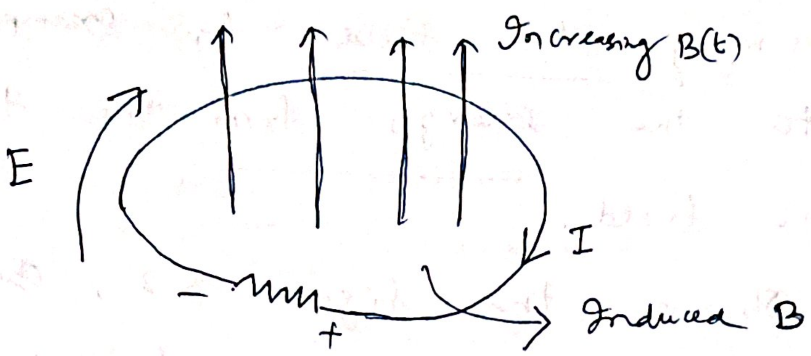


Fig 5.3 : Induced emf due to a stationary loop in a time-varying B field

This emf induced by the time-varying current (producing the time-varying B field) in a stationary loop is often referred to as transformer emf in power analysis, since it is due to transformer action. By applying Stoke's theorem to the middle term in eqn (5.6), we obtain

$$\int_S (\nabla \times E) . ds = -\int_S \frac{\partial B}{\partial t} . ds \quad \text{——} (5.7)$$

For the two integrals to be equal, their integrands must be equal; that is

$$\boxed{\nabla \times E = -\frac{\partial B}{\partial t}} \quad \text{——} (5.8)$$

→ This is one of the Maxwell's equations for time-varying fields. It shows that the time-varying E field is not conservative ($\nabla \times E \neq 0$).

→ This does not imply that the principles of energy conservation are violated. This work done in taking a charge about a closed path in a time-varying electric field, for example, is due to the energy from the time-varying magnetic field.

→ Observe that figure 5.3, obeys Lenz's law: the induced current I flows such as to produce a magnetic field that opposes the change in B(t).

B. Moving Loop in Static B field (Motional EMF)

When a conducting loop is moving in a static 'B' field, an emf is induced in the loop. We recall that force on charge moving with uniform velocity 'U' in a magnetic field 'B' is

$$F_m = Q\,(U \times B) \quad \text{——— (5.9)}$$

We define the motional electric field as

$$E_m = \frac{F_m}{Q} = U \times B \quad \text{——— (5.10)}$$

If we consider a conducting loop, moving with uniform velocity 'U' as consisting of large number of free electrons, the emf induced in the loop is

$$V_{emf} = \oint_L E_m \cdot dl = \oint_L (U \times B) \cdot dl \quad \text{——— (5.11)}$$

This type of emf is called motional emf

or flux-cutting emf because it is due to motional action. It is the kind of emf found in electrical machines such as motors, generators, and alternators. Fig. 5.4 illustrates a two-pole d.c machine with one armature coil and a two-bar commutators. We can observe that voltage is generated as the coil rotates within the magnetic field. (166)

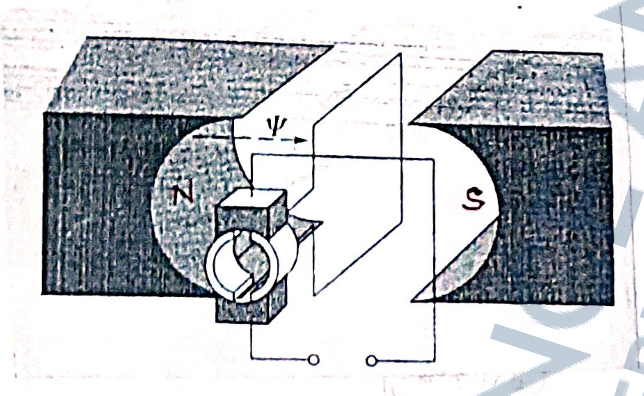


Fig 5.4 : A direct - current Machine

C. Moving Loop in time-varying field

In case, a moving conducting loop is in a time-varying magnetic field, Both transformer and motional emf are present. Combining eqn (5.6) and (5.11) gives total emf

$$V_{emf} = \oint_L E \cdot dl = -\int_S \frac{\partial B}{\partial t} \cdot ds + \oint_L (U \times B) \cdot dl \quad (5.12)$$

