Integration

Integration is the reverse process of differentiation. Let $\frac{d}{dx} f(x) = g(x)$ then $f(x)$ is called integral of $g(x)$ and we write it as $\int g(x)\, dx = f(x)$

Here $\int g(x)\, dx$ is read as integral of $g(x)$ w.r.t x.

Sometimes integral is called as anti-derivative or primitive.

Also $\frac{d}{dx}(f(x) + c) = g(x)$

$\therefore \int g(x)\, dx = f(x) + c$

Here c is called constant of integration

Here $g(x)$ is called integrand

This process of integration is called <u>indefinite integration</u>.

— G.F.B Riemann discovered integration and it is called Riemann integral.

<u>~~From~~ Some Standard results</u>

1. $\frac{d}{dx}\frac{x^{n+1}}{n+1} = \frac{1}{n+1}\cdot(n+1)x^n = x^n$

$$\Rightarrow \boxed{\int x^n\,dx = \frac{x^{n+1}}{n+1}}$$

Provided $n \neq -1$

(2) $\frac{d}{dx}\log x = \frac{1}{x}$

$$\Rightarrow \boxed{\int \frac{1}{x}\,dx = \log|x| \text{ or } \ln|x|}$$

(3) $\frac{d}{dx}\sin x = \cos x$

$$\Rightarrow \boxed{\int \cos x\,dx = \sin x}$$

(4). $\frac{d}{dx} - \cos x = \sin x$

$$\Rightarrow \boxed{\int \sin x \, dx = -\cos x}$$

(5.) $\frac{d}{dx} \tan x = \sec^2 x$

$$\Rightarrow \boxed{\int \sec^2 x \, dx = \tan x}$$

(6.) $\frac{d}{dx} - \cot x = \operatorname{cosec}^2 x$

$$\Rightarrow \boxed{\int \operatorname{cosec}^2 x \, dx = -\cot x}$$

(7) $\frac{d}{dx} \sec x = \sec x \cdot \tan x$

$$\Rightarrow \boxed{\int \sec x \cdot \tan x \, dx = \sec x}$$

(8.) $\frac{d}{dx} - \operatorname{cosec} x = \operatorname{cosec} x \cdot \cot x$

$$\Rightarrow \boxed{\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x}$$

(9). $\frac{d}{dx} e^x = e^x$

$\Rightarrow$ $$\int e^x \, dx = e^x$$

(10) $\frac{d}{dx} \frac{a^x}{\ln a} = a^x$

$\Rightarrow$ $$\int a^x . dx = \frac{a^x}{\ln a}$$

(11.) $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$

$\Rightarrow$ $$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x \text{ or } -\cos^{-1} x$$

(12). $\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$

$\Rightarrow$ $$\int \frac{1}{1+x^2} dx = \tan^{-1} x \text{ or } -\cot^{-1} x$$

(13) $\frac{d}{dx} \sec^{-1}x = \frac{1}{|x|\sqrt{x^2-1}}$

$$\Rightarrow \int \frac{1}{|x|\sqrt{x^2-1}}\,dx = \sec^{-1}x \text{ or } -\operatorname{cosec}^{-1}x.$$

(14) $$\int K f(x)\,dx = K\int f(x)\,dx$$

where K is constant.

(15) $$\int dx = \int 1\cdot dx = \int x^0\,dx$$

$$= \frac{x^{0+1}}{0+1}$$

$$\therefore \int dx = x$$

(16.) $$\int (f(x) \pm g(x))\,dx$$

$$= \int f(x)\,dx \pm \int g(x)\,dx$$

* **Definite integration**

Let $\frac{d}{dx} f(x) = g(x)$ then

$$\int_a^b g(x)\,dx = f(x)\Big]_a^b = f(b) - f(a)$$

This process is called definite integration. The above result is called <u>fundamental theorem of calculus</u>

Here a and b are called limits of integration.

a is called lower } limit.
b is called upper

Ex:- $\int_0^1 \frac{1}{\sqrt{1-x^2}} dx$

$= \sin^{-1}x \Big]_0^1$

$= \sin^{-1}1 - \sin^{-1}0$

$= \pi/2 - 0 = \pi/2$

Ex $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1}x + C$

Integration of absolute value function

1. $\int_{-5}^{5} |x| \, dx$

$= \int_{-5}^{0} |x| \, dx + \int_{0}^{5} |x| \, dx$

$= \int_{-5}^{0} -x \, dx + \int_{0}^{5} x \, dx$

$= -\left.\frac{x^2}{2}\right]_{-5}^{0} + \left.\frac{x^2}{2}\right]_{0}^{5}$

$= -\left(0 - \frac{25}{2}\right) + \left(\frac{25}{2} - 0\right)$

$= \frac{25}{2} + \frac{25}{2} = 25$

2. $\int_0^3 |x-2|\, dx$

Put

$x-2 = y$

$\frac{dy}{dx} = 1$

$\Rightarrow dy = dx$

$x = 0 \Rightarrow y = -2$

$x = 3 \Rightarrow y = 1$

$= \int_0^3 (x-2)\, dx$

$= \int_0^3 x\, dx - 2$

$= \int_{-2}^{1} |y|\, dy$

$= \int_{-2}^{0} |y|\, dy + \int_0^1 |y| . dy$

$= \int_{-2}^{0} -y\, dy + \int_0^1 y\, dy$

$= -\frac{y^2}{2}\Big]_{-2}^{0} + \frac{y^2}{2}\Big]_0^1$

$= \left(0 - \left(-\frac{4}{2}\right)\right) + \left(\frac{1}{2} - 0\right)$

$= 2 + \frac{1}{2}$

$= \frac{5}{2}$

3. Integration of greatest integer function

1. $\int_{n}^{n+1} [x]\, dx$ where n is integer

$= \int_{n}^{n+1} n\, dx$ $\left(\because n \leq x < n+1 \text{ then } [x] = n \right)$

$= n \int_{n}^{n+1} 1 . dx$

$= n . (x) \Big]_{n}^{n+1}$

$= n (\cancel{n}+1-\cancel{n})$

$= n$ (Ans)

2. $\therefore$ $\boxed{\int_{n}^{n+1} [x]\, dx = n}$

2. $\int_{-1}^{2} [x]\, dx$

$= \int_{-1}^{0} [x]\, dx + \int_{0}^{1} [x]\, dx + \int_{1}^{2} [x]\, dx$

$= \cancel{x} + 0 + \cancel{x}$

$= 0$

Integration by substitution

Sometimes by substitution, the integral can be put in a convinient form and then it can be integrated

(i) $\int (ax+b)^8 \, dx$

Put $ax+b = t$

$\frac{dt}{dx} = a$

$\Rightarrow dt = a \, dx$

$\Rightarrow dx = \frac{dt}{a}$

$= \int t^8 \cdot \frac{dt}{a}$

$= \frac{1}{a} \int t^8 \, dt$

~~$= \frac{1}{a}\left(\frac{t^9}{9} + c\right)$~~

~~$= \frac{t^9}{9a}$ (Ans)~~

~~$= \frac{1}{9a}\left\{(ax+b)^9 + c\right\}$~~

~~$= \frac{(ax+9)^9}{9a} +$~~

$= \frac{1}{9a} t^9 + c$

$= \frac{(ax+b)^9}{9a} + c$

(Ans)

2.

$$\int \cos Kx \, dx$$ where K is Constant

$$= \int \cos t \cdot \frac{dt}{K}$$

Put
$Kx = t$
$\frac{dt}{dx} = K$
$dx = \frac{dt}{K}$

$$= \frac{1}{K} \int \cos t \, dt$$

$$= \frac{1}{K} \sin t + C$$

$$= \frac{\sin Kx}{K} + C$$ (Ans)

3.

$$\int \sin Kx \, dx = -\frac{\cos Kx}{K} + C$$

Proof

$$\int \sin Kx \, dx$$

Put
$Kx = t$
$\Rightarrow \frac{dt}{dx} = K$
$\Rightarrow dx = \frac{dt}{K}$

$$= \int \sin t \, \frac{dt}{K}$$

$$= \frac{1}{K} \int \sin t \, dt$$

$$= \frac{1}{K} (-\cos t) + C$$

$$= -\frac{\cos Kx}{K} + C$$

4. $\int e^{kx}\,dx = \frac{e^{kx}}{k} + C$

Proof. $\int e^{kx}\,dx$

$= \int e^{t} \cdot \frac{dt}{k}$

$= \frac{1}{k}\int e^{t}\,dt$

$= \frac{1}{k} e^{t} + C = \frac{e^{kx}}{k} + C$

Put $kx = t$

$\Rightarrow \frac{dt}{dx} = k$

$\Rightarrow dx = \frac{dt}{k}$

5. $\int a^{kx}\,dx = \frac{a^{kx}}{k \ln a} + C$

proof

$= \int a^{kx}\,dx$

$= \int a^{t} \frac{dt}{k}$

$= \frac{1}{k}\int a^{t}\,dt$

$= \frac{1}{k} \cdot \frac{a^{t}}{\ln a} + C$

$= \frac{a^{kx}}{k \ln a} + C$

Put $kx = t$

$\Rightarrow \frac{dt}{dx} = k$

$\Rightarrow dx = \frac{dt}{k}$

6. $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$

$= \int \frac{-dt}{t}$

$= -\int \frac{dt}{t}$

$= -\ln|t| + C$

Put $\cos x = t$

$\frac{dt}{dx} = -\sin x$

$\Rightarrow -dt = \sin x\,dx$

$= -\ln|\cos x| + C$

$= \ln|\sec x| + C$

$\therefore \boxed{\int \tan x \, dx = \ln|\sec x| + C = \ln \sec x + C}$

7. $\int \cot x \, dx =$

Put $\sin x = t$

$\Rightarrow \frac{dt}{dx} = \cos x$

$\Rightarrow dt = \cos x \, dx$

$= \int \frac{\cos x}{\sin x} dx$

$= \int \frac{dt}{t}$

$= \ln t + C$

$= \ln \sin x + C$

$\therefore \boxed{\int \cot x \, dx = \ln \sin x + C}$

8. ~~Rule~~ <u>Rule</u>: whenever the derivative of denominator is some constant multiple of numerator then put d.r as some variable t.

8. $\int \sec x \, dx$

$= \int \frac{\sec x (\sec x + \tan x)}{(\sec x + \tan x)} dx$

Put $\sec x + \tan x = t$

$\Rightarrow \frac{dt}{dx} = \sec x \tan x + \sec^2 x$

$= \sec x (\sec x + \tan x)$

$\Rightarrow dt = \sec x (\sec x + \tan x) \, dx$

$= \int \frac{dt}{t}$

$= \ln t + C$

$= \ln (\sec x + \tan x) + C$

~~$\int \sec x \, dx = \ln$~~

$= \ln \left(\frac{1}{\cos x} + \frac{\sin x}{\cos x} \right) + C$

$= \ln \left(\frac{1 + \sin x}{\cos x} \right) + C$

$= \ln \left\{ \frac{(\cos \frac{x}{2} + \sin \frac{x}{2})^2}{\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}} \right\} + C$

$= \ln \frac{\cos \frac{x}{2} + \sin \frac{x}{2}}{\cos \frac{x}{2} - \sin \frac{x}{2}} + C$

$= \ln \frac{1 + \tan \frac{x}{2}}{1 - \tan \frac{x}{2}} + C$

$= \ln \tan\left(\frac{\pi}{4}+\frac{x}{2}\right) + C$

$\therefore \int \sec x \, dx$

$= \ln(\sec x + \tan x) + C$

or

$\ln \tan\left(\frac{\pi}{4}+\frac{x}{2}\right) + C$

9. $\int \operatorname{cosec} x \, dx$

$= \int \frac{\operatorname{cosec} x(\operatorname{cosec} x - \cot x)}{(\operatorname{cosec} x - \cot x)} dx$

Put

$\operatorname{cosec} x - \cot x = t$

$\Rightarrow \frac{dt}{dx} = - \operatorname{cosec} x \cdot \cot x + \operatorname{cosec}^2 x$

$= \operatorname{cosec} x(\operatorname{cosec} x - \cot x)$

$\Rightarrow dt = \operatorname{cosec} x(\operatorname{cosec} x - \cot x) dx$

$= \int \frac{dt}{t}$

$= \ln t + C$

$= \ln(\operatorname{cosec} x - \cot x) + C$

$= \ln\left(\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right) + C$

$$= \ln\left(\frac{1-\cos x}{\sin x}\right) + C$$

$$= \ln\left(\frac{2\sin^2\frac{x}{2}}{2\sin\frac{x}{2}\cdot\cos\frac{x}{2}}\right) + C$$

$$= \ln \tan\frac{x}{2} + C$$

$\therefore$

$$\int \operatorname{cosec} x \, dx = \ln(\operatorname{cosec} x - \cot x) + C$$

or $= \ln \tan\frac{x}{2} + C$

Note:-

$\operatorname{cosec} x + \cot x$ can be multiplied in n.r and d.r

10. $\int \sin^2 x \, dx$

$$= \int \left(\frac{1-\cos 2x}{2}\right) dx$$

$$= \frac{1}{2}\int (1-\cos 2x)\, dx$$

$$= \frac{1}{2}\left\{\int 1 \cdot dx - \int \cos 2x \, dx\right\}$$

$$= \frac{1}{2}\left\{x - \frac{\sin 2x}{2}\right\} + C$$

11. $\int \cos^2 x \, dx$

$= \int \frac{1+\cos 2x}{2} \, dx$

$= \frac{1}{2} \int (1+\cos 2x) \, dx$

$= \frac{1}{2} \left\{ \int 1 . dx + \int \cos 2x \, dx \right\}$

$= \frac{1}{2} \left\{ x + \frac{\sin 2x}{2} \right\} + C$

12. $\int \tan^2 x \, dx$

$= \int (\sec^2 x - 1) \, dx$

$= \int \sec^2 x \, dx - \int 1 . dx$

$= (\tan x - x) + C$

13. $\int \cot^2 x \, dx$

$= \int (\operatorname{cosec}^2 x - 1) \, dx$

$$= \int \operatorname{cosec}^2 x \, dx - \int 1 \, dx$$

$$= (-\cot x - x) + C$$

14. $\int \sin^3 x \, dx$

Put

$\sin 3x = 3\sin x - 4\sin^3 x$

$\Rightarrow 4\sin^3 x = 3\sin x - \sin 3x$

$\Rightarrow \sin^3 x = \dfrac{3\sin x - \sin 3x}{4}$

$$= \int \frac{3\sin x - \sin 3x}{4} \, dx$$

$$= \frac{1}{4} \int (3\sin x - \sin 3x) \, dx$$

$$= \frac{1}{4} \left\{ \int 3\sin x \, dx - \int \sin 3x \, dx \right\}$$

$$= \frac{1}{4} \left\{ 3(-\cos x) + \frac{\cos 3x}{3} \right\}$$

$$= \left(\frac{\cos 3x}{12} - \frac{3\cos x}{4} \right) + C$$

Alternative method

$$\int \sin^3 x \, dx$$

$$= \int \sin^2 x \cdot \sin x \, dx$$

$$= \int (1 - \cos^2 x) \sin x \, dx$$

$$= \int (1-t^2)(-dt)$$

Put
$\cos x = t$
$\frac{dt}{dx} = -\sin x$
$\rightarrow -dt = \sin x \, dx$

$$= \int (t^2-1) \, dt$$

$$= \int t^2 \, dt - \int 1 \cdot dt$$

$$= \left(\frac{t^3}{3} - t\right) + C$$

$$= \left(\frac{\cos^3 x}{3} - \cos x\right) + C$$

Integration by parts

$$\int f(x) \cdot g(x) \, dx = f(x) \int g(x) \, dx - \int \left(\frac{d}{dx} f(x)\right)\left(\int g(x) \, dx\right) dx$$

i.e Integral of (first)(2nd)

= (1st) (Integral of 2nd) − Integral of product of derivative of 1st and integral of 2nd

ILATE

- I → Inverse Trigonometric function.
- L → Logarithim function
- A → Algebric function
- T → Trigonometric function
- E → Exponential function.