Continuity Equation

From the principle of charge conservation, the time rate of decrease of charge within a given volume must be equal to the net outward current flow through the surface of the volume. Thus current I_{out} coming out of the closed surface is

$$I_{out} = \oint J \cdot ds = -\frac{dQ_{in}}{dt} \quad \text{—} (1)$$

where Q_{in} is the total charge enclosed by the surface. Invoking divergence theorem, we write

$$\oint J \cdot ds = \int_V (\nabla \cdot J)\, dv \quad \text{—} (2)$$

But $$-\frac{dQ_{in}}{dt} = -\frac{d}{dt}\int_V \rho_v\, dv = -\int_V \frac{\partial \rho_v}{\partial t}\, dv \quad \text{—} (3)$$

Putting eqn (2) & (3) in eqn (1), we have

$$\int_V (\nabla \cdot J)\, dv = -\int_V \frac{\partial \rho_v}{\partial t}\, dv$$

$$\Rightarrow \boxed{\nabla \cdot J = -\frac{\partial \rho_v}{\partial t}} \quad \text{—} (4)$$

Which is called the continuity of current equation or simply continuity equation. Thus continuity eqn derived from principle of conservation of charge.

→ There can be no accumulation of charge at any point. Total charge leaving a volume is same as total charge entering it. Kirchhoff's law follows from this.

Displacement Current

We shall now reconsider Maxwell's curl equation for magnetic fields (Ampere's circuit law) for time-varying conditions.

For static EM fields, we recall that

$$\nabla \times H = J \quad \text{--- (5.13)}$$

[Refer eqn (4.20)]

But the divergence of the curl of any vector field is identically zero. Hence eqn (5.13) becomes

$$\nabla \cdot (\nabla \times H) = 0 = \nabla \cdot J \quad \text{--- (5.14)}$$

But from the continuity eqn

$$\nabla \cdot J = -\frac{\partial \rho_v}{\partial t} \quad \text{--- (5.15)}$$

So eqn (5.14) & (5.15) are incompatible for time-varying conditions. We must modify eqn (5.13) to agree with eqn (5.15).

To do this, we add a term to eqn (5.13) so that it becomes

$$\nabla \times H = J + J_d \quad \text{--- (5.16)}$$

where J_d is to be determined and defined.

Again, the divergence of the curl of

any vector is zero. Hence eqn (5.16) becomes

$$\nabla\cdot(\nabla\times H) = 0 = \nabla\cdot J + \nabla\cdot J_d \quad \text{—— (5.17)}$$

In order eqn (5.17) to agree with eqn (5.15),

Maxwell's 1st eqn

$$\nabla\cdot J_d = -\nabla\cdot J = \frac{\partial \rho_v}{\partial t} = \frac{\partial}{\partial t}(\nabla\cdot D)$$

$$= \nabla\cdot\left(\frac{\partial D}{\partial t}\right) \quad \text{—— (5.18a)}$$

$$\Rightarrow \nabla\cdot J_d = \nabla\cdot\left(\frac{\partial D}{\partial t}\right)$$

$$\Rightarrow \boxed{J_d = \frac{\partial D}{\partial t}} \quad \text{—— (5.18 b)}$$

Substituting 5.18(b) onto eqn (5.16), we have

$$\boxed{\nabla\times H = J + \frac{\partial D}{\partial t}} \quad \text{—— (5.19)}$$

This is the Maxwell's eqn (based on Ampere's circuit law) for time-varying field.

(Note:- Now verify Divergence of Curl = 0 }

$$\nabla\cdot(\nabla\times H) = \nabla\cdot J + \nabla\cdot\frac{\partial D}{\partial t}$$

from eqn (5.19)

$$= \nabla\cdot J + \frac{\partial}{\partial t}(\nabla\cdot D)$$

$$= \nabla\cdot J + \frac{\partial}{\partial t}\rho_v$$

$$= -\frac{\partial \rho_v}{\partial t} + \frac{\partial \rho_v}{\partial t}$$

$$= 0 \quad)$$

The term $J_d = \frac{\partial D}{\partial t}$ is known as displacement current density and J is the conduction current density ($J = \sigma E$).

The insertion of J_d into eqn (5.13) was one of the major contributions of Maxwell. Without the term J_d, the propagation of EM waves would be impossible.