Ex: $\int x \sin px \, dx$

$= x \cdot \int \sin px \, dx - \int \left(\frac{d}{dx} x\right)\left(\int \sin px \, dx\right) dx$

(Integrating by parts)

$= x\left(-\frac{\cos px}{p}\right) - \int (1)\left(-\frac{\cos px}{p}\right) dx$

$= -\frac{x \cos px}{p} + \frac{1}{p}\int \cos px \, dx$

$= -\frac{x \cos px}{p} + \frac{\sin px}{p^2} + C$

(Ans)

Ex :- $\int \log x \, dx$

$= \int \log x \cdot 1 \, dx$

$= \log x \int 1 \cdot dx - \int \left(\frac{d}{dx} \log x\right)\left(\int 1 \cdot dx\right) dx$

(Integrating by parts)

$= (\log x)(x) - \int \left(\frac{1}{x} \cdot x\right) \cdot dx$

$= (\log x)(x) - \int 1 \cdot dx$

$\int \log x \, dx = x \log x - x + C$

$\int \log x \, dx = x(\log x - 1) + C$

(Ans)

Note :- When there is one function like $\log x$, $\sin^{-1} x$ etc we have to multiply 1 to make 2 functions & apply by parts rule.

Ex :- $\int \tan^{-1} x \, dx$

$= \int \tan^{-1} x \cdot 1 \, dx$

$= \tan^{-1} x \int 1 \cdot dx - \int \left(\frac{d}{dx}(\tan^{-1} x)\right)\left(\int 1 \cdot dx\right) dx$

(Integrating by parts)

$= (\tan^{-1} x)(x) - \int \left(\frac{1}{1+x^2}\right) \cdot (x) \, dx$

$$= x\tan^{-1}x - \int \left(\frac{x}{1+x^2}\right) dx$$

$$= x\tan^{-1}x - \int \frac{dt}{2t}$$

Put $1+x^2 = t$

$\Rightarrow \frac{dt}{dx} = 2x$

$\Rightarrow \frac{dt}{2} = x\,dx$

$$= x\tan^{-1}x - \frac{1}{2}\int \frac{1}{t} \cdot dt$$

$$= x\tan^{-1}x - \frac{1}{2}\ln t + C$$

$$= x\tan^{-1}x - \frac{\ln(1+x^2)}{2} + C$$

(Ans)

Ex :- $\int e^x\left(f(x) + f'(x)\right) dx$

$$= \int e^x f(x)\, dx + \int e^x f'(x)\, dx$$

$$= \int f(x)\, e^x\, dx + \int e^x f'(x)\, dx$$

$$= f(x)\int e^x dx - \int\left(\frac{d}{dx} f(x)\right)\left(\int e^x dx\right) dx$$

$$+ \int e^x f'(x)\, dx$$

(Integrating by parts)

$$= f(x) \cdot e^x - \int f'(x)\, e^x\, dx - \int e^x f'(x)\, dx$$

$$= e^x f(x) + C$$

$$\therefore \boxed{\int e^x \left(f(x) + f'(x)\right) dx = e^x f(x) + c}$$

Ex:
$$\int e^x \left(\tan x - \ln \cos x\right) dx$$

$$= \int e^x \left(-\ln \cos x + \tan x\right) dx$$

$$= \int e^x(-\ln \cos x)\, dx + \int e^x \tan x \, dx$$

$$= \int (-\ln \cos x) e^x \, dx + \int e^x \tan x \, dx$$

$$= (-\ln \cos x) \int e^x \, dx - \int \left(\frac{d}{dx}(-\ln \cos x)\right)\left(\int e^x \, dx\right) dx$$

$$+ \int e^x \tan x \, dx$$

(Integrating by parts)

$$= (-\ln \cos x) . (e^x) - \int \tan x \, e^x \, dx$$

$$+ \int e^x \tan x \, dx .$$

$= - \ln \cos x . e^x + C$

$= - e^x \ln \cos x + C$

(Ans)

Note :-

Sometimes it is very difficult to put the given question in the above form $e^x (f(x) + f'(x))$. In such cases we integrate by parts taking numerator as first function and $\frac{1}{\text{denominator}}$ as 2nd function

Topic

Ex :- $\int e^x \frac{(x-1)^2}{(x+1)^2} dx$

$= \int e^x (x-1)^2 . \frac{1}{(x+1)^2} . dx$

$= e^x (x-1)^2 \int \frac{1}{(x+1)^2} dx - \int \left(\frac{d}{dx} e^x (x-1)^2 \right) \left(\int \frac{1}{(x+1)^2} dx \right) dx$

(Integrating by parts)

$= e^x (x-1)^2 . \left(\frac{-1}{x+1} \right) + \int \left\{ (x-1)^2 e^x + e^x 2(x-1) \right\} \frac{1}{1+x} dx$

Put $x+1 = t \Rightarrow \frac{dt}{dx} = 1 \Rightarrow dx = dt$

$\therefore \int \frac{1}{(x+1)^2} dx = \int \frac{1}{t^2} dt = \int t^{-2} \cdot dt$

$= \frac{t^{-2+1}}{-2+1} = -t^{-1} = \frac{-1}{t} = \frac{-1}{x+1}$

$= -e^x \frac{(x-1)^2}{x+1} + \int e^x \frac{(x^2-1)}{1+x} \cdot dx$

$= -e^x \frac{(x-1)^2}{x+1} + \int (x-1) e^x \, dx$

$= -\frac{e^x (x-1)^2}{(x+1)} + \int x e^x \, dx - \int e^x \, dx$

$= -\frac{e^x (x-1)^2}{x+1} + x \cdot \int e^x \, dx - \int \left(\frac{d(x)}{dx}\right)\left(\int e^x dx\right) dx$

$- e^x$

$= -\frac{e^x (x-1)^2}{x+1} + x e^x - \int 1 \cdot e^x \, dx$

$- e^x$

$= -\frac{e^x (x-1)^2}{x+1} + x e^x - e^x - e^x + c$

$= -\frac{e^x (x-1)^2}{x+1} + x e^x - 2e^x + C$

(Ans)

$$= e^x\left(-\frac{(x-1)^2}{x+1} + x - 2\right) + c$$

(Ans)

$$= e^x\left(\frac{-x^2 - 1 + 2x + x^2 + x - x - 2}{x+1}\right) + c$$

$$= e^x\left(\frac{x-3}{x+1}\right) + c$$

Ex: Find $\int e^x \sin x \, dx$

Soln

~~$\int \sin x$~~

Let $I = \int e^x \sin x \, dx$

$$= \int \sin x \, e^x \, dx$$

$$= \sin x \int e^x dx - \int \left(\frac{d}{dx}\sin x\right)\left(\int e^x dx\right)(dx)$$

$$= \sin x \cdot e^x - \int \cos x \cdot e^x \, dx$$

$$= \sin x \cdot e^x - \left\{\cos x \cdot \int e^x dx - \int\left(\frac{d}{dx}\cos x\right)\left(\int e^x dx\right) dx\right.$$

$$= \sin x \cdot e^x - \left\{\cos x \, e^x + \int \sin x \, e^x \, dx\right.$$

$= e^x \sin x - e^x \cos x - \int e^x \sin x \, dx$

$= e^x \sin x - e^x \cos x - I$

$\Rightarrow 2I = e^x \sin x - e^x \cos x$

$\Rightarrow I = \dfrac{e^x \sin x - e^x \cos x}{2}$

$= \dfrac{e^x}{2}(\sin x - \cos x) + C$

(Ans)

(Exponential, Trigonometry 2 product যদি, it will be repeated, put the function $= I$)

Alternative method

Short question

$\int e^x \sin x \, dx = I$

$= \int e^x \left\{ \dfrac{\sin x - \cos x}{2} + \dfrac{\sin x + \cos x}{2} \right\} dx$

$= \int e^x \{ f(x) + f'(x) \} dx$

where $f(x) = \dfrac{\sin x - \cos x}{2}$

$= e^x f(x) + C = e^x \left(\dfrac{\sin x - \cos x}{2}\right) + C$ (Ans)

OR (3 marks) $\int e^x \sin x \, dx$

$= \int e^x \left\{ \frac{\sin x - \cos x}{2} + \frac{\sin x + \cos x}{2} \right\} dx$

$= \int \left(\frac{\sin x - \cos x}{2} \right) e^x dx + \int \left(\frac{\sin x + \cos x}{2} \right) e^x dx$

$= \frac{\sin x - \cos x}{2} \cdot \int e^x dx - \int \left(\frac{d}{dx} \frac{\sin x - \cos x}{2} \right) \left(\int e^x dx \right) dx$

$+ \int \left(\frac{\sin x + \cos x}{2} \right) e^x dx$

$= \left(\frac{\sin x - \cos x}{2} \right) \cdot (e^x) - \int \frac{\cos x + \sin x}{2} \cdot e^x \cdot dx$

$+ \int \frac{\cos x + \sin x}{2} e^x dx$

$= e^x \left(\frac{\sin x - \cos x}{2} \right)$ (Ans)

Ask doubt before.

Ex: 8 $\int x^2 \sin x \, dx$

$= x^2 \int \sin x \, dx - \int \left(\frac{d}{dx} (x^2) \right) \left(\int \sin x \, dx \right) dx$

$= x^2 (-\cos x) - \int 2x \cdot (-\cos x) \cdot dx$

$= -x^2 \cos x + 2 \int x \cos x \, dx$

$$= -x^2 \cos x + 2\left\{x \int \cos x\,dx - \int \left(\frac{d}{dx}(x)\right)\left(\int \cos x\,dx\right) dx\right.$$

$$= -x^2 \cos x + 2\left\{x (\sin x) - \int 1 . (\sin x)\,dx\right\}$$

$$= -x^2 \cos x + 2\left\{x \sin x - \int \sin x\,dx\right)$$

$$= -x^2 \cos x + 2\left\{x \sin x + \cos x\right\} + C$$

$$= -x^2 \cos x + 2x \sin x + 2 \cos x + C$$

__Ex-9__ $\int x^6 \cos x\,dx$

It can be solved by the above method but we have to integrate by parts 6 times which is a time-killing process. So we can solve it by a new method given below.

+1 or −1	x^6 and its derivatives	$\cos x$ and its integrals
		$\cos x$
+1	x^6	$\sin x$
−1	$6x^5$	$-\cos x$
+1	$30x^4$	$-\sin x$
−1	$120x^3$	$\cos x$
+1	$360x^2$	$\sin x$
−1	$720x$	$-\cos x$
+1	720	
−1	0	$-\sin x$

$\therefore \int x^6 \cos x \, dx$

$= +1 \cdot x^6 \sin x + (-1)\, 6x^5 (-\cos x)$

$+ 1\, (30x^4)(-\sin x) + (-1)\, 120x^3 \cos x + (+1)\, 360x^2 \sin x$

$+ (-1)\, 720x\, (-\cos x) + (1)(720) \cdot (-\sin x)$

$+ C$

$= x^6 \sin x + 6x^5 \cos x - 30x^4 \sin x - 120x^3 \cos x$

$+ 360x^2 \sin x + 720x \cos x - 720 \sin x + C$

9. Important integrals

Rule

If a^2-x^2 form then put $x = a\sin\theta$

or $a\cos\theta$

If a^2+x^2 form then put $x = a\tan\theta$

or $a\cot\theta$

~~or $a\sinh\theta$~~

or $a\sinh\theta$

If x^2-a^2 form then put $x = a\sec\theta$

or $a\operatorname{cosec}\theta$

~~or $a\cosh\theta$~~

or $a\cosh\theta$

1. $\int \frac{1}{x^2+a^2}\,dx$

$= \int \frac{a\sec^2\theta\,d\theta}{a^2\sec^2\theta}$

Put

$x = a\tan\theta$

$\Rightarrow \frac{dx}{d\theta} = a\sec^2\theta$

$\Rightarrow dx = a\sec^2\theta\,d\theta$

$$= \int \frac{1}{a} d\theta = \frac{1}{a} \int 1 \, d\theta$$

$$= \frac{1}{a} \cdot \theta + C$$

$$= \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$$

$x^2 + a^2$
$= a^2 \tan^2\theta + a^2$
$= a^2 \sec^2\theta$

$\frac{x}{a} = \tan\theta$

$\Rightarrow \theta = \tan^{-1}\left(\frac{x}{a}\right)$

$$\therefore \int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$$

2. $$\int \frac{1}{x^2 - a^2} dx$$

$$= \int \frac{a \sec\theta \tan\theta \, d\theta}{a^2 \tan^2\theta}$$

$$= \frac{1}{a} \int \frac{\sec\theta}{\tan\theta} d\theta$$

$$= \frac{1}{a} \int \operatorname{cosec}\theta \, d\theta$$

$$= \frac{1}{a} \ln(\operatorname{cosec}\theta - \cot\theta) + C$$

$$= \frac{1}{a} \ln\left(\frac{x}{\sqrt{x^2 - a^2}} - \frac{a}{\sqrt{x^2 - a^2}}\right) + C$$

$$= \frac{1}{a} \ln\left|\frac{x - a}{\sqrt{x^2 - a^2}}\right| + C$$

Put

$x = a \sec\theta$

$\Rightarrow dx = a \sec\theta \tan\theta \, d\theta$

$x^2 - a^2$
$= a^2 \sec^2\theta - a^2$
$= a^2 \tan^2\theta$

$\frac{x}{a} = \sec\theta$

$\Rightarrow \theta = \sec^{-1}\left(\frac{x}{a}\right)$

$\tan\theta = \sqrt{\sec^2\theta - 1}$
$= \sqrt{\frac{x^2}{a^2} - 1}$
$= \frac{\sqrt{x^2 - a^2}}{a}$

$\cot\theta = \frac{a}{\sqrt{x^2 - a^2}}$

$\operatorname{cosec}\theta = \sqrt{1 + \cot^2\theta}$

$$= \frac{1}{a} \ln\left(\sqrt{\frac{x-a}{x+a}} \right) + C$$

$$= \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + C$$

$$= \sqrt{1 + \frac{a^2}{x^2-a^2}}$$

$$= \frac{x}{\sqrt{x^2-a^2}}$$

$$\therefore \boxed{\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + C}$$

3. $$\int \frac{dx}{\sqrt{a^2-x^2}}$$

$$= \int \frac{a \cos\theta \, d\theta}{a \cos\theta}$$

$$= \frac{1}{a} \int \sec\theta \, d\theta$$

$$= \frac{1}{a} \ln |\sec\theta + \tan\theta| + C$$

$$= \frac{1}{a} \ln \left| \frac{a}{\sqrt{a^2-x^2}} + \frac{x}{\sqrt{a^2-x^2}} \right| + C$$

$$= \frac{1}{a} \ln \left| \frac{a+x}{\sqrt{a^2-x^2}} \right| + C$$

$$= \frac{1}{a} \ln \left| \sqrt{\frac{a+x}{a-x}} \right| + C$$

Put

$x = a \sin\theta$

$\Rightarrow dx = a \cos\theta \, d\theta$

$a^2 - x^2 = a^2 \cos^2\theta$

$\frac{x}{a} = \sin\theta$

$\Rightarrow \theta = \sin^{-1}\left(\frac{x}{a}\right)$

$\cos\theta = \sqrt{1 - \frac{x^2}{a^2}}$

$= \frac{\sqrt{a^2-x^2}}{a}$

$\sec\theta = \frac{a}{\sqrt{a^2-x^2}}$

$\tan\theta = \frac{x}{\sqrt{a^2-x^2}}$

$$= \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right| + C$$

$$\therefore \boxed{\int \frac{dx}{a^2-x^2} = \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right| + C}$$