Based on the displacement current density, we define the displacement current as

$$\boxed{I_d = \int J_d \cdot ds = \int \frac{\partial D}{\partial t} \cdot ds} \quad \text{(5.20)}$$

We must bear in mind that displacement current is a result of time-varying electric field.

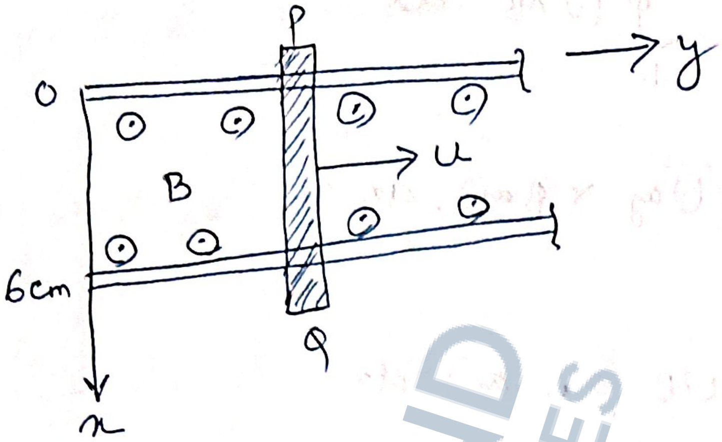
Example 5.1 :-

A conducting bar can slide freely over two conducting rails as shown in figure. Calculate the induced voltage in the bar

(a) If the bar is stationary at $y = 8$ cm and $B = 4 \cos 10^6 t \, a_z$ mWb/m²

(b) If the bar slides at a velocity $u = 20 a_y \frac{m}{s}$ and $B = 4 a_z$ mWb/m²

(c) If the bar slides at a velocity $u = 20 a_y$ m/s and $B = 4\cos(10^6 t - y)\, a_z \ \frac{mWb}{m^2}$.



Ans:- (a) In this case, we have transformer emf given by

$$V_{emf} = -\int \frac{\partial B}{\partial t} \cdot ds$$

$$= -\int_{y=0}^{0.08}\int_{x=0}^{0.06} \frac{\partial}{\partial t}(4\cos 10^6 t)\times 10^{-3}\, dx\, dy$$

a_z ds normal

$ds = dx\,dy$

$y = 8cm = 0.08m$

$x = 6cm = 0.06m$

$$= +4\times10^{-3}\times10^{6}\int_{y=0}^{0.08}\int_{x=0}^{0.06} \sin 10^6 t\; dx\, dy$$

$$= 4\times10^{3} \times \sin 10^6 t \times [x]_0^{0.06}\ [y]_0^{0.08}$$

$$= 4\times10^{3}\times(0.06)\times(0.08)\times \sin 10^6 t$$

$$\Rightarrow \boxed{V_{emf} = 19.2 \sin 10^6 t \ \text{Volt}}$$

(b) This is the case of motional emf

$$V_{emf} = \oint_L (U \times B) \cdot dl$$

$$= \int_{x=l}^{0} (U a_y \times B a_z) \cdot dx\, a_x$$

$$= \int_{x=l}^{0} UB (a_x \cdot a_x)\, dx$$

$$= UB [x]_l^0$$

$$= -UBl$$

$$= -20 \times 4 \times 0.06 \times 10^{-3}$$

$\because B = 4 \frac{mWb}{m^2}$

$$\boxed{V_{emf} = -4.8\ mV}$$

(C) Both transformer emf and motional emf are present in this case

$$V_{emf} = -\int \frac{\partial B}{\partial t} ds + \oint_L (U \times B) dl$$

$$= \int_{x=0}^{0.06} \int_0^y 4 \times 10^{-3} \times 10^6 \sin(10^6 t - y') dy'\, dx$$

$\because \frac{\partial}{\partial t} \cos(10^6 t - y) = -10^6 \sin(10^6 t - y)$

$$+ \int_{0.06}^{0} [20 a_y \times 4 \times 10^{-3} \cos(10^6 t - y) a_z] \cdot dx\, a_x$$

$$\Rightarrow V_{emf} = 4\times10^{3}\times0.06\cos(10^6t-y')\Big|_0^y$$
$$-\ 80\times10^{-3}\times0.06\cos(10^6t-y)$$

$$= 240\left[\cos(10^6t-y) - \cos(10^6t)\right] - 4.8\times10^{-3}\cos(10^6t-y)$$

$$\Rightarrow V_{emf} \simeq 240\left[\cos(10^6t-y) - \cos 10^6t\right]$$

($\because$ motional emf is negligible compared with transformer emf)