Note :- 2nd and 3rd integrals can be evaluated by using partial fraction

4. $$\int \frac{1}{\sqrt{x^2+a^2}}\, dx$$

Put $x = a\tan\theta$

$\Rightarrow dx = a\sec^2\theta\, d\theta$

$\sqrt{x^2+a^2}$

$= \sqrt{a^2\tan^2\theta + a^2}$

$= a\sec\theta$

$\therefore \frac{\sqrt{x^2+a^2}}{a} = \sec\theta$

$\frac{x}{a} = \tan\theta$

$$= \int \frac{a\sec^2\theta\, d\theta}{a\sec\theta}$$

$$= \int \sec\theta\, d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left| \frac{\sqrt{x^2+a^2}}{a} + \frac{x}{a} \right| + C$$

$$= \ln\left| \frac{\sqrt{x^2+a^2}+x}{a} \right| + C$$

$$= \ln\left(\sqrt{x^2+a^2}+x\right) - \ln|a| + C$$

$= \ln|x+\sqrt{x^2+a^2}| + K$

where $K = C - \ln|a|$

5. $\int \frac{dx}{\sqrt{x^2-a^2}}$

Put $x = a\sec\theta$

$\Rightarrow dx = a\sec\theta\tan\theta\, d\theta$

$\sqrt{x^2-a^2}$

$= \sqrt{a^2\sec^2\theta - a^2}$

$= a\tan\theta$

$\sec\theta = \frac{x}{a}$

$\tan\theta = \frac{\sqrt{x^2-a^2}}{a}$

$= \int \frac{a\sec\theta\tan\theta\, d\theta}{a\tan\theta}$

$= \int \sec\theta\, d\theta$

$= \ln|\sec\theta+\tan\theta| + C$

$= \ln\left|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right| + C$

$= \ln\left|\frac{x+\sqrt{x^2-a^2}}{a}\right| + C$

$= \ln|x+\sqrt{x^2-a^2}| - \ln|a| + C$

$= \ln|x+\sqrt{x^2-a^2}| + K$

(where $K = C - \ln|a|$)

$\therefore \int \frac{dx}{\sqrt{x^2-a^2}} = \ln|x+\sqrt{x^2-a^2}| + K$

6. $\int \frac{dx}{\sqrt{a^2-x^2}}$

Put

$x = a\sin\theta$

$\Rightarrow dx = a\cos\theta\, d\theta$

$\sqrt{a^2-x^2}$

$= a\cos\theta$

~~$\Rightarrow \cos\theta = a^2-x^2$~~

$\frac{x}{a} = \sin\theta$

$\Rightarrow \theta = \sin^{-1}(\frac{x}{a})$

$= \int \frac{a\cos\theta\, d\theta}{a\cos\theta} = \int 1 . d\theta$

$= \theta + C$

$= \sin^{-1}\frac{x}{a} + C$

$\therefore \boxed{\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C}$

7. $\int \sqrt{a^2-x^2}\, dx$

$x = a\sin\theta$

$\Rightarrow dx = a\cos\theta\, d\theta$

$\sqrt{a^2-x^2} = a\cos\theta$

$\frac{x}{a} = \sin\theta$

$\Rightarrow \theta = \sin^{-1}\frac{x}{a}$

$\cos\theta = \frac{\sqrt{a^2-x^2}}{a}$

$= \int a\cos\theta \ a\cos\theta\, d\theta$

$= \int a^2 \cos^2\theta\, d\theta$

$= a^2 \int \frac{1+\cos 2\theta}{2}\, d\theta$

$= \frac{a^2}{2} \int (1+\cos 2\theta)\, d\theta$

$= \frac{a^2}{2} \left\{ \int d\theta + \int \cos 2\theta\, d\theta \right\}$

$$= \frac{a^2}{2}\left\{\theta + \frac{\sin 2\theta}{2}\right\} + C$$

~~$$= \frac{a^2}{2}\left\{\sin^{-1}\frac{x}{a} + \frac{\sin 2 \cdot \sin^{-1}\frac{x}{a}}{2}\right\} + C$$~~

$$= \frac{a^2}{2}\left\{\theta + \sin\theta \cdot \cos\theta\right\} + C$$

$$= \frac{a^2}{2}\left\{\sin^{-1}\frac{x}{a} + \frac{x}{a} \cdot \frac{\sqrt{a^2-x^2}}{a}\right\} + C$$

$$= \frac{a^2}{2}\left\{\sin^{-1}\frac{x}{a} + \frac{x\sqrt{a^2-x^2}}{a^2}\right\} + C$$

$$= \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{a^2-x^2} + C$$

8. $\int \sqrt{a^2+x^2}\, dx$

$= \int a\sec\theta \; a\sec^2\theta\, d\theta$

$= a^2\int \sec^3\theta\, d\theta$ ——(i)

put

$x = a\tan\theta$

$\Rightarrow dx = a\sec^2\theta\, d\theta$

$\sqrt{a^2+x^2}$

$= a\sec\theta$

$\frac{x}{a} = \tan\theta$

$\frac{\sqrt{a^2+x^2}}{a} = \sec\theta$

Consider $\int \sec^3\theta \, d\theta$

Let I

$= \int \sec^3\theta \, d\theta$

$= \int \sec\theta . \sec^2\theta \, d\theta$

$= \sec\theta \int \sec^2\theta \, d\theta - \int \left(\frac{d}{d\theta}(\sec\theta)\right)\left(\int \sec^2\theta d\theta\right) d\theta$

$= \sec\theta \tan\theta - \int \sec\theta \tan\theta . \tan\theta \, d\theta$

(Integrating by parts)

$= \sec\theta \tan\theta - \int \sec\theta \tan^2\theta \, d\theta$

$= \sec\theta \tan\theta - \int \sec\theta (\sec^2\theta - 1) \, d\theta$

$= \sec\theta \tan\theta - \int (\sec^3\theta - \sec\theta) \, d\theta$

$= \sec\theta \tan\theta - \int \sec^3\theta \, d\theta + \int \sec\theta \, d\theta$

$= \sec\theta \tan\theta - I + \ln|\sec\theta + \tan\theta|$

$\Rightarrow 2I = \sec\theta \tan\theta + \ln|\sec\theta + \tan\theta|$

$\Rightarrow I = \dfrac{\sec\theta \tan\theta + \ln|\sec\theta + \tan\theta|}{2}$

From (i),

$$\int \sqrt{a^2+x^2}\, dx$$

$$= \frac{a^2}{2}\left(\sec\alpha \tan\alpha + \ln|\sec\alpha + \tan\alpha|\right) + C$$

$$= \frac{a^2}{2}\left\{\frac{\sqrt{a^2+x^2}}{a}\cdot\frac{x}{a} + \ln\left|\frac{\sqrt{a^2+x^2}}{a} + \frac{x}{a}\right|\right) + C$$

$$= \frac{x\sqrt{a^2+x^2}}{2} + \frac{a^2}{2}\ln\left|\frac{\sqrt{a^2+x^2}+x}{a}\right| - \frac{a^2}{2}\ln|a| + C$$

$$= \frac{x\sqrt{a^2+x^2}}{2} + \frac{a^2}{2}\ln\left|\sqrt{a^2+x^2}+x\right| + K$$

$$\left(\text{where } K = C - \frac{a^2}{2}\ln|a|\right)$$

9. $\int \sqrt{x^2-a^2}\, dx$

Put $x = a\sec\theta$

$dx = a\sec\theta\tan\theta\, d\theta$

$\sqrt{x^2-a^2} = a\tan\theta$

$\sec\theta = \frac{x}{a}$

$\tan\theta = \frac{\sqrt{x^2-a^2}}{a}$

$$= \int a\tan\theta \cdot a\sec\theta\tan\theta\, d\theta$$

$$= a^2\int \sec\theta \cdot \tan^2\theta\, d\theta$$

$$= a^2\int \sec\theta(\sec^2\theta - 1)\, d\theta$$

$$= a^2 \int (\sec^3\theta - \sec\theta)\, d\theta$$

$$= a^2 \left\{ \int \sec^3\theta \, d\theta - \int \sec\theta \, d\theta \right\}$$

$$= a^2 \left\{ \frac{\sec\theta \cdot \tan\theta + \ln|\sec\theta + \tan\theta|}{2} - \ln|\sec\theta + \tan\theta| \right\} + C$$

$$= a^2 \left\{ \frac{\sec\theta \cdot \tan\theta + \ln|\sec\theta + \tan\theta| - 2\ln|\sec\theta + \tan\theta|}{2} \right\} + C$$

$$= a^2 \left\{ \frac{\sec\theta \cdot \tan\theta - \ln|\sec\theta + \tan\theta|}{2} \right\} + C$$

$$= \frac{a^2}{2} \left\{ \frac{x}{a} \cdot \frac{\sqrt{x^2 - a^2}}{a} - \ln\left| \frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a} \right| \right\} + C$$

$$= \frac{a^2}{2} \left\{ \frac{x\sqrt{x^2 - a^2}}{a^2} - \ln\left| \frac{x + \sqrt{x^2 - a^2}}{a} \right| \right\} + C$$

$$= \frac{x\sqrt{x^2 - a^2}}{2} - \frac{a^2}{2} \ln|x + \sqrt{x^2 - a^2}| + \frac{a^2}{2} \ln|a| + C$$

$$= \frac{x\sqrt{x^2 - a^2}}{2} - \frac{a^2}{2} \ln|x + \sqrt{x^2 - a^2}| + K$$

(where $K = \frac{a^2}{2} \ln|a| + C$)

Evaluate $\int \frac{dx}{\sqrt{x^2+2x+5}}$

Ans: $\int \frac{dx}{\sqrt{x^2+2x+5}} = \int \frac{dx}{\sqrt{(x)^2+2(1)(x)+1-1+5}}$

$= \int \frac{dx}{\sqrt{(x+1)^2+2^2}}$

$= \int \frac{2\sec^2\theta \, d\theta}{2\sec\theta}$ (= $\int \sec\theta \, d\theta$)

$= \ln|\sec\theta + \tan\theta| + C$

$= \ln\left|\frac{\sqrt{x^2+2x+5}}{2} + \frac{x+1}{2}\right| + C$

$= \ln\left|\frac{x+1+\sqrt{x^2+2x+5}}{2}\right| + C$

$= \ln|x+1+\sqrt{x^2+2x+5}| - \ln 2 + C$

$= \ln|x+1+\sqrt{x^2+2x+5}| + K$

(where $K = C - \ln 2$)

Put $x+1 = 2\tan\theta$

$\Rightarrow dx = 2\sec^2\theta \, d\theta$

$\tan\theta = \frac{x+1}{2}$

$\sqrt{x^2+2x+5}$

$= \sqrt{(x+1)^2+2^2}$

$= 2\sec\theta$

$\Rightarrow \sec\theta = \frac{\sqrt{x^2+2x+5}}{2}$

(2) Evaluate $\int_2^3 \frac{dx}{\sqrt{5x-6-x^2}}$

Consider

Ans. $\int \frac{dx}{\sqrt{5x-6-x^2}}$

$= \int \frac{dx}{\sqrt{-(x^2-5x+6)}} = \int \frac{dx}{\sqrt{-\left(x^2-2\cdot\frac{5}{2}\cdot x+\frac{25}{4}-\frac{25}{4}+6\right)}}$

$= \int \frac{dx}{\sqrt{-\left\{\left(x-\frac{5}{2}\right)^2-\frac{1}{4}\right\}}}$

$= \int \frac{dx}{\sqrt{-\left\{\left(x-\frac{5}{2}\right)^2-\left(\frac{1}{2}\right)^2\right\}}}$

$= \int \frac{dx}{\sqrt{\left(\frac{1}{2}\right)^2-\left(\frac{x-5}{2}\right)^2}}$

$= \int \frac{\frac{1}{2}\cos\theta \cdot d\theta}{\frac{1}{2}\cos\theta}$

$= \theta = \sin^{-1}(2x-5)$

~~$= \sin^{-1}(2x-5) + C$~~

Put

$\frac{x-5}{2} = \frac{1}{2}\sin\theta$

$\Rightarrow dx = \frac{1}{2}\cos\theta\, d\theta$

$\sqrt{5x-6-x^2}$

$= \sqrt{\left(\frac{1}{2}\right)^2-\left(x-\frac{5}{2}\right)^2}$

$= \sqrt{\frac{1}{4}-\frac{1}{4}\sin^2\theta}$

$= \frac{1}{2}\cos\theta$

$\frac{2x-5}{2} = \frac{1}{2}\sin\theta$

$\Rightarrow \sin\theta = 2x-5$

$\theta = \sin^{-1}(2x-5)$

$$\int_2^3 \frac{dx}{\sqrt{5x-6-x^2}}$$

$$= \sin^{-1}(2x-5)\Big]_2^3$$

$$= \sin^{-1} 1 - \sin^{-1}(-1)$$

$$= \frac{\pi}{2} + \sin^{-1} 1$$

$$= \frac{\pi}{2} + \frac{\pi}{2}$$

$$= \pi$$ (Ans)

Integration of rational functions using partial fractions

Step-1

First see that the degree of numerator is less than the degree denominator

If the degree of n.r ≥ ~~d.r~~ degree of d.r then perform actual division

Step -2

Factorise the d.r

There are 4 cases

Case : i

(a) Linear non-repeated factors in the d.r

Case-ii

Linear repeated factors in the d.r

Case-iii

Quadratic non repeated factors in the d.r

Case -iv

Quadratic repeated factors in the d.r.

Case-I

Linear non. repeated factors in the d.r

1) Find $\int \frac{x^2\,dx}{(x+1)(x+2)(x-3)}$

Q-1 $\int \frac{x^2\,dx}{(x+1)(x+2)(x-3)}$

Ans

Let $\frac{x^2}{(x+1)(x+2)(x-3)}$ = ~~$\frac{A}{(x+1)} + \frac{B}{x+2}$~~

$= \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x-3}$

$= \frac{A(x+2)(x-3) + B(x+1)(x-3) + C(x+1)(x+2)}{(x+1)(x+2)(x-3)}$

$\therefore \quad x^2 = A(x+2)(x-3) + B(x+1)(x-3) + C(x+1)(x+2)$

Put $x = -1$ and we get

$1 = A(-1+2)(-1-3) = -4A$

$\Rightarrow A = -\frac{1}{4}$

Put $x = -2$

we get

$4 = B(-1)\cdot(-5) = 5B$

$\Rightarrow B = \frac{4}{5}$

Putting $x = 3$

$$9 = C(4)(5) = 20C$$

$$\Rightarrow C = \frac{9}{20}$$

$$\therefore \frac{x^2}{(x+1)(x+2)(x-3)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x-3}$$

$$= \frac{-1}{4(x+1)} + \frac{4}{5(x+2)} + \frac{9}{20(x-3)}$$

$$\therefore \int \frac{x^2\,dx}{(x+1)(x+2)(x-3)}$$

$$= \int \frac{-1}{4(x+1)}dx + \int \frac{4}{5(x+2)}dx + \int \frac{9}{20(x-3)}dx$$

$$= -\frac{1}{4}\int \frac{dx}{x+1} + \frac{4}{5}\int \frac{dx}{x+2} + \frac{9}{20}\int \frac{dx}{x-3}$$

$$= -\frac{1}{4}\ln|x+1| + \frac{4}{5}\ln|x+2| + \frac{9}{20}\ln|x-3| + C$$

(Ans)

Johnson question

2. Find $\int \frac{2x^4+3x^3-x^2+x-1}{(x^3-x)} dx$

Ans :

$$\begin{array}{l|l|l} x^3-x & 2x^4+3x^3-x^2+x-1 & 2x+3 \\ & \underset{(-)}{2x^4} \quad \underset{(+)}{-2x^2} & \\ \hline & 3x^3+x^2+x-1 & \\ & \underset{(-)}{3x^3} \quad \underset{(+)}{-3x} & \\ \hline & x^2+4x-1 & \end{array}$$

$\therefore \int \frac{2x^4+3x^3-x^2+x-1}{x^3-x}$

$= \int 2x+3 + \frac{x^2+4x-1}{x^3-x}$

$= 2x+3 + \frac{x^2+4x-1}{x(x-1)(x+1)}$

Let $\frac{x^2+4x-1}{x(x-1)(x+1)}$

$$= \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$$

$$= \frac{A(x-1)(x+1) + Bx(x+1) + Cx(x-1)}{x(x-1)(x+1)}$$

$$\therefore \; x^2 + 4x - 1 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1)$$

Putting $x = 0$

we get $-1 = A(-1).(1) = -A$

$$\Rightarrow \boxed{A = 1}$$

Putting $x = 1$

$4 = 2B$

$\Rightarrow B = 2$

Putting $x = -1$

~~$C = -1.(-2) = 2$~~

$-4 = 2C$

$\Rightarrow C = -2$

$$\therefore \; \frac{x^2 + 4x - 1}{x(x-1)(x+1)} = \frac{1}{x} + \frac{2}{x-1} - \frac{2}{x+1}$$

$$\therefore \frac{2x^4 + 3x^3 - x^2 + x - 1}{(x^3 - x)}$$

$$= 2x + 3 + \frac{1}{x} + \frac{2}{x-1} - \frac{2}{x+1}$$

$$\therefore \int \frac{2x^4 + 3x^3 - x^2 + x - 1}{x^3 - x} dx$$

$$= \int (2x + 3) dx + \int \frac{1}{x} dx + \int \frac{2}{x-1} dx - \int \frac{2}{x+1} dx$$

$$= x^2 + 3x + \ln|x| + 2\ln|x-1| - 2 \cdot \ln|x+1| + C$$

Case - II

Linear repeated factors in the d.r

Find → $\int \frac{x^2 dx}{(x-1)^2 (x-2)}$

Sol^n : Let $\frac{x^2}{(x-1)^2 (x-2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-2)}$

$$= \frac{A(x-1)(x-2) + B(x-2) + C(x-1)^2}{(x-1)^2 \cdot (x-2)}$$

$$\therefore x^2 = A(x-1)(x-2) + B(x-2) + C(x-1)^2$$

Putting $x = 1$, we get

$$-1 = B(-1)$$

$$\Rightarrow \boxed{B = -1}$$

Putting $x = 2$

$$4 = C \cdot 1$$

$$\Rightarrow \boxed{C = 4}$$

Putting $x = 0$, we get

$$0 = 2A - 2B + C$$

$$\Rightarrow 0 = 2A + 2 + 4$$

$$\Rightarrow 2A = -6$$

$$\Rightarrow \boxed{A = -3}$$

$$\therefore \frac{x^2}{(x-1)^2(x-2)} = \frac{-3}{x-1} - \frac{1}{(x-1)^2} + \frac{4}{x-2}$$

$$\therefore \int \frac{x^2\,dx}{(x-1)^2(x-2)}$$

$$= -3\int \frac{1}{x-1}\,dx - \int \frac{1\,dx}{(x-1)^2} + 4\int \frac{1}{x-2}\,dx$$

$$= -3\ln|x-1| + \frac{1}{x-1} + 4\ln|x-2| + C$$

$$\int \frac{dx}{(x-1)^2} = \int \frac{dt}{t^2} = \int t^{-2}\,dt = -\frac{1}{t} = \frac{-1}{x-1}$$

Put $x-1 = t$

$dx = dt$

Case - III

Quadratic non-repeated factors in the d.r

Find $\int \frac{2x^2+x+3}{(x^2+2)(x-1)}\,dx$

Soln: Let $\frac{2x^2+x+3}{(x^2+2)(x-1)} = \frac{Ax+B}{x^2+2} + \frac{C}{x-1}$

$$= \frac{(Ax+B)(x-1)+C(x^2+2)}{(x^2+2)(x-1)}$$

$\therefore$ $2x^2+x+3 = (Ax+B)(x-1)+C(x^2+2)$

$= Ax^2 - Ax + Bx - B + Cx^2 + 2C$

$= x^2(A+C) + x(B-A) + (2C-B)$

Equating the co-efficients of like powers of x from both sides, we get

$A+C = 2$ — (i)

$1 = B-A$ — (ii)

$3 = 2C-B$ — (iii)

From Eqn (i) and (ii)

$$B + C = 3$$

$$2C - B = 3$$

$$3C = 6$$

$$\Rightarrow \boxed{C = 2}$$

From eqn (i)

$$A + C = 2$$

$$\Rightarrow A + 2 = 2$$

$$\Rightarrow \boxed{A = 0}$$

$$B - A = 1$$

$$\Rightarrow B - 0 = 1$$

$$\Rightarrow \boxed{B = 1}$$

$$\frac{2x^2 + x + 3}{(x^2+2)(x-1)} = \frac{1}{x^2+2} + \frac{2}{x-1}$$

$$\int \frac{2x^2+x+3}{(x^2+2)(x-1)}\,dx = \int \frac{1}{x^2+2}\,dx + 2\int \frac{1}{x-1}\,dx$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + 2\ln|x-1| + C$$

$$\left(\because \int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} \right)$$ (Ans)

~~$\int \frac{1}{x^2+2}\,dx$~~ Put $x^2+2 =$

Taux → integram 2 cms in a integral

Case - IV

Quadratic repeated factors in the d. r

Find: $\int \frac{x^3 - 3x^2 + 2x - 3}{(x^2+1)^2} dx$

Soln

Let $\frac{x^3 - 3x^2 + 2x - 3}{(x^2+1)^2} = \frac{Ax+B}{(x^2+1)} + \frac{Cx+D}{(x^2+1)^2}$

$= \frac{(Ax+B)(x^2+1) + Cx+D}{(x^2+1)^2}$

$\therefore x^3 - 3x^2 + 2x - 3 = (Ax+B)(x^2+1) + (Cx+D)$

$= Ax^3 + Ax + Bx^2 + B + Cx + D$

$= Ax^3 + Bx^2 + x(A+C) + (B+D)$

Equating the co-efficient of like powers of x from both sides, we get

$\boxed{A = 1}$ ——

$\boxed{B = -3}$ ——

$A + C = 2 \Rightarrow 1 + C = 2 \Rightarrow \boxed{C = 1}$

$B + D = -3 \Rightarrow -3 + D = -3 \Rightarrow \boxed{D = 0}$

$\therefore \frac{x^3 - 3x^2 + 2x - 3}{(x^2+1)^2} = \frac{x-3}{x^2+1} + \frac{x}{(x^2+1)^2}$

$$\therefore \int \frac{x^3 - 3x^2 + 2x - 3}{(x^2+1)^2}\, dx$$

$$= \int \frac{x-3}{x^2+1}\, dx + \int \frac{x}{(x^2+1)^2}\, dx$$

$$= \int \frac{x\, dx}{x^2+1} - 3\int \frac{dx}{x^2+1} + \int \frac{x}{(x^2+1)^2}\, dx$$

$$= \int \frac{x\, dx}{x^2+1} - 3\tan^{-1} x + \int \frac{x}{(x^2+1)^2}\, dx$$

$$= \int \frac{dt}{2t} - 3\tan^{-1} x + \int \frac{dt}{2t^2}$$

(Put $x^2+1 = t$

$\Rightarrow dt = 2x\, dx$

$\Rightarrow \frac{dt}{2} = x\, dx$)

$$= \frac{1}{2} \ln|t| - 3\tan^{-1} x + \frac{1}{2}\left(-\frac{1}{t}\right) + C$$

$$= \frac{1}{2} \ln|x^2+1| - 3\tan^{-1} x - \frac{1}{2(x^2+1)} + C$$

<u>Problem</u>

$$\int \frac{3x^2 + x - 2}{(x-2)^4 (1-x)}\, dx$$

<u>Soln</u>: Let $x - 2 = y$

$\Rightarrow x = y + 2 \qquad \Rightarrow dx = dy$

Now

$$\frac{3x^2+x-2}{(x-2)^4(1-2x)}$$

$$= \frac{3(y+2)^2+(y+2)-2}{y^4(1-2y-4)}$$

$$3(y^2+4y+4) \quad 3y^2+12y+12$$

$$= \frac{3y^2+13y+12}{-y^4\cdot(2y+3)}$$

$$
\begin{array}{r|l|l}
3+2y & 12+13y+3y^2 & 4+\frac{5}{3}y-\frac{1}{9}y^2+\frac{2}{27}y^3 \\
 & 12 + 8y & \\
 & (-)\quad(-) & \\
\hline
 & 5y+3y^2 & \\
 & 5y+\frac{10}{3}y^2 & \\
 & (-)\quad(-) & \\
\hline
 & -\frac{1}{3}y^2 & \\
 & -\frac{1}{3}y^2-\frac{2}{9}y^3 & \\
 & (+)\quad(+) & \\
\hline
 & \frac{2}{9}y^3 & \\
 & \frac{2}{9}y^3+\frac{4}{27}y^4 & \\
 & (-)\quad(-) & \\
\hline
 & -\frac{4}{27}y^4 &
\end{array}
$$

$$\therefore \frac{3y^2+13y+12}{-y^4(3+2y)} = \frac{-1}{y^4}\cdot\left(4+\frac{5}{3}y-\frac{1}{9}y^2+\frac{2}{27}y^3-\frac{4y^4}{27(3+2y)}\right)$$

$$= \frac{-4}{y^4} - \frac{5}{3y^3} + \frac{1}{9y^2} - \frac{2}{27y} + \frac{4}{27} \times \frac{1}{(3+2y)}$$

$$= \int \frac{3x^2+x-2}{(x-2)^4 (1-2x)} dx$$

$$= \int \frac{3y^2+13y+12}{-y^4 (2y+3)} dy$$

$$= \int \left(-\frac{4}{y^4} - \frac{5}{3y^3} + \frac{1}{9y^2} - \frac{2}{27y} + \frac{4}{27} \times \frac{1}{3+2y} \right) dy$$

$$= \frac{4}{3y^3} + \frac{5}{6y^2} - \frac{1}{9y} - \frac{2}{27} \log|y| + \frac{2}{27} \log(3+2y) + C$$

$$= \frac{4}{3(x-2)^3} + \frac{5}{6(x-2)^2} - \frac{1}{9(x-2)} - \frac{2}{27} \log|x-2| + \frac{2}{27} \log|2x-1| + c$$

(Ans)

Integration of powers of Trigonometric functions.

Q: $\int \sin^4 x \, dx$

Let $I = \int \sin^4 x \, dx$

$$= \int \sin^3 x \cdot \sin x \, dx$$

$$= \sin^3 x \int \sin x \, dx - \int \left(3 \sin^2 x \cos x \cdot \int \sin x \, dx\right) dx$$

$$= \sin^3 x(-\cos x) + 3 \int \sin^2 x \cos^2 x \, dx$$

$$= -\cos x \sin^3 x + 3 \int \sin^2 x \cos^2 x \, dx$$

$$= -\cos x \sin^3 x + 3 \int \sin^2 x (1 - \sin^2 x) \, dx$$

$$= -\cos x \sin^3 x + 3 \int \sin^2 x \, dx - 3 \int \sin^4 x \, dx$$

$$= -\cos x \sin^3 x + 3 \int \sin^2 x \, dx - 3I$$

$$\Rightarrow 4I = -\cos x \sin^3 x + 3 \int \sin^2 x \, dx$$

$$\Rightarrow I = -\frac{\cos x \sin^3 x}{4} + \frac{3}{4} \int \sin^2 x \, dx$$

$$= -\frac{\cos x \sin^3 x}{4} + \frac{3}{4} \int \frac{1 - \cos 2x}{2} \, dx$$

$$= \frac{-\cos x \sin^3 x}{4} + \frac{3}{8} \left[x - \int \cos 2x \, dx\right]$$

$$= -\frac{\cos x \sin^3 x}{4} + \frac{3}{8} \left[x - \frac{\sin 2x}{2}\right] + C$$

(Ans)

Rule : Take $\sin x$ or $\cos x$ with dx

The above method is called reduction formula.

2. $\int \sin^5 x \, dx$

$= \int \sin^4 x \cdot \sin x \, dx$

$= \int (1-\cos^2 x)^2 \cdot \sin x \, dx$

$= \int (1-t^2)^2 (-dt)$

$= -\int (1+t^4-2t^2) \, dt$

Put $\cos x = t$
$\Rightarrow dt = -\sin x \, dx$
$\Rightarrow -dt = \sin x \, dx$

$= -\left[\int 1 \, dt + \int t^4 \, dt - 2\int t^2 \, dt\right]$

$= -\left[t + \frac{t^5}{5} - \frac{2t^3}{3}\right] + C$

$= -\left[\cos x + \frac{\cos^5 x}{5} - \frac{2\cos^3 x}{3}\right] + C$

$= -\cos x - \frac{\cos^5 x}{5} + \frac{2}{3}\cos^3 x + c$

Note : If we integrate the even power of $\sin x$ or $\cos x$ then we apply only reduction formula

If we integrate the odd power of $\sin x$ or $\cos x$ then we apply reduction formula and method of

Substitution both. Here method of substitution is better to apply.

Rule

2. When we integrate the powers of $\tan x$, $\cot x$, $\operatorname{cosec} x$, $\sec x$ we take $\tan^2 x$, $\cot^2 x$, $\operatorname{cosec}^2 x$, $\sec^2 x$ respectively with dx

Q → $\int \sec^4 x \, dx$

$= \int \sec^2 x \cdot \sec^2 x \, dx$

$= \int (1 + \tan^2 x) \sec^2 x \, dx$

$= \int (1 + t^2) \, dt$

$= t + \frac{t^3}{3} + c$

$= \tan x + \frac{\tan^3 x}{3} + c$

Put $\tan x = t$

$dt = \sec^2 x \, dx$

Note: When we integrate the even power of $\sec x$ or $\operatorname{cosec} x$ etc we apply both reduction formula and method of substitution.

When we integrate the odd powers of sec x or cosec x we apply only reduction formula. In even powers use substitution because it is easier than reduction.

Q.) $\int \tan^4 x \, dx$

$= \int \tan^2 x \cdot \tan^2 x \, dx$

$= \int (\sec^2 x - 1) \tan^2 x \, dx$

$= \int \sec^2 x \tan^2 x \, dx - \int \tan^2 x \, dx$

$= \int \sec^2 x \tan^2 x \, dx - \int (\sec^2 x - 1) \, dx$

$= \int \sec^2 x \cdot \tan^2 x \, dx - \tan x + x$

Put $\tan x = t$

$dt = \sec^2 x \, dx$

$= \int t^2 \cdot dt - \tan x + x$

$= \frac{t^3}{3} - \tan x + x + C$

$= \frac{\tan^3 x}{3} - \tan x + x + C$

Q → $\int \tan^5 x \, dx$

$= \int \tan^3 x \tan^2 x \, dx$

$= \int \tan^3 x (\sec^2 x - 1) \, dx$

$= \int \tan^3 x \sec^2 x \, dx - \int \tan^3 x \, dx$

$= \int \tan^3 x \sec^2 x \, dx - \int \tan x \cdot \tan^2 x \, dx$

$= \int \tan^3 x \cdot \sec^2 x \, dx - \int \tan x (\sec^2 x - 1) \, dx$

$= \int \tan^3 x \cdot \sec^2 x \cdot dx - \int \tan x \sec^2 x \, dx + \int \tan x \, dx$

$= \int t^3 \, dt - \int t \, dt + \ln|\sec x| + c$

$= \frac{t^4}{4} \, dt - \frac{t^2}{2} \, dt + \ln|\sec x| + c$

Put $\tan x = t$ in first and 2nd integral $dt = \sec^2 x \, dx$

$= \frac{\tan^4 x}{4} - \frac{\tan^2 x}{2} + \ln|\sec x| + c$

tan ↓ De → $\int$ n v reduce

Integrals of type $\int \sin^m x \cdot \cos^n x \, dx$

where m or n or both of them are odd integers.

Rule : In this case if the power of cos x is odd then put $\sin x = t$

If power of sin x is odd then put $\cos x = t$.

If both are odd then put $\sin x = t$ or $\cos x = t$

Ex : $\int \sin^4 x \cdot \cos^7 x \, dx$

$= \int \sin^4 x \cdot (\cos^2 x)^3 \cdot \cos x \, dx$ | Put $\sin x = t$, $dt = \cos x \, dx$, $\cos^2 x = 1 - t^2$

$= \int t^4 \cdot (1-t^2)^3 \cdot dt$

$= \int t^4 (1 - 3t^2 + 3t^4 - t^6) \, dt$

$= \int t^4 \, dt - 3\int t^6 \, dt + 3\int t^8 \, dt - \int t^{10} \, dt$

$= \frac{t^5}{5} - \frac{3t^7}{7} + \frac{3t^9}{9} - \frac{t^{11}}{11} + C$

$= \frac{\sin^5 x}{5} - \frac{3\sin^7 x}{7} + \frac{\sin^9 x}{3} - \frac{\sin^{11} x}{11} + C$

Task → $\int \sin^5 x \cos^4 x \, dx$

Integrals of type $\int \sin^m x \cdot \cos^n x \, dx$

where m+n is -ve integer

In this case convert the integrand in terms of tanx and secx and proceed by putting tanx = t or secx = t according to the question given.