By trigonometry identity

$$\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$\Rightarrow V_{emf} = 240\times(-2)\times\sin\left(\frac{10^6t-y+10^6t}{2}\right)\times \sin\left(\frac{\cancel{10^6t}-y-\cancel{10^6t}}{2}\right)$$

$$\boxed{\Rightarrow V_{emf} = 480\sin\left(10^6t-\frac{y}{2}\right)\sin\left(\frac{y}{2}\right)\ \text{Volt}}$$

* Example :- (5.2) [Page 193]

A parallel-plate capacitor with plate area of $5\ cm^2$ and plate separation of $3\ mm$ has a voltage $50\sin10^3t$ V applied to its plates. Calculate the displacement current assuming $\epsilon = 2\epsilon_0$.

Ans:- (Example 5.2)

$$D = \epsilon E = \epsilon \frac{V}{d} \qquad \left(\because E = \frac{V}{d}\right)$$

$$J_d = \frac{\partial D}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\epsilon V}{d}\right) = \frac{\epsilon}{d}\frac{\partial V}{\partial t}$$

$$I_d = J_d . S = \frac{\epsilon}{d}\frac{\partial V}{\partial t}(S) = \frac{\epsilon S}{d}\frac{\partial V}{\partial t}$$

$$\therefore I_d = \frac{2\epsilon_0 \times S}{d}\frac{\partial}{\partial t}(50 \sin 10^3 t)$$

$$= 2 \times \frac{10^{-9}}{36\pi} \times \frac{5 \times 10^{-4}}{3 \times 10^{-3}} \times 50 \times 10^3 \cos(10^3 t)$$

$$= \left(\frac{2 \times 5 \times 50}{108\pi}\right) \times 10^{-7} \cos 10^3 t$$

$$= 1.4736 \times 10^{-7} \cos 10^3 t$$

$$\boxed{I_d = 147.36 \cos 10^3 t \ nA}$$

(Displacement current) (Ans)

Maxwell's Equations in Final forms

The Maxwell's equations for static electric and magnetic fields were presented in Table 4.1 of Magnetostatic fields. The more generalized forms of these equations are those for time-varying conditions as shown in Table 5.1.

Table 5.1 Generalized Forms of Maxwell's Equation

Differential Form	Integral Form	Remarks
$\nabla \cdot D = \rho_v$	$\oint_S D \cdot ds = \int_V \rho_v \, dv$	Gauss's law
$\nabla \cdot B = 0$	$\oint B \cdot ds = 0$	Non existence of isolated magnetic charge
$\nabla \times E = -\frac{\partial B}{\partial t}$	$\oint E \cdot dl = -\frac{\partial}{\partial t}\int_S B \cdot ds$	Faraday's law
$\nabla \times H = J + \frac{\partial D}{\partial t}$	$\oint H \cdot dl = \int_S \left(J + \frac{\partial D}{\partial t}\right) ds$	Ampere's Circuit law

From the Table 5.1, we notice that the divergence equations remain the same, while the curl equations have been modified. The integral form of Maxwell's equations depicts the underlying physical laws, whereas the differential form is used more frequently in solving problems. For a field to qualify as an electromagnetic field, it must satisfy all four Maxwell's equations.

Time - Varying Potentials

For static EM fields, we obtained the electric scalar potential as

$$V = \int_v \frac{\rho_v \, dv}{4\pi\epsilon R} \quad -(5.21)$$

and the magnetic vector potential as

$$A = \int_v \frac{\mu J \, dv}{4\pi R} \quad -(5.22)$$

We would like to examine what happens to these potentials when the fields are time varying.

→ From Table 5.1, $\nabla \cdot B = 0$, which still holds good for time-varying fields.

Recalling Magnetic scalar & vector potential concepts,

$$B = \nabla \times A \quad -(5.23)$$

holds for time-varying conditions.