Q → $\int \sec^{\frac{4}{3}} x \, \operatorname{cosec}^{\frac{8}{3}} x \, dx$

$= \int \cos^{-\frac{4}{3}} x \cdot \sin^{-\frac{8}{3}} x \, dx$

$= \int \frac{\cos^{-4} x \cdot \sin^{-\frac{8}{3}} x}{\cos^{-\frac{8}{3}} x} \, dx$

$= \int \frac{\sec^4 x}{\tan^{\frac{8}{3}} x} \, dx = \int \frac{\sec^2 x \cdot \sec^2 x \, dx}{\tan^{\frac{8}{3}} x}$

Put $\tan x = t$

$dt = \sec^2 x \, dx$

$\sec^2 x = 1 + t^2$

$= \int \frac{(1+t^2) \, dt}{t^{\frac{8}{3}}}$

$= \int \frac{1}{t^{\frac{8}{3}}} \, dt + \int \frac{1}{t^{\frac{2}{3}}} \, dt$

$$= \int t^{-\frac{8}{3}}\, dt + \int t^{-\frac{2}{3}}\, dt$$

$$= -\frac{3t^{-\frac{5}{3}}}{5} + 3t^{\frac{1}{3}} + C$$

$$= -\frac{3}{5t^{\frac{5}{3}}} + 3t^{\frac{1}{3}} + C$$

$$= \frac{-3}{5\tan^{\frac{5}{3}} x} + 3\tan^{\frac{1}{3}} x + C$$

Integrals of type $\int \sin^m x \cdot \cos^n x \, dx$

when both m and n are even integers

Here we apply De-moivre's theorem to simplify the integrand

Ex: $\int \sin^4 x \cdot \cos^2 x \, dx$

Put $y = \cos x + i\sin x$

$\frac{1}{y} = \cos x - i\sin x$

$y + \frac{1}{y} = 2\cos x$

$y - \frac{1}{y} = 2i\sin x$

$\Rightarrow \frac{1}{2}\left(y + \frac{1}{y}\right) = \cos x$

and $\frac{1}{2i}\left(y - \frac{1}{y}\right) = \sin x$

$\therefore \sin^4 x \cdot \cos^2 x$

$= \left\{\frac{1}{2i}\left(y - \frac{1}{y}\right)\right\}^4 \cdot \left\{\frac{1}{2}\left(y + \frac{1}{y}\right)\right\}^2$

$= \frac{1}{16}\left(y - \frac{1}{y}\right)^4 \cdot \frac{1}{4}\left(y + \frac{1}{y}\right)^2$

$= \frac{1}{64}\left(y - \frac{1}{y}\right)^2 \cdot \left(y^2 - \frac{1}{y^2}\right)^2$

$= \frac{1}{64}\left\{y^2 + \frac{1}{y^2} - 2\right\}\left\{y^4 + \frac{1}{y^4} - 2\right\}$

$= \frac{1}{64}\left[y^6 + \frac{1}{y^2} - 2y^2 + y^2 + \frac{1}{y^6} - \frac{2}{y^2} - 2y^4 - \frac{2}{y^4} + 4\right]$

$= \frac{1}{64}\left[\left(y^6 + \frac{1}{y^6}\right) - 2\left(y^4 + \frac{1}{y^4}\right) - \left(y^2 + \frac{1}{y^2}\right) + 4\right]$

$= \frac{1}{64}\left[2\cos 6x - 2\cdot 2\cos 4x - 2\cos 2x + 4\right]$

$\left(\begin{aligned} \because\ y &= \cos x + i\sin x \\ y^6 &= \cos 6x + i\sin 6x \\ \frac{1}{y^6} &= \cos 6x - i\sin 6x \end{aligned}\right)$

$$\left(y^6 + \frac{1}{y^6} = 2\cos 6x \quad \text{etc} \right)$$

II

$$\int \sin^4 x \cos^2 x \, dx$$

$$= \frac{2}{64} \left[\int (\cos 6x - 2\cos 4x - \cos 2x + 2) \, dx \right]$$

$$= \frac{1}{32} \left[\frac{\sin 6x}{6} - 2\frac{\sin 4x}{2} - \frac{\sin 2x}{2} + 2x \right] + C$$

$\int \sin x$

Task → 1. $\int \sin^n x \, dx$
2. Read below
3. Induction → Inequality
4. Tan

Task given

$$\int \sin^5 x \cos^4 x \, dx$$

$$= \int \sin^4 x \cos^4 x \cdot \sin x \, dx$$

$$= \int (1 - \cos^2 x)^2 \cdot \cos^4 x \sin x \, dx$$

$$= \int (1 - t^2)^2 \cdot t^4 \cdot (-dt)$$

$$= -\int (1 + t^4 - 2t^2) t^4 \, dt$$

$$= -\int (t^4 + t^8 - 2t^6) \, dt$$

Put $\cos x = t$

$dt = -\sin x \, dx$

$$= -\left[\frac{t^5}{5} + \frac{t^9}{9} - 2\frac{t^7}{7}\right] + C$$

$$= -\frac{t^5}{5} - \frac{t^9}{9} + \frac{2t^7}{7} + C$$

$$= \frac{2\cos^7 x}{7} - \frac{\cos^9 x}{9} - \frac{\cos^5 x}{5} + C \quad \text{(Ans)}$$

Consider formula

Q → $\int \sin^n x \, dx$

Ans:- Let $I = \int \sin^{n-1} x \; \sin x \, dx$

$$= \sin^{n-1} x \int \sin x \, dx - \int \left(\frac{d}{dx} \sin^{n-1} x\right) \left(\int \sin x \, dx\right) dx$$

$$= \sin^{n-1} x \,(-\cos x) + \int (n-1) \sin^{n-2} x \, \cos x \cdot \cos x \, dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \cdot \cos^2 x \, dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \,(1 - \sin^2 x)\, dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \, dx - (n-1) \int \sin^n x \, dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \, dx - (n-1) \cdot I$$

$$\Rightarrow I + (n-1) I = -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \, dx$$

$$\Rightarrow nI = -\sin^{n-1}x \cos x + (n-1)\int \sin^{n-2}x \, dx$$

$$\Rightarrow I = \frac{-\sin^{n-1}x \cos x}{n} + \left(\frac{n-1}{n}\right)\int \sin^{n-2}x \, dx$$

Integrals of type $\int \tan^m x \sec^n x \, dx$

where n is even ~~then Pos~~

Here put $\tan x = t$

Ex: $\int \tan^5 x \sec^4 x \, dx$

Put $\tan x = t$

$dt = \sec^2 x \, dx$

$\sec^2 x = 1 + t^2$

$= \int \tan^5 x \sec^2 x \sec^2 x \, dx$

$= \int t^5 (1+t^2) \, dt$

$= \int t^5 \, dt + \int t^7 \, dt$

$= \frac{t^6}{6} + \frac{t^8}{8} + C$

$= \frac{\tan^6 x}{6} + \frac{\tan^8 x}{8} + C$

Integrals of type $\int \tan^m x \sec^n x \, dx$

where m and n are odd

Here put $\sec x = t$

Ex: $\int \sec^3 x \cdot \tan^5 x \, dx$

$= \int \sec x \tan x \cdot \sec^2 x \tan^4 x \, dx$

$$= \int \sec^6 x \tan^4 x \, (\sec x \tan x \, dx)$$

$$= \int t^2 \cdot (t^2-1)^2 \cdot dt$$

Put $\sec x = t$, $dt = \sec x \tan x$, $\tan^2 x = t^2 - 1$

$$= \int t^2 (t^4 + 1 - 2t^2) \, dt$$

$$= \int (t^6 + t^2 - 2t^4) \, dt$$

$$= \frac{t^7}{7} + \frac{t^3}{3} - \frac{2t^5}{5} + C$$

$$= \frac{\sec^7 x}{7} + \frac{\sec^3 x}{3} - \frac{2\sec^5 x}{5} + C$$

Note: If in above integral m is even and n is odd then express $\tan^m x$ in terms of $\sec x$ and integrate.

In case cosec and cot, if power of cot is odd put $\csc\theta = t$. If power of cot is even put $\cot\theta = t$, if both are odd — [illegible] t put [illegible]. Both powers are even put [illegible] power — $= t$ [illegible]

Integrals of type $\int \sin mx \cos nx \, dx$

Or $\int \sin mx \sin nx \, dx$ or $\int \cos mx \cos nx \, dx$

Here we use $\sin mx \cos nx = \frac{1}{2}\{\sin(m+n)x + \sin(m-n)x\}$

$\sin mx \sin nx = -\frac{1}{2}\{\cos(m+n)x - \cos(m-n)x\}$

$\cos mx \cos nx = \frac{1}{2}\{\cos(m+n)x + \cos(m-n)x\}$

Ex: $\int \sin 6x \cos 5x \, dx$

$= \int \frac{1}{2}(\sin 11x + \sin x)\, dx$

$= \frac{1}{2}\left[\int \sin 11x \, dx + \int \sin x \, dx\right]$

$= \frac{1}{2}\left[-\frac{\cos 11x}{11} - \cos x\right] + C$

$= -\frac{\cos 11x}{22} - \frac{\cos x}{2} + C$

Integrals of type $\int \frac{dx}{a+b\cos x}$ or

$\int \frac{dx}{a+b\sin x}$ or $\int \frac{dx}{a\sin x + b\cos x}$ or $\int \frac{dx}{a\sin x + b\cos x + c}$

Here put $\boxed{\tan\frac{x}{2} = t}$

$\therefore \sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

$= \frac{2t}{1+t^2}$

$\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{1-t^2}{1+t^2}$

$dt = \sec^2 \frac{x}{2} \left(\frac{1}{2} \times dx\right)$

$= \left(1 + \tan^2 \frac{x}{2}\right) \frac{1}{2} dx$

$= \frac{(1 + t^2)}{2} dx$

$\Rightarrow dx = \frac{2\,dt}{(1+t^2)}$

Ex: $\int \frac{1}{a + b\cos x} dx$ where $a^2 > b^2$ or $a > b$

Sol^n $\int \frac{1}{a + b\left(\frac{1-t^2}{1+t^2}\right)} \times \frac{2\,dt}{(1+t^2)}$

$= 2\int \frac{dt}{a + b(1-t^2)}$

Put $\tan\frac{x}{2} = t$

$\cos x = \frac{1-t^2}{1+t^2}$

$dx = \frac{2\,dt}{1+t^2}$

Sol^n $\int \frac{1}{\left(a + b\left(\frac{1-t^2}{1+t^2}\right)\right)} \times \frac{2\,dt}{1+t^2}$

$= \int \frac{(1+t^2)}{a(1+t^2) + b(1-t^2)} \times \frac{2\,dt}{(1+t^2)}$

$= 2 \int \frac{dt}{(a+b) + t^2(a-b)}$

$= \frac{2}{a-b} \int \frac{dt}{t^2 + \frac{a+b}{a-b}}$

$$= \frac{2}{a-b} \int \frac{dt}{t^2 + \left(\sqrt{\frac{a+b}{a-b}}\right)^2}$$

~~$x = \frac{a+b}{a-b} \tan$~~

$\left(\because \frac{a+b}{a-b} = \frac{(a+b)^2}{a^2-b^2} > 0 \right.$

$$= \frac{2}{a-b} \times \frac{1}{\sqrt{\frac{a+b}{a-b}}} \cdot \tan^{-1}\left(\frac{t}{\sqrt{\frac{a+b}{a-b}}}\right) + C$$

$$= \frac{2}{\frac{(a-b)}{\sqrt{a-b}}} \times \frac{\sqrt{a-b}}{\sqrt{a+b}} \cdot \tan^{-1}\left(t\sqrt{\frac{a-b}{a+b}}\right) + C$$

$$I = \frac{2}{\sqrt{a^2-b^2}} \cdot \tan^{-1}\left(\sqrt{\frac{a-b}{a+b}} \cdot \tan\frac{x}{2}\right) + C$$

$\because \int \frac{dx}{x^2+A^2} = \frac{1}{A} \tan^{-1}\left(\frac{x}{A}\right)$

Ex: $\int \frac{dx}{a+b\cos x}$ where $a^2 < b^2$

Put $\tan\frac{x}{2} = t$

$\cos x = \frac{1-t^2}{1+t^2}$

$dx = \frac{2dt}{1+t^2}$

$$= \int \frac{1}{a+b\cdot\left(\frac{1-t^2}{1+t^2}\right)} \times \frac{2dt}{1+t^2}$$

$$= \int \frac{2\,dt}{a(1+t^2) + b(1-t^2)}$$

$$= 2\int \frac{dt}{(a+b) + t^2(a-b)}$$

$$= \frac{2}{a-b} \int \frac{dt}{t^2 + \frac{a+b}{a-b}}$$

Here $\frac{a+b}{a-b} = \frac{(a+b)^2}{a^2-b^2} < 0$

Put $\frac{a+b}{a-b} = -B^2$

$$= \frac{2}{a-b} \cdot \int \frac{dt}{t^2 - B^2}$$

$$= \frac{2}{a-b} \cdot \frac{1}{2B} \ln\left|\frac{t-B}{t+B}\right| + C$$

$$= \frac{1}{a-b} \frac{1}{\sqrt{\frac{a+b}{b-a}}} \ln\left|\frac{\tan\frac{x}{2} - \sqrt{\frac{a+b}{b-a}}}{\tan\frac{x}{2} + \sqrt{\frac{a+b}{b-a}}}\right| + C \quad \left(\because \int \frac{dx}{x^2 - A^2} = \frac{1}{2A} \ln\left|\frac{x-A}{x+A}\right| \right)$$

$$= \frac{-1}{\sqrt{b^2-a^2}} \ln\left|\frac{\tan\frac{x}{2} - \sqrt{\frac{b+a}{b-a}}}{\tan\frac{x}{2} + \sqrt{\frac{a+b}{b-a}}}\right| + C$$

$$= \frac{1}{\sqrt{b^2-a^2}} \ln\left|\frac{\tan\frac{x}{2} + \sqrt{\frac{a+b}{b-a}}}{\tan\frac{x}{2} - \sqrt{\frac{b+a}{b-a}}}\right| + C$$

$$= \frac{1}{\sqrt{b^2-a^2}} \ln\left|\frac{\sqrt{b-a}\tan\frac{x}{2} + \sqrt{a+b}}{\sqrt{b-a}\tan\frac{x}{2} - \sqrt{a+b}}\right| + C$$

Alternative method for $\int \frac{dx}{a\sin x + b\cos x}$

Put $a = r\cos\alpha$

$b = r\sin\alpha$

$\therefore r = \sqrt{a^2+b^2}$

$\tan\alpha = \frac{b}{a}$

$\Rightarrow \alpha = \tan^{-1}\left(\frac{b}{a}\right)$

$$\int \frac{dx}{a\sin x + b\cos x}$$

$$= \int \frac{dx}{r\cos\alpha \sin x + r\sin\alpha \cos x}$$

$$= \int \frac{dx}{r\sin(x+\alpha)}$$

$$= \frac{1}{r}\int \operatorname{cosec}(x+\alpha)\,dx$$

Put $x+\alpha = t$, $dt = dx$

$$= \frac{1}{r}\int \operatorname{cosec} t \, dt$$

$$= \frac{1}{r}\ln|\operatorname{cosec} t - \cot t| + C$$

$$= \frac{1}{r}\ln|\operatorname{cosec}(x+\alpha) - \cot(x+\alpha)| + C$$

where $r = \sqrt{a^2+b^2}$

$\alpha = \tan^{-1}\left(\frac{b}{a}\right)$

Integrals of type $\int \left(\frac{d + e\cos x + f\sin x}{a + b\cos x + c\sin x}\right) dx$

Here we express the numerator as

$L \cdot (\text{Dr}) + M \cdot (\text{Derivative of denominator}) + N$

L, M, N will be found by comparing the co-efficients of $\cos x$, $\sin x$ and constant terms. Then the integral is reduced to sum of 3 integrals.

$$\int \frac{L \cdot D.r}{D.r} dx + \int \frac{M \cdot \text{Derivative of } d.r}{D.r} dx + \int \frac{N}{D.r} dx$$

First integral $= Lx$

Second " $= M \log (D.r)$

Third " ~~const~~ is found by

putting $\tan \frac{x}{2} = t$

Integrals of type $\int \left(\frac{p \cos x + q \sin x}{a \sin x + b \cos x} \right) dx$

Here we express the n.r as

$L.(D.r) + M.(\text{Derivative of } D.r)$

~~L and~~ M can be found by comparing the co-efficient of $\cos x$, $\sin x$. Then the integral is reduced to sum of two integrals

$$\int \frac{L.Dr}{D.r} dx + \int \frac{M \cdot \text{Derivative of } d.r}{d.r} dx$$

First integral $= Lx$

Second " $= M \log|D.r|$.

Ex. → $\int \left(\frac{2\sin x + 3\cos x}{3\sin x + 4\cos x}\right) dx$

Ans :

Now $2\sin x + 3\cos x$

$= L.(3\sin x + 4\cos x) + M.(3\cos x - 4\sin x)$

$= \sin x\,(3L - 4m) + \cos x\,(4L + 3m)$

Comparing the Co-efficient of $\cos x$ and $\sin x$, we get

$3L - 4m = 2$ ——

$4L + 3m = 3$ ——

$12L - 16m = 8$

$12L + 9m = 9$

(−) (+) (−)

$-25m = -1$

$\Rightarrow m = \frac{1}{25}$

$L = \frac{2+4m}{3}$

$= \frac{2 + \frac{4}{25}}{3}$

$= \frac{54}{25} \times \frac{1}{3} = \rightarrow$ $L = \frac{18}{25}$

$\therefore 2\sin x + 3\cos x = \frac{18}{25}(3\sin x + 4\cos x) + \frac{1}{25}(3\cos x - 4\sin x)$

$\Rightarrow \frac{2\sin x + 3\cos x}{3\sin x + 4\cos x} = \frac{18}{25} + \frac{1}{25}\left(\frac{3\cos x - 4\sin x}{3\sin x + 4\cos x}\right)$

$$\Rightarrow \int \frac{2\sin x + 3\cos x}{3\sin x + 4\cos x}\,dx = \frac{18}{25}x + \frac{1}{25}\log|3\sin x + 4\cos x| + C$$

Ex - 2

$$\int \frac{2+3\cos\theta}{\sin\theta + 2\cos\theta + 3}$$

Sol^n :

Now

$2 + 3\cos\theta$

$= L(\sin\theta + 2\cos\theta + 3) + M.(\cos\theta - 2\sin\theta)$

$+ N$

$= \sin\theta(L - 2M) + \cos\theta(2L + M) + (3L + N)$

Comparing $\sin\theta$, $\cos\theta$ and Constant term

$$L - 2M = 0$$
$$2L + M = 3$$
$$3L + N = 2$$

$L = 2M$, $\quad 2L + M = 3 \Rightarrow 4m + m = 3$

$\Rightarrow m = \frac{3}{5}$

$= 2.\frac{3}{5}$

$= \frac{6}{5}$, $\quad N = 2 - 3L = 2 - 3.\frac{6}{5} = \frac{-8}{5}$

Solving $L = \frac{6}{5}$, $M = \frac{3}{5}$, $N = \frac{-8}{5}$

$\therefore 2 + 3\cos\theta = \frac{6}{5}(\sin\theta + 2\cos\theta + 3) + \frac{3}{5}(\cos\theta - 2\sin\theta) - \frac{8}{5}$

$$\Rightarrow \frac{2+3\cos\theta}{\sin\theta+2\cos\theta+3} = \frac{6}{5} + \frac{3}{5}\left(\frac{\cos\theta-2\sin\theta}{\sin\theta+2\cos\theta+3}\right) - \frac{8}{5}\times\frac{1}{(\sin\theta+2\cos\theta+3)}$$

$$\Rightarrow \int \frac{2+3\cos\theta}{\sin\theta+2\cos\theta+3}\,d\theta = \frac{6}{5}\theta + \frac{3}{5}\ln|\sin\theta+2\cos\theta+3| - \frac{8}{5}\int\frac{d\theta}{\sin\theta+2\cos\theta+3}$$

Consider

$$\int \frac{d\theta}{\sin\theta+2\cos\theta+3}$$

Put $\tan\frac{\theta}{2} = t$

$\sin\theta = \frac{2t}{1+t^2}$

$\cos\theta = \frac{1-t^2}{1+t^2}$

$d\theta = \frac{2dt}{1+t^2}$

$$= \int \frac{2dt}{2t+2-2t^2+3+3t^2}$$

$$= 2\int \frac{dt}{t^2+2t+5}$$

$$= 2\int \frac{dt}{(t+1)^2+2^2}$$

$$= 2\cdot\frac{1}{2}\tan^{-1}\left(\frac{t+1}{2}\right)$$

$$\left(\int\frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right)$$

$$= \tan^{-1}\left(\frac{\tan\frac{\theta}{2}+1}{2}\right)$$

$$\therefore \int \frac{2+3\cos\theta}{\sin\theta+2\cos\theta+3}\, d\theta$$

$$= \frac{6}{5}\theta + \frac{3}{5}\ln|\sin\theta + 2\cos\theta+3| - \frac{8}{5}\cdot \tan^{-1}\left(\frac{1+\tan\frac{\theta}{2}}{2}\right)$$

$$+ C \quad \text{(Ans)}$$

Ex → 3 (Imp)

$$\int \frac{x^2+1}{x^4+1}\, dx$$

$$= \int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\, dx \quad \left(\text{Dividing n.r and d.r of integrand by } x^2\right)$$

$$= \int \frac{\left(1+\frac{1}{x^2}\right) dx}{x^2+\frac{1}{x^2}-2+2}$$

$$= \int \frac{\left(1+\frac{1}{x^2}\right) dx}{\left(x-\frac{1}{x}\right)^2+2}$$

Put $x-\frac{1}{x} = t$

$\Rightarrow dt = \left(1+\frac{1}{x^2}\right)dx$

$$= \int \frac{dt}{t^2+2}$$

$$= \int \frac{dt}{t^2+(\sqrt{2})^2}$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{t}{\sqrt{2}}\right) + C \quad \left(\because \int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\right)$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x-\frac{1}{x}}{\sqrt{2}}\right) + C$$

$$= \frac{1}{\sqrt{2}} \cdot \tan^{-1}\left(\frac{x^2-1}{\sqrt{2}x}\right) + C$$

Ex : 4 → $\int \frac{x^2-1}{x^4+1}\,dx$

$$= \int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,dx$$

(dividing n.r and d.r of integrand by x^2)

$$= \int \frac{\left(1-\frac{1}{x^2}\right)dx}{\left(x^2+\frac{1}{x^2}+2\right)-2}$$

$$= \int \frac{\left(1-\frac{1}{x^2}\right)dx}{\left(x+\frac{1}{x}\right)^2-2}$$

Put $x+\frac{1}{x}=t$

$dt=\left(1-\frac{1}{x^2}\right)dx$

$$= \int \frac{dt}{t^2-(\sqrt{2})^2}$$

$$= \frac{1}{2\sqrt{2}} \ln\left|\frac{t-\sqrt{2}}{t+\sqrt{2}}\right| + C$$

$$= \frac{1}{2\sqrt{2}} \ln\left|\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right| + C$$

$$= \frac{1}{2\sqrt{2}} \cdot \ln\left|\frac{x^2+1-\sqrt{2}x}{x^2+1+\sqrt{2}x}\right| + C$$

(Ans)

Ex-5 $\int \frac{1}{1+x^4}\,dx$

$$= \frac{1}{2}\int \left(\frac{x^2+1}{1+x^4} - \frac{x^2-1}{1+x^4}\right) dx$$

$$= \frac{1}{2}\left[\int \frac{x^2+1}{1+x^4}\,dx - \int \frac{x^2-1}{1+x^4}\,dx\right]$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{2}\,x}\right) - \frac{1}{2\sqrt{2}}\ln\left|\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right|\right] + C$$

Ex-6 $\int \frac{x^2}{1+x^4}\,dx$

$$= \frac{1}{2}\int \left(\frac{x^2+1}{1+x^4} + \frac{x^2-1}{1+x^4}\right) dx$$

$$= \frac{1}{2}\left[\int \frac{x^2+1}{1+x^4}\,dx + \int \frac{x^2-1}{1+x^4}\,dx\right]$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{2}x}\right) + \frac{1}{2\sqrt{2}}\ln\left|\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right|\right] + C$$

(Ans)

Ex 7 → $\int_0^{\pi/4} \sqrt{\tan\theta}\, d\theta$

V. imp

$= \int_0^1 x \cdot \frac{2x\,dx}{1+x^4}$

$= \int_0^1 \frac{2x^2\,dx}{1+x^4}$

$= \int_0^1 \left[\frac{x^2+1}{1+x^4} + \frac{x^2-1}{1+x^4}\right] dx$

$= \int_0^1 \frac{x^2+1}{1+x^4}\,dx + \int_0^1 \frac{x^2-1}{1+x^4}\,dx$

$= \left[\frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x^2-1}{\sqrt{2}x}\right) + \frac{1}{2\sqrt{2}} \ln\left|\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right|\right]_0^1$

$= \left[0 + \frac{1}{2\sqrt{2}} \ln\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right] - \left[\frac{1}{\sqrt{2}}\left(\frac{-\pi}{2}\right) + \frac{1}{2\sqrt{2}}\, 0\right]$

$= \frac{1}{2\sqrt{2}} \ln\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right) + \frac{\pi}{2\sqrt{2}}$

---

put $\tan\theta = x^2$

$\sec^2\theta\, d\theta = 2x\,dx$

$\Rightarrow d\theta = \frac{2x\,dx}{\sec^2\theta}$

$= \frac{2x\,dx}{1+\tan^2\theta}$

$= \frac{2x\,dx}{1+x^4}$

$\theta = 0 \Rightarrow x = 0$

$\theta = \frac{\pi}{4} \Rightarrow x = 1$

my hint, why $x \neq -1$ because, $0, \frac{\pi}{4}$ এর $\sqrt{\tan\theta}$ এর value 0 to 1

∴ x এর value 0 to 1

not equal to −1

$$= \frac{1}{2\sqrt{2}} \ln \frac{(\sqrt{2}-1)^2}{1} + \frac{\pi\sqrt{2}}{4}$$

$$= \frac{1}{\sqrt{2}} \ln(\sqrt{2}-1) + \frac{\pi\sqrt{2}}{4}$$ **Ans**

~~Task~~

$$\int_0^{\pi/4} \sqrt{\cot\theta}\, d\theta$$

In the last page of this khata / may be in the cover

Put

$\cot\theta = x^2$

$-\operatorname{cosec}^2\theta\, d\theta = 2x\, dx$

$\Rightarrow d\theta = -\frac{2x\,dx}{\operatorname{cosec}^2\theta}$

$= \frac{-2x\,dx}{1+x^4}$

$\theta = 0, \Rightarrow x = \infty$

$\theta = \frac{\pi}{4} \Rightarrow x = 1$

$$= \int_\infty^1 x \cdot \frac{-2x\,dx}{1+x^4}$$

$$= \int_\infty^1 -\frac{2x^2\,dx}{1+x^4}$$

$$= \int_\infty^1 -\left[\frac{x^2+1}{1+x^4} + \frac{x^2-1}{1+x^4}\right] dx$$

$$= \int_\infty^1 -\left(\frac{x^2+1}{1+x^4}\right) + \int_\infty^1 -\left(\frac{x^2-1}{1+x^4}\right) dx$$

$$= \int_\infty^1 -\frac{1}{\sqrt{2}} \tan^{-1}\frac{x^2-1}{\sqrt{2}x} + \int_\infty^1 -\frac{1}{2\sqrt{2}} \ln\left|\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right|$$

$$= \left[-\frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x^2-1}{\sqrt{2}x}\right) - \frac{1}{2\sqrt{2}} \ln\left|\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right|\right]_\infty^1$$

Another process

$$\int_0^{\pi/4} \sqrt{\cot\theta}\, d\theta = \int_0^{\pi/4} \sqrt{\tan(\tfrac{\pi}{2}-\theta)}\, d\theta$$

$$= \int_{\pi/2}^{\pi/4} -\sqrt{\tan A}\, dA$$

Put

$\frac{\pi}{2} - \theta = A$

$\Rightarrow -d\theta = dA$

$\Rightarrow d\theta = -dA$

$\theta = 0 \Rightarrow A = \frac{\pi}{2}$

$\theta = \pi/4, \Rightarrow A = \pi/4$

$$= \int_{\pi/4}^{0} \sqrt{\tan x}\, dx$$

$\tan A = x^2$

$\theta = 0, \Rightarrow x = 0$

$\theta = \frac{\pi}{4} \Rightarrow x = 1$

By the below formula.

$$= \left[\frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x^2-1}{\sqrt{2}x}\right) + \frac{1}{2\sqrt{2}} \ln \left| \frac{x^2 - \sqrt{2}x + 1}{x^2 + \sqrt{2}x + 1} \right| \right]_1^0$$

$$= \left[\frac{1}{\sqrt{2}} \left(\frac{-\pi}{2}\right) + 0 \right] - \left[0 + \frac{1}{2\sqrt{2}} \ln\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right) \right]$$

$$= -\frac{\pi}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} \ln\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)$$

$$= -\frac{\pi\sqrt{2}}{4} - \frac{1}{2\sqrt{2}} \ln (\sqrt{2}-1)^2$$

$$= -\left[\frac{\pi\sqrt{2}}{4} + \frac{1}{\sqrt{2}} \ln(\sqrt{2}-1)\right]$$ (Ans)

Integration of irrational functions

Rule :- In case we find that an algebraic expression involves fractional powers of x or of $ax+b$ then we should put x or $ax+b = t^n$ where n is the L.C.M of d.r of various fractional powers.