From Maxwell's eqn

$$\nabla \times E = -\frac{\partial B}{\partial t} = -\frac{\partial}{\partial t}(\nabla \times A)$$

∵ Referring eqn (3.46)

$$V = \frac{Q}{4\pi\epsilon_0 r} = \int \frac{\rho_v dv}{4\pi\epsilon_0 r}$$

&

$$Q = \int \rho_v \, dv$$

∵ From Maxwell's eqn

$$\nabla \cdot B = 0 \quad -①$$

We know Divergence of Curl = 0

$$\nabla \cdot (\nabla \times A) = 0 \quad ②$$

Equating ① & ②

$$B = \nabla \times A$$

where A is the magnetic vector potential

$$\Rightarrow \nabla \times E = -\nabla \times \left(\frac{\partial A}{\partial t}\right)$$

$$\Rightarrow \nabla \times \left(E + \frac{\partial A}{\partial t}\right) = 0 \quad \text{——} (5.24)$$

Since the curl of gradient of a scalar field is identically zero, from eqn (5.24), we have

$$E + \frac{\partial A}{\partial t} = -\nabla V \quad \text{——} (5.25)$$

$$\Rightarrow \boxed{E = -\nabla V - \frac{\partial A}{\partial t}} \quad \text{——} (5.26)$$

From eqn (5.23) & (5.26), we can determine the vector fields B and E, provided the potentials A & V are known.

From Maxwell's 1st eqn (same for static & time-varying fields)

$$\nabla \cdot D = \rho_v$$

$$\Rightarrow \nabla \cdot (\epsilon E) = \rho_v$$

$$\Rightarrow \nabla \cdot E = \frac{\rho_v}{\epsilon}$$

Putting the value of E from eqn (5.26)

$$\Rightarrow \nabla \cdot \left(-\nabla V - \frac{\partial A}{\partial t}\right) = \frac{\rho_v}{\epsilon}$$

$$\Rightarrow -\nabla^2 V - \frac{\partial}{\partial t}(\nabla \cdot A) = \frac{\rho_v}{\epsilon}$$

$$\Rightarrow \boxed{\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot A) = -\frac{\rho_v}{\epsilon}} \quad \text{——} (5.27)$$

Taking curl of eqn (5.23), we have

$$\nabla \times B = \nabla \times (\nabla \times A)$$

$$\Rightarrow \nabla \times (\mu H) = \nabla \times (\nabla \times A) \quad \Big| \because B = \mu H$$

$$\Rightarrow \mu (\nabla \times H) = \nabla \times (\nabla \times A)$$

$$\Rightarrow \mu \left(J + \frac{\partial D}{\partial t} \right) = \nabla \times (\nabla \times A) \quad \Big| \because \nabla \times H = J + \frac{\partial D}{\partial t}$$

$$\Rightarrow \mu (J) + \mu \frac{\partial}{\partial t}(\epsilon E) = \nabla \times (\nabla \times A) \quad \Big| \because D = \epsilon E$$

$$\Rightarrow \mu J + \mu \epsilon \frac{\partial}{\partial t}\left[-\nabla V - \frac{\partial A}{\partial t} \right] = \nabla \times (\nabla \times A) \quad \Big| \text{Using eqn (5.26)}$$

$$\Rightarrow \nabla \times (\nabla \times A) = \mu J - \mu \epsilon \nabla\left(\frac{\partial V}{\partial t}\right) - \mu \epsilon \frac{\partial^2 A}{\partial t^2} \quad (5.28)$$

From the vector identity

$$\nabla \times (\nabla \times A) = \nabla (\nabla \cdot A) - \nabla^2 A \quad (5.29)$$

Putting eqn (5.29) into eqn (5.28)

$$\nabla(\nabla \cdot A) - \nabla^2 A = \mu J - \mu \epsilon \nabla\left(\frac{\partial V}{\partial t}\right) - \mu \epsilon \frac{\partial^2 A}{\partial t^2}$$

$$\Rightarrow \nabla^2 A - \nabla(\nabla \cdot A) = -\mu J + \mu \epsilon \nabla\left(\frac{\partial V}{\partial t}\right) + \mu \epsilon \frac{\partial^2 A}{\partial t^2} \quad (5.30)$$

$$\Rightarrow \nabla^2 A - \mu \epsilon \frac{\partial^2 A}{\partial t^2} = -\mu J + \nabla\left(\nabla \cdot A + \mu \epsilon \frac{\partial V}{\partial t}\right) \quad (5.31)$$