Ex ⇒ $$\int \frac{dx}{(1+x)^{\frac{1}{2}} - (1+x)^{\frac{1}{3}}}$$

L.C.M of 2 and 3 = 6

Put $1+x = t^6$

$dx = 6t^5\, dt$

$$= \int \frac{6t^5\, dt}{t^3 - t^2}$$

$$= 6 \int \frac{t^3\, dt}{t-1}$$

$$= 6\int\left[(t^2+t+1)+\frac{1}{t-1}\right]dt$$

$$\begin{array}{r} t-1\,)\,t^3\,(\,t^2+t+1 \\ \underline{t^3-t^2} \\ t^2 \\ \underline{t^2-t} \\ t-1 \\ \underline{t-1} \\ 1 \end{array}$$

$$= 6\left[\frac{t^3}{3}+\frac{t^2}{2}+t+\ln|t-1|\right]+C$$

$$= 6\left[\frac{(1+x)^{\frac{3}{6}}}{3}+(1+x)^{\frac{2}{3}}\right.$$

$$= 6\left[\frac{(1+x)^{\frac{1}{2}}}{3}+\frac{(1+x)^{\frac{1}{3}}}{2}+(1+x)^{\frac{1}{6}}+\ln|(1+x)^{\frac{1}{6}}-1|\right]+C$$

$$= 2(1+x)^{\frac{1}{2}}+3(1+x)^{\frac{1}{3}}+6(1+x)^{\frac{1}{6}}+6\ln|(1+x)^{\frac{1}{6}}-1|+C$$

Integrals of the form $\int \frac{\phi(x)\,dx}{(px+q)(\sqrt{ax+b})}$

or $\int \frac{\phi(x)\,dx}{(px^2+qx+r)\sqrt{ax+b}}$

Here put $ax+b = t^2$

Ex

$$\int \frac{dx}{(x+2)(\sqrt{x-1})}$$

$$= \int \frac{2t\,dt}{(3+t^2)(t)}$$

Put
$x-1 = t^2$
$dx = 2t\,dt$
$x = t^2+1$

$$= 2\int \frac{dt}{3+t^2}$$

$$= 2\int \frac{dt}{t^2+(\sqrt{3})^2}$$

$$= 2 \cdot \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{t}{\sqrt{3}}\right) + C$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{\sqrt{x-1}}{\sqrt{3}}\right) + C$$

Integrals of the form $\int \frac{\phi(x)\,dx}{(px+q)\left(\sqrt{ax^2+bx+c}\right)}$

Here put $px+q = \frac{1}{t}$

Q→ $\int \frac{dx}{(x+1)\left(\sqrt{1+2x-x^2}\right)}$

Put

$x+1 = \frac{1}{t}$

$dx = \frac{-1}{t^2}\,dt$

$\Rightarrow x = \frac{1-t}{t}$

$1+2x-x^2$

$= 1+2\cdot\left(\frac{1-t}{t}\right)-\left(\frac{1-t}{t}\right)^2$

$= \frac{t^2+2t-2t^2-1+2t-t^2}{t^2}$

$= \frac{4t-2t^2-1}{t^2}$

$= \frac{2}{t^2}\left(2t-t^2-\frac{1}{2}\right)$

$= \frac{2}{t^2}\left\{\frac{1}{2}-(t-1)^2\right\}$

$= \frac{2}{t^2}\left\{\left(\frac{1}{\sqrt{2}}\right)^2-(t-1)^2\right\}$

$$= \int \frac{-\frac{1}{t^2}\,dt}{\frac{1}{t}\cdot\frac{\sqrt{2}}{t}\sqrt{\left(\frac{1}{\sqrt{2}}\right)^2-(t-1)^2}}$$

$$= \frac{-1}{\sqrt{2}}\int \frac{dt}{\sqrt{\left(\frac{1}{\sqrt{2}}\right)^2-(t-1)^2}}$$

$$= \frac{-1}{\sqrt{2}}\sin^{-1}\left(\frac{t-1}{\frac{1}{\sqrt{2}}}\right)+C$$

$$= \frac{-1}{\sqrt{2}}\sin^{-1}\left\{\sqrt{2}\,(t-1)\right\}+C$$

$$= \frac{-1}{\sqrt{2}}\sin^{-1}\left\{\sqrt{2}\left(\frac{1}{1+x}-1\right)\right\}+C$$

$$= \frac{-1}{\sqrt{2}}\sin^{-1}\left(\sqrt{2}\times\frac{(1-1-x)}{1+x}\right)$$

$$= \frac{-1}{\sqrt{2}}\sin^{-1}\left(\frac{-\sqrt{2}x}{x+1}\right)+C$$

$$= \frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{\sqrt{2}x}{x+1}\right)+C \quad \text{(Ans)}$$

Integrals of the form $\int \frac{dx}{P\sqrt{Q}}$.

where P and Q are both quadratic expressions. ~~Of~~ and P can be resolved into linear factor.

In this case split $\frac{1}{P}$ into partial fractions and then the problems is reduced to above discussed type

Q →

$$\int \frac{dx}{(x^2-3x+2)\sqrt{x^2-7x+12}}$$

$$= \int \frac{dx}{(x-2)(x-1)\sqrt{x^2-7x+12}}$$

$$= \int \left(\frac{1}{x-2} - \frac{1}{x-1}\right) \frac{dx}{\sqrt{x^2-7x+12}}$$

$$= \int \frac{dx}{(x-2)\sqrt{x^2-7x+12}} - \int \frac{dx}{(x-1)(\sqrt{x^2-7x+12})}$$

$$= \int \frac{-\frac{1}{t^2}\,dt}{\frac{1}{t}\cdot\frac{\sqrt{2}}{t}\cdot\sqrt{(t-\frac{3}{4})^2-(\frac{1}{4})^2}} - \int \frac{-\frac{1}{v^2}\,dv}{\frac{1}{v}\cdot\frac{\sqrt{6}}{v}\cdot\sqrt{(v-\frac{5}{12})^2-(\frac{1}{12})^2}}$$

$$= \frac{-1}{\sqrt{2}}\int \frac{dt}{\sqrt{\left(t-\frac{3}{4}\right)^2-\left(\frac{1}{4}\right)^2}} + \frac{1}{\sqrt{6}}\int \frac{dv}{\sqrt{\left(v-\frac{5}{12}\right)^2-\left(\frac{1}{12}\right)^2}}$$

$$= \frac{-1}{\sqrt{2}} \ln\left| t-\frac{3}{4} + \sqrt{\left(t-\frac{3}{4}\right)^2-\left(\frac{1}{4}\right)^2}\right| + \frac{1}{\sqrt{6}} \ln\left|\left(v-\frac{5}{12}\right)+\sqrt{\left(v-\frac{5}{12}\right)^2-\left(\frac{1}{12}\right)^2}\right| + C$$

$$= \frac{-1}{\sqrt{2}} \ln\left| \frac{1}{x-2} - \frac{3}{4} + \sqrt{\left(\frac{1}{x-2}-\frac{3}{4}\right)^2 - \frac{1}{16}}\right|$$

$$+ \frac{1}{\sqrt{6}} \ln\left| \frac{1}{x-1} - \frac{5}{12} + \sqrt{\left(\frac{1}{x-1}-\frac{5}{12}\right)^2 - \frac{1}{144}}\right| + C$$

$$= \frac{-1}{\sqrt{2}} \ln\left| \frac{4-3x+6}{4(x-2)} + \sqrt{\frac{(10-3x)^2}{(x-2)^2 16} - \frac{1}{16}}\right|$$

$$+ \frac{1}{\sqrt{6}} \ln\left| \frac{12-5x+5}{(x-1)12} + \sqrt{\frac{(17-5x)^2}{144(x-1)^2} - \frac{1}{144}}\right| + C$$

$$= \frac{-1}{\sqrt{2}} \ln\left| \frac{10-3x}{4(x-2)} + \frac{\sqrt{100+9x^2-60x-x^2-4+4x}}{4(x-2)}\right|$$

$$+ \frac{1}{\sqrt{6}} \ln\left| \frac{17-5x}{(x-1)12} + \frac{\sqrt{289+25x^2-170x-x^2-1+2x}}{12(x-1)}\right| + C$$

$$= \frac{-1}{\sqrt{2}} \ln\left| \frac{10-3x+\sqrt{8x^2-56x+96}}{4(x-2)}\right| + \frac{1}{\sqrt{6}} \ln\left|\frac{17-5x+\sqrt{24x^2-168x+288}}{12(x-1)}\right| + C$$

$$= \frac{-1}{\sqrt{2}} \ln|10-3x+\sqrt{8x^2-56x+96}| + \frac{1}{\sqrt{2}} \ln 4 + \frac{1}{\sqrt{2}} \ln|x-2|$$

$$+ \frac{1}{\sqrt{6}} |17-5x+\sqrt{24x^2-168x+288}| - \frac{1}{\sqrt{6}} \ln 2 - \frac{1}{\sqrt{6}} \ln|x-1| + C$$

$$= \frac{-1}{\sqrt{2}} \left[\ln|10-3x+\sqrt{8x^2-56x+96}| + \ln|x-2|\right] + \frac{1}{\sqrt{6}}\left[\ln|17-5x+\sqrt{24x^2-168x+288}| - \ln|x-1|\right] + K$$

(where $K = C - \frac{1}{\sqrt{2}}\ln 4 - \frac{1}{\sqrt{6}}\ln 2$)

Put

$x-2 = \frac{1}{t}$

$\Rightarrow dx = \frac{-1}{t^2} dt$

$x = \frac{1+2t}{t}$

$x^2 - 7x + 12$

$= \frac{(1+2t)^2}{t^2} - 7\cdot\left(\frac{1+2t}{t}\right) + 12$

$= \frac{1+4t^2+4t-7t-14t^2+12t^2}{t^2}$

$= \frac{2t^2-3t+1}{t^2}$

$= \frac{2}{t^2}\left(t^2 - \frac{3}{2}t + \frac{1}{2}\right)$

$= \frac{2}{t^2}\left\{\left(t-\frac{3}{4}\right)^2 - \left(\frac{1}{4}\right)^2\right\}$

Put

$x-1 = \frac{1}{v}$

$dx = \frac{-1}{v^2} dv$

$x = \frac{1+v}{v}$

$x^2 - 7x + 12$

$= \frac{(1+v)^2}{v^2} - 7\cdot\left(\frac{1+v}{v}\right) + 12$

$= \frac{1+v^2+2v-7v-7v^2+12v^2}{v^2}$

$= \frac{6v^2-5v+1}{v^2}$

$= \frac{6}{v^2}\left\{v^2 - \frac{5}{6}v + \frac{1}{6}\right\}$

$= \frac{6}{v^2}\left\{\left(v-\frac{5}{12}\right)^2 - \left(\frac{1}{12}\right)^2\right\}$

Integrals of the form $\int \frac{dx}{P\sqrt{Q}}$ when P and Q are quadratic expressions and P is perfect square. Here

Put $\sqrt{P} = \frac{1}{t}$

Ex →

$$\int \frac{dx}{(x^2-6x+9)(\sqrt{x^2-6x+14})}$$

$$= \int \frac{dx}{(x-3)^2 (\sqrt{(x-3)^2-5})}$$

Put $x-3 = \frac{1}{t}$

$\Rightarrow dx = \frac{-1}{t^2} dt$

$x = 3+\frac{1}{t} = \frac{3t+1}{t}$

$1-5t^2 = u$

$$= \int \frac{\frac{-1}{t^2} dt}{\frac{1}{t^2} \cdot \sqrt{\frac{1}{t^2}-5}}$$

$$= -\int \frac{t\,dt}{\sqrt{1-5t^2}}$$

$$= -\int \frac{-\frac{1}{5} x\,dx}{x}$$

Put

$1-5t^2 = x^2$

$-10t\,dt = 2x\,dx$

$\Rightarrow t\,dt = -\frac{2}{10} x\,dx$

$= -\frac{1}{5} x\,dx$

$$= \frac{1}{5}\int dx$$

$$= \frac{1}{5}x + C$$

$$= \frac{1}{5}\sqrt{1-5t^2} + C$$

$$= \frac{1}{5}\sqrt{1-5t^2} + C$$

$$= \frac{1}{5}\sqrt{1-5\left(\frac{1}{x-3}\right)^2} + C$$

$$= \frac{1}{5}\frac{\sqrt{x^2+9-6x-5}}{x-3} + C$$

$$= \frac{1}{5}\frac{\sqrt{x^2-6x+4}}{x-3} + C$$

Integrals of the form $\int \frac{dx}{(ax^2+b)\sqrt{cx^2+d}}$

Here we put $x = \frac{1}{t}$ and then put the expression in t under radical $= z^2$

Ex:- $\int \frac{dx}{(1+x^2)(\sqrt{1-x^2})}$

put $x = \frac{1}{t}$

$dx = \frac{-1}{t^2}dt$

$$= \int \frac{-\frac{1}{t^2}dt}{\left(1+\frac{1}{t^2}\right)\sqrt{1-\frac{1}{t^2}}}$$

$$= \int \frac{t\left(1 - \frac{1}{t^2}\right) dt}{\frac{t^2+1}{t^2} \cdot \sqrt{t^2-1}}$$

$$= -\int \frac{t \, dt}{(t^2+1)\sqrt{t^2-1}}$$

Put

$t^2 - 1 = z^2$

$2t \frac{dt}{dz} = 2z \, dz$

$\Rightarrow t \, dt = z \, dz$

$$= -\int \frac{z \, dz}{(2+z^2) \cdot z}$$

$$= -\int \frac{dz}{z^2+(\sqrt{2})^2}$$

$$= -\frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{z}{\sqrt{2}}\right) + C$$

$$= -\frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{\sqrt{t^2-1}}{\sqrt{2}}\right) + C$$

$$= -\frac{1}{\sqrt{2}} \tan^{-1}\left(\sqrt{\frac{\frac{1}{x^2}-1}{2}}\right) + C$$

$$= -\frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{\sqrt{1-x^2}}{\sqrt{2}x}\right) + C$$

Integrals of the form $\int \frac{dx}{(ax^2+bx+c)(\sqrt{px^2+qx+r})}$

where ax^2+bx+c neither perfect square nor can be resolved in to linear factors

Here we put $\sqrt{\frac{px^2+qx+r}{ax^2+bx+c}} = t$

(Q—h)

Q→ $\int_0^{\frac{\pi}{2}} \cos\theta \cdot \cos 2\theta \cdot \cos 4\theta \, d\theta$

$= \int_0^{\frac{\pi}{2}} \frac{\cos\theta}{2} \cdot \left(\cos 6\theta + \cos(2\theta)\right) d\theta$

$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \left\{\left(\cos\theta \cos 6\theta\right) + \left(\cos\theta \cdot \cos 2\theta\right)\right\} d\theta$

$= \frac{1}{4}\int_0^{\frac{\pi}{2}} \left(\cos 7\theta + \cos 5\theta + \cos 3\theta + \cos\theta\right) d\theta$

$= \frac{1}{4}\left[\frac{\sin 7\theta}{7} + \frac{\sin 5\theta}{5} + \frac{\sin 3\theta}{3} + \sin\theta\right]_0^{\frac{\pi}{2}}$

$$= \frac{1}{4}\left[\left\{-\frac{1}{7} + \frac{1}{5} - \frac{1}{3} + 1\right\} - \left\{0+0+0\right\}\right]$$

$$= \frac{1}{4} \times \frac{76}{105} = \frac{19}{105} \quad \text{(Ans)}$$

$$= \cancel{\frac{76}{420}}$$

Q-)

$$\int \left\{\sqrt{\tan x} + \sqrt{\cot x}\right\} dx$$

$$= \int \left(\frac{\sqrt{\sin x}}{\sqrt{\cos x}} + \frac{\sqrt{\cos x}}{\sqrt{\sin x}}\right) dx$$

$$= \int \frac{\sin x + \cos x}{\sqrt{\cos x \sin x}} \, dx$$

$$= \sqrt{2} \int \frac{\sin x + \cos x}{\sqrt{\sin 2x}} \, dx$$

$$= \sqrt{2} \int \frac{dt}{\sqrt{1-t^2}}$$

$$= \sqrt{2} \sin^{-1}(t) + C$$

Put

$\sin x - \cos x = t$

$\Rightarrow (\cos x + \sin x) = \frac{dt}{dx}$

$t^2 = 1 - \sin 2x$

$\Rightarrow \sin 2x = 1 - t^2$

$= \sqrt{2} \sin^{-1}(\sin x - \cos x) + C$

Properties of definite integrals

1. $\int_a^b f(x)\,dx$

$= \int_a^b f(t)\,dt$

<u>Proof</u> : Let $\frac{d}{dx} g(x) = f(x)$

$\therefore \int_a^b f(x)\,dx = g(x)\Big]_a^b$

$= g(b) - g(a)$

by fundamental theorem of Calculus.

Also $\frac{d}{dt} g(t) = f(t)$

$\therefore \int_a^b f(t)\,dt = g(t)\Big]_a^b$

$= g(b) - g(a)$

$\therefore \int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

Alternative method

$$\int_a^b f(x)\,dx$$

$$= \int_a^b f(t)\,dt$$

Put $x = t$

$dx = dt$

$x = a \Rightarrow t = a$

$x = b \Rightarrow t = b$

(2) $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

Proof :-

Let $\frac{d}{dx} g(x) = f(x)$

$$\therefore \int_a^b f(x)\,dx = g(x)\Big]_a^b$$

$$= g(b) - g(a)$$

(by fundamental theorem)

Also $$\int_b^a f(x)\,dx = g(x)\Big]_b^a$$

$$= g(a) - g(b)$$

$$\therefore \int_a^b f(x)\,dx = g(b) - g(a) = -\{g(a) - g(b)\} = -\int_b^a f(x)\,dx$$

(3) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

Proof :-

Let $\frac{d}{dx} g(x) = f(x)$

$\therefore \int_a^b f(x)\,dx = g(x)\Big]_a^b$

$= g(b) - g(a)$

by fundamental theorem

$R.H.S = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

$= g(x)\Big]_a^c + g(x)\Big]_c^b$

$= g(c) - g(a) + g(b) - g(c)$

$= g(b) - g(a)$

$= \int_a^b f(x)\,dx$

$= L.H.S$

(Proved)

4. V. imp.

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

<u>Proof</u>

$$\int_0^a f(x)\,dx$$

Put

$x = a - t$

$dx = -dt$

$x = 0 \Rightarrow t = a$

$x = a \Rightarrow t = 0$

$$= \int_a^0 f(a-t)(-dt)$$

$$= -\int_a^0 f(a-t)\,dt$$

$$= \int_0^a f(a-t)\,dt \quad \text{(by property 2)}$$

$$= \int_0^a f(a-x)\,dx \quad \text{(by property 1)}$$

(proved)

5. $$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx \quad \text{if } f \text{ is even function}$$

$$= 0 \quad \text{if } f \text{ is odd function.}$$

Proof :- Let

$$I = \int_{-a}^{a} f(x)\, dx$$

$$= \int_{-a}^{0} f(x)\, dx + \int_{0}^{a} f(x)\, dx$$

(by property 3).