Putting the Lorentz's condition for potentials, i.e $\boxed{\nabla \cdot A = -\mu \epsilon \frac{\partial V}{\partial t}}$ in eqn (5-31), ——— (5-32)

we have

$$\boxed{\nabla^2 A - \mu \epsilon \frac{\partial^2 A}{\partial t^2} = -\mu J} \quad \text{——— (5-33)}$$

& Also putting eqn (5-32) in eqn (5-27)

we have

$$\nabla^2 V + \frac{\partial}{\partial t}\left(-\mu \epsilon \frac{\partial V}{\partial t}\right) = -\frac{\rho_v}{\epsilon}$$

$$\Rightarrow \boxed{\nabla^2 V - \mu \epsilon \frac{\partial^2 V}{\partial t^2} = \frac{-\rho_v}{\epsilon}} \quad \text{——— (5-34)}$$

Eqn (5-33) & (5-34) are called the wave eqn's, which will be discussed in the next chapter.

It can be shown that solution of eqn (5-33) & (5-34) are

$$A = \int_v \frac{\mu [J]\, dv}{4\pi R} \quad \text{——— (5-35)}$$

$$V = \int_v \frac{[\rho_v]\, dv}{4\pi \epsilon R} \quad \text{——— (5-36)}$$

The term $[\rho_v]$ or $[J]$ means that the time t in $\rho_v(x, y, z, t)$ or $[J(x, y, z, t)]$ is replaced by the retarded time (t') given by

$$t' = t - \frac{R}{u} \quad \text{---------} (5\text{-}37)$$

where $R = |r - r'|$ is the distance betn the source point r' and the observation point r ~~and~~ and

$$u = \frac{1}{\sqrt{\mu\epsilon}} \quad \text{---------} (5\text{-}38)$$

is the velocity of wave propagation. In free space, $u = c \approx 3\times10^8 \frac{m}{sec}$ is the speed of light in vacuum.

→ Potentials V & A in eqn (5-36) & (5-35) are, respectively, called the retarded electric scalar potential and retarded magnetic vector potential.

→ Given ρ_v & J, V & A can be determined by using eqn (5-36) & (5-35); from V & A, E & B can be determined by using eqn (5-26) & (5-23), respectively.

Time-Harmonic Fields

A time-harmonic field is one that varies periodically or sinusoidally with time. Sinusoids are easily expressed in phasors, which are more convenient to work with.

Review of concepts of Phasors:

→ A phasor is a complex number that contains the amplitude and the phase of a sinusoidal oscillation. As a complex number, a phasor Z can be represented as

$$Z = x + jy = r\angle\phi \qquad \text{—— (5.39)}$$

or

$$Z = r e^{j\phi} = r(\cos\phi + j\sin\phi) \qquad \text{—— (5.40)}$$

where $j = \sqrt{-1}$, x is the real part of Z, y is the imaginary part of Z, r is the magnitude of Z, given by

$$r = |Z| = \sqrt{x^2 + y^2} \qquad \text{—— (5.41)}$$

and ϕ is the phase of Z, given by

$$\phi = \tan^{-1}\left(\frac{y}{x}\right) \qquad \text{—— (5.42)}$$

The phasor Z can be represented in rectangular form as $Z = x + jy$ or in polar form as $Z = r\angle\phi = re^{j\phi}$. The two form of representing Z are related in eqn (5.39) to (5.42) and illustrated in fig 5.5.

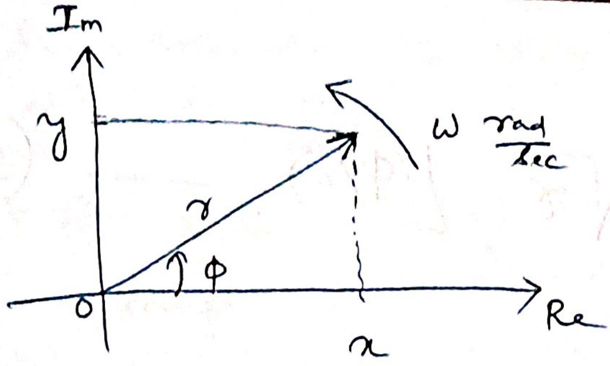


Fig 5.5: Representation of a phasor $Z = x + jy = r\angle\phi$

→ Addition & Subtraction of Phasors are better performed in rectangular form, multiplication and division are better done in Polar form.

→ Given complex numbers

$$Z = x + jy = r\angle\phi, \quad Z_1 = x_1 + jy_1 = r_1\angle\phi_1 \text{ and}$$

$$Z_2 = x_2 + jy_2 = r_2\angle\phi_2,$$ the following basic properties should be noted.

Addition :-

$$Z_1 + Z_2 = (x_1 + x_2) + j(y_1 + y_2) \quad \text{—— (5.43a)}$$

Subtraction :-

$$Z_1 - Z_2 = (x_1 - x_2) + j(y_1 - y_2) \quad \text{—— (5.43b)}$$

Multiplication :-

$$Z_1 Z_2 = r_1 r_2 \angle\phi_1 + \phi_2 \quad \text{—— (5.43c)}$$

Division :-

$$\frac{Z_1}{Z_2} = \frac{r_1}{r_2}\angle\phi_1 - \phi_2 \quad \text{—— (5.43d)}$$

Square root:

$$\sqrt{Z} = \sqrt{r} \angle(\phi/2) \quad \text{——} \quad (5.43e)$$

Complex Conjugate:

$$Z^* = x - jy = r \angle -\phi = r e^{-j\phi} \quad - (5.43f)$$

To introduce time element, we let

$$\phi = \omega t + \theta \quad \text{——} \quad (5.44)$$

where θ may be a function of time or space coordinates or a constant. The Real (Re) and imaginary (Im) part of

$$r e^{j\phi} = r e^{j(\omega t + \theta)} = r e^{j\theta} e^{j\omega t}$$

are respectively, given by

$$Re(r e^{j\phi}) = r \cos(\omega t + \theta) \quad \text{——} \quad (5.45)$$

and $$Im(r e^{j\phi}) = r \sin(\omega t + \theta) \quad \text{——} \quad (5.46)$$

Thus a sinusoidal current $I(t) = I_0 \cos(\omega t + \theta)$, for example, equals the real part of $\underline{I_0 e^{j\theta} e^{j\omega t}}$.

→ The complex term $I_0 e^{j\theta}$, which results from dropping the time factor $e^{j\omega t}$ in $\underline{I(t)}$, is called the <u>phasor current</u>, denoted by $\underline{I_s}$; that is

$$I_s = I_0 e^{j\theta} = I_0 \angle \theta \quad \text{——} \quad (5.47)$$

Where the subscript 'S' denotes the phasor form of $I(t)$.

Thus $I(t) = I_0 \cos(\omega t + \theta)$, the instantaneous form, can be expressed as

$$I(t) = Re\left(I_s e^{j\omega t}\right) \quad -(5.48)$$

$\because$ $Re\left(I_s e^{j\omega t}\right)$

$= \cancel{Re\left(I_0 \cdot e^{j\omega t}\right)}$

$= Re\left(I_0 e^{j\theta} \cdot e^{j\omega t}\right)$

$= Re\left(I_0 \cdot e^{j(\theta + j\omega t)}\right)$

$= I_0 \cos(\theta + \omega t)$

$= I_0 \cos(\omega t + \theta)$

→ In general, a phasor could be scalar or vector.

→ If a vector $A(x, y, z, t)$ is a time-harmonic field, the phasor form of A is $A_s(x, y, z)$; the two quantities are related as

$$\boxed{A = Re\left(A_s e^{j\omega t}\right)} \quad —(5.49)$$

For example, If $A = A_0 \cos(\omega t - \beta x) a_y$, we can write A as

$$A = Re\left(A_0 e^{-j\beta x} a_y \, e^{j\omega t}\right) \quad -(5.50)$$

Comparing (5.49) and (5.50), we have

$$A_s = A_0 e^{-j\beta x} a_y \quad —(5.51)$$

Notice from eq^n (5.49) that

$$\frac{\partial A}{\partial t} = \frac{\partial}{\partial t} Re\left(A_s e^{j\omega t}\right)$$

$$= Re\left(j\omega A_s e^{j\omega t}\right) \quad —(5.52)$$

Thus taking time derivative of instantaneous quantity is equivalent to ~~multiplication~~ multiplying its phasor form by $j\omega$. [Compare Eqn (5-49) & Eqn (5-52)]

i.e

$$\frac{\partial A}{\partial t} \rightarrow j\omega A_s \qquad (5\text{-}53)$$

Similarly, $\left(\because \frac{\partial}{\partial t} \text{ equivalent to } j\omega \right)$

$$\int A \, \partial t \rightarrow \frac{A_s}{j\omega} \qquad (5\text{-}54)$$

→ Notice the basic difference between the instantaneous form $A(x,y,z,t)$ and its phasor form $A_s(x,y,z)$; the former is time dependent and real, whereas the latter is time invariant and generally complex. It is easy to work with A_s and obtain A form A_s whenever necessary by using eqn (5-49)

→ We shall now apply phasor concept to time-varying EM fields. The field quantities $E(x,y,z,t)$, $D(x,y,z,t)$, $H(x,y,z,t)$, $B(x,y,z,t)$, $J(x,y,z,t)$, and $\rho_v(x,y,z,t)$ and their derivatives can be expressed in phasor form.

In Phasor form, Maxwell's equations for time-harmonic EM fields are presented in Table 5.2. From Table 5.2, note that the time factor $e^{j\omega t}$ disappears because it is associated with every term and therefore factors out, resulting in time independent eq^ns.

Table 5.2 Time - Harmonic Maxwell's equations

Assuming Time factor $e^{j\omega t}$

Point form	Integral form
$\nabla \cdot D_s = \rho_{vs}$	$\oint D_s \cdot ds = \int \rho_{vs} \, dv$
$\nabla \cdot B_s = 0$	$\oint B_s \cdot ds = 0$
$\nabla \times E_s = -j\omega B_s$	$\oint E_s \cdot dl = -j\omega \int B_s \cdot ds$
$\left(\because \frac{\partial}{\partial t} = j\omega \right)$	
$\nabla \times H_s = J_s + j\omega D_s$	$\oint H_s \cdot dl = \int (J_s + j\omega D_s) \cdot ds$

Example 5.2 :- Evaluate the complex numbers

(a) $Z_1 = \dfrac{j(3-j4)^*}{(-1+j6)(2+j)^2}$ (b) $Z_2 = \left[\dfrac{1+j}{4-j8}\right]^{\frac{1}{2}}$

Ans :-
(a)

$$Z_1 = \frac{j(3-j4)^*}{(-1+j6)(2+j)^2}$$

Let

$$Z_3 = j = 0+1.j = 1\angle 90^\circ$$

$$Z_4 = (3-j4)^* = 3+j4 = 5\angle 53.13^\circ$$

Note:- In calculator converter Rectangular to polar

$$Z_5 = (-1+j6) = \sqrt{37}\angle 99.46^\circ$$

$$Z_6 = (2+j)^2 = \left(\sqrt{5}\angle 26.56^\circ\right)^2 = 5\angle 2\times 26.56$$

$$\Rightarrow Z_6 = 5\angle 53.13^\circ$$

Hence,

$$Z_1 = \frac{Z_3 Z_4}{Z_5 Z_6} = \frac{(1\angle 90^\circ)(5\angle 53.13^\circ)}{(\sqrt{37}\angle 99.46^\circ)(5\angle 53.13^\circ)}$$

$$= \frac{1}{\sqrt{37}}\angle 90^\circ - 99.46^\circ$$

$$= 0.1644\angle -9.46^\circ$$

Note:- In calculator convert polar to rectangular

$$\boxed{Z_1 = 0.1622 - j0.027}$$

(Ans)

(b) $$Z_2 = \left[\frac{1+j}{4-j8}\right]^{\frac{1}{2}}$$

$$= \left[\frac{\sqrt{2}\ \angle 45^\circ}{4\sqrt{5}\ \angle -63.4^\circ}\right]^{\frac{1}{2}}$$

$$= \left(\frac{\sqrt{2}}{4\sqrt{5}}\right)^{\frac{1}{2}} \angle \frac{(45^\circ + 63.4^\circ)}{2}$$

$$\left(\because \sqrt{z} = \sqrt{r}\ \angle \phi/2\right)$$

$$Z_2 = 0.3976\ \angle 54.2^\circ$$ (Ans)