In the first integral put $x = -t$

$\therefore \quad dx = -dt$

$x = -a \Rightarrow t = a$

$x = 0 \Rightarrow t = 0$

$$\therefore I = -\int_{a}^{0} f(-t)\, dt + \int_{0}^{a} f(x)\, dx$$

$$= \int_{0}^{a} f(-t)\, dt + \int_{0}^{a} f(x)\, dx$$

(By 2nd property).

$$= \int_{0}^{a} f(-x)\, dx + \int_{0}^{a} f(x)\, dx.$$

(by property 1)

If f is even then $f(-x) = f(x)$

~~odd~~

$\therefore I = \int_0^a f(x)\,dx + \int_0^a f(x)\,dx$

$= 2\int_0^a f(x)\,dx$ — (proved)

If f is odd then $f(-x) = -f(x)$

$\therefore I = \int_0^a -f(x)\,dx + \int_0^a f(x)\,dx$

$= -\int_0^a f(x)\,dx + \int_0^a f(x)\,dx$

$= 0$ (proved)

(6). $\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx$ if $f(2a-x) = f(x)$

$= 0$ if $f(2a-x) = -f(x)$

Proof :

Let $I = \int_0^{2a} f(x)\,dx$

$= \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx$ (by 3rd property)

In the second integral put $x = 2a - t$

$dx = -dt$

$x = a \Rightarrow t = a$

$x = 2a \Rightarrow t = 0$

$$\therefore I = \int_0^a f(x)\,dx + \int_a^0 f(2a-t)(-dt)$$

$$= \int_0^a f(x)\,dx - \int_a^0 f(2a-t)\,dt$$

$$= \int_0^a f(x)\,dx + \int_0^a f(2a-t)\,dt$$

(by property 2)

$$= \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$$

(by property 1)

If $f(2a-x) = f(x)$, then

$$I = 2\int_0^a f(x)\,dx$$

If $f(2a-x) = -f(x)$, then

$I = 0$ (Proved)

Examples

(12-j) — 1. (ii) Imp. III (U) L.U.S.E. L6

Solⁿ →

Let $I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx$

$= \int_0^{\pi/2} \frac{\sqrt{\sin(\frac{\pi}{2}-x)}}{\sqrt{\sin(\frac{\pi}{2}-x)} + \sqrt{\cos(\frac{\pi}{2}-x)}}\,dx$ (by property 4)

$= \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}}\,dx$

$\therefore \quad 2I = I + I$

$= \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx + \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx$

$= \int_0^{\pi/2} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx$

$= \int_0^{\frac{\pi}{2}} 1 \cdot dx = \left. x \right]_0^{\frac{\pi}{2}}$

$= \frac{\pi}{2} - 0$

$= \frac{\pi}{2}$

(ii) $I = \frac{\pi}{4}$ (Ans)

$\Rightarrow \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx = \frac{\pi}{4}$

Higher Secondary Vol-II, last properties of Definite integral

Prove that

$\int_0^{\pi/2} \frac{dx}{1+\tan^3 x} = \frac{\pi}{4}$

Proof Let $I = \int_0^{\pi/2} \frac{dx}{1+\tan^3 x}$

$= \int_0^{\pi/2} \frac{\cos^3 x \, dx}{\sin^3 x + \cos^3 x}$

$= \int_0^{\pi/2} \frac{\cos^3(\frac{\pi}{2} - x)}{\sin^3(\frac{\pi}{2} - x) + \cos^3(\frac{\pi}{2} - x)} dx$

$= \int_0^{\pi/2} \frac{\sin^3 x}{\cos^3 x + \sin^3 x} dx$

$\therefore \; 2I = I + I$

$$= \int_0^{\pi/2} \frac{\cos^3 x \, dx}{\sin^3 x + \cos^3 x} + \int_0^{\pi/2} \frac{\sin^3 x}{\sin^3 x + \cos^3 x} dx$$

$$= \int_0^{\pi/2} 1 \cdot dx$$

$$= [x]_0^{\pi/2}$$

$$= \frac{\pi}{2} - 0$$

$$\Rightarrow 2I = \frac{\pi}{2}$$

$$\Rightarrow I = \frac{\pi}{4}$$ (Proved)

$$\Rightarrow \int_0^{\pi/2} \frac{dx}{1+\tan^3 x} = \frac{\pi}{4}$$

Q → V. good Prove that $\int_0^{\pi/2} \ln \sin x \, dx = -\frac{\pi}{2} \ln 2$

Proof Let $I = \int_0^{\pi/2} \ln \sin x \, dx$

$$= \int_0^{\pi/2} \ln \sin\left(\frac{\pi}{2} - x\right) dx \quad \text{(by property 4)}$$

$$= \int_0^{\frac{\pi}{2}} \ln \cos x \, dx$$

$\therefore$ $\underline{2I}$

$$= I + I$$

$$= \int_0^{\frac{\pi}{2}} \ln \sin x \, dx + \int_0^{\frac{\pi}{2}} \ln \cos x \, dx$$

$$= \int_0^{\frac{\pi}{2}} (\ln \sin x + \ln \cos x) \, dx$$

$$= \int_0^{\frac{\pi}{2}} \ln \sin x . \cos x \, dx$$

$$= \int_0^{\pi/2} \ln \frac{\sin 2x}{2} \, dx$$

$$= \int_0^{\pi/2} \ln \sin 2x \, dx - \int_0^{\frac{\pi}{2}} \ln 2 \, dx$$

$$2I = \int_0^{\frac{\pi}{2}} \ln \sin 2x \, dx - \frac{\pi}{2} . \ln 2 \quad \text{——— (i)}$$

Consider $\int_0^{\frac{\pi}{2}} \ln \sin 2x \, dx$

Put
$2x = t$
$dx = \frac{dt}{2}$
$x=0 \Rightarrow t=0$
$x=\frac{\pi}{2} \Rightarrow t=\pi$

$$= \int_0^{\pi} \ln \sin t \, \frac{dt}{2}$$

$$= \frac{1}{2} \int_0^{\pi} \ln \sin t \, dt$$

$$= \frac{1}{2} \cdot 2 \int_0^{\pi/2} \ln \sin t \, dt \quad \text{(by Property 6)}$$

$$= \int_0^{\frac{\pi}{2}} \ln \sin t \, dt$$

$$= \int_0^{\frac{\pi}{2}} \ln \sin x \, dx \quad \text{(by property 1)}$$

$$= I$$

From eqn (i),

$$2I = I - \frac{\pi}{2} \ln 2$$

$$\Rightarrow I = -\frac{\pi}{2} \ln 2$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \ln \sin x \, dx = -\frac{\pi}{2} \ln 2 \quad \text{(Proved)}$$

12-(j)

1.(iii) Let→ $I = \int_0^1 \frac{\log(1+x)}{1+x^2}\,dx$

Put
$x = \tan\theta$
$dx = \sec^2\theta\,d\theta$
$x = 0 \Rightarrow \theta = 0$
$x = 1 \Rightarrow \theta = \frac{\pi}{4}$

$= \int_0^{\pi/4} \frac{\log(1+\tan\theta)}{(\sec^2\theta)} \sec^2\theta\,d\theta$

$I = \int_0^{\pi/4} \log(1+\tan\theta)\,d\theta$

$= \int_0^{\pi/4} \log\left(1+\tan\left(\frac{\pi}{4}-\theta\right)\right) d\theta$ (By pr - 4)

$= \int_0^{\pi/4} \log\left(1+\frac{1-\tan\theta}{1+\tan\theta}\right) d\theta$

$= \int_0^{\pi/4} \log\left(\frac{2}{1+\tan\theta}\right) d\theta$

$= \int_0^{\pi/4} \log 2\,d\theta - \int_0^{\pi/4} \log(1+\tan\theta)\,d\theta$

$= \frac{\pi}{4}\log 2 - I$

$\Rightarrow 2I = \frac{\pi}{4}\log 2$

$\Rightarrow I = \frac{\pi}{8}\log 2$ (Ans)

Gamma function

Capital → Γ
Small – γ

Gamma function is denoted by

Γx or $\Gamma(x)$.

① Now $\Gamma(n+1) = n!$ when $n = 0, 1, 2, \ldots$

$\Gamma 1 = 0! = 1$

$\Gamma 2 = 1! = 1$

$\Gamma 3 = 2! = 1 \times 2 = 2$

$\Gamma 4 = 3! = 1 \times 2 \times 3 = 6$

derived

② $\Gamma(n+1) = n\,\Gamma n$ ← (General formula)

$\Gamma \frac{1}{2} = \sqrt{\pi}$

$\Gamma \frac{5}{2} = \frac{3}{2}\Gamma\frac{3}{2} = \frac{3}{2}\,\frac{1}{2}\,\Gamma\frac{1}{2} = \frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}$

$= \frac{3\sqrt{\pi}}{4}$

$\Gamma \frac{7}{2} = \frac{5}{2}\Gamma\frac{5}{2} = \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2} \times \sqrt{\pi}$

$= \frac{15}{8}\sqrt{\pi}$

$$\int_0^{\frac{\pi}{2}} \sin^p\theta \cdot \cos^q\theta \, d\theta$$

$$= \frac{\Gamma\left(\frac{p+1}{2}\right) \Gamma\left(\frac{q+1}{2}\right)}{2\,\Gamma\left(\frac{p+q+2}{2}\right)}$$

Ex:

$$\int_0^{\frac{\pi}{2}} \sin^4\theta \cos^6\theta \, d\theta$$

$$= \frac{\Gamma\left(\frac{5}{2}\right) \cdot \Gamma\left(\frac{7}{2}\right)}{2 \cdot \Gamma(6)}$$

$$= \frac{\frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \sqrt{\pi}}{2 \cdot 5!}$$

$$= \frac{3 \times \pi \times 5 \times 3}{32 \times 2 \times 1 \times 2 \times 3 \times 4 \times 5}$$

$$= \frac{3\pi}{512} \quad \text{(Ans)}$$

$$\int_0^{\frac{\pi}{2}} \cos^9 x \, dx$$

$$= \int_0^{\frac{\pi}{2}} \sin^0 x \cdot \cos^9 x \, dx$$

$$= \frac{\Gamma\frac{1}{2} \cdot \Gamma 5}{2\,\Gamma\frac{11}{2}}$$

$$= \frac{\sqrt{\pi} \cdot 1 \times 2 \times 3 \times 4}{2 \times \frac{9}{2} \times \frac{7}{2} \times \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2}\sqrt{\pi}}$$

$$= \frac{16 \times 3 \times 4 \times 2}{9 \times 7 \times 5 \times 3}$$

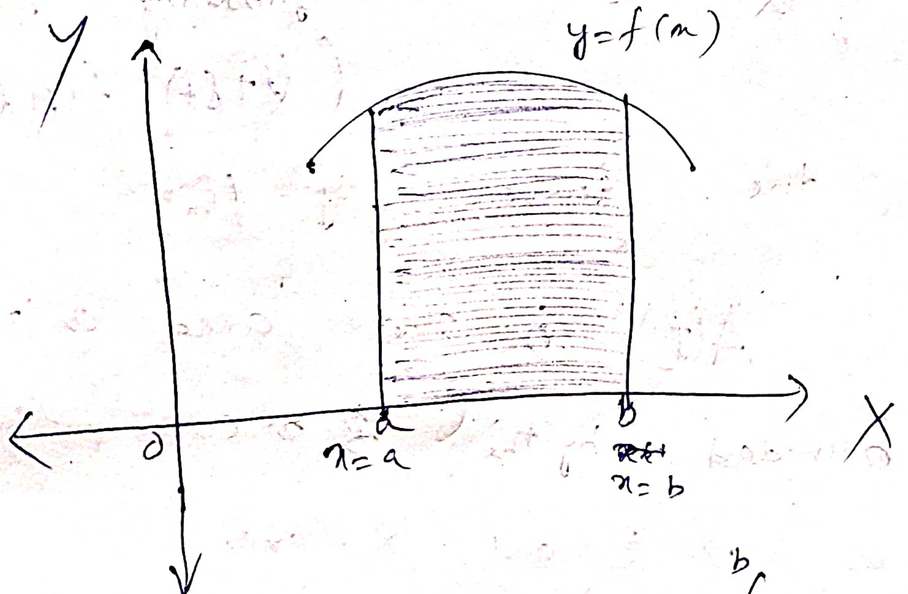
$$= \frac{128}{315} \text{ (Ans)}$$

Area

Quadrature

The process of finding the area is called quadrature.

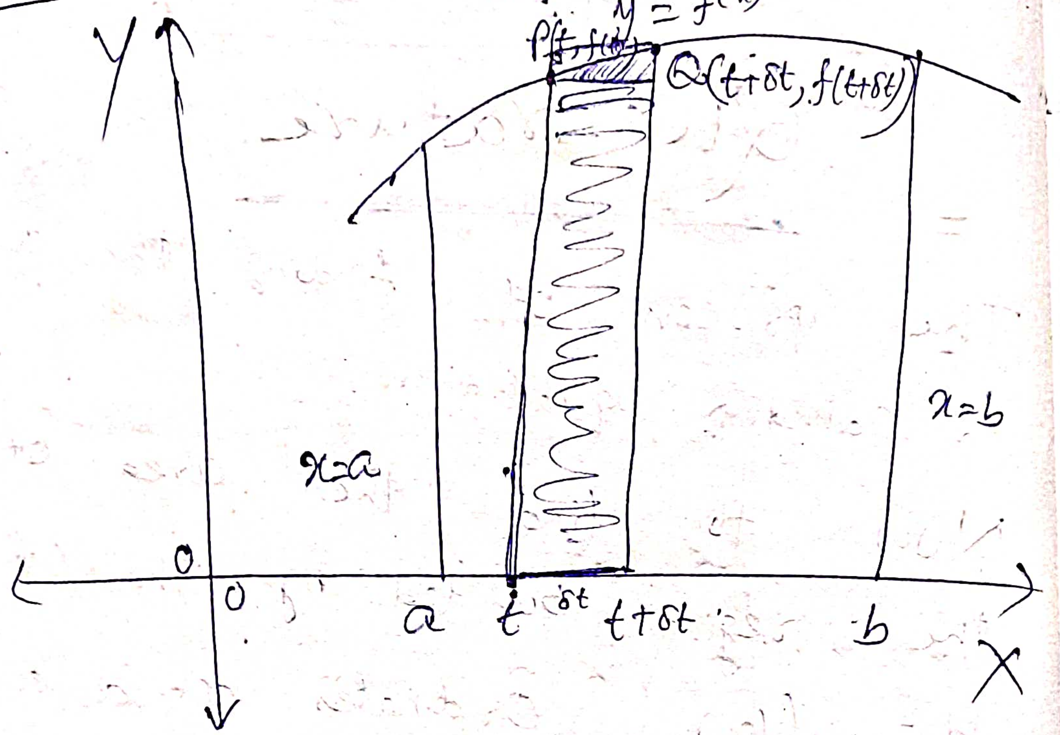
Now to find the area of the region bounded by the curve $y = f(x)$, ordinates $x = a$ and $x = b$ and X-axis ($y = 0$)



The required area $= \int_a^b y\,dx$

$= \int_a^b f(x)\,dx$

Proof →



Let P be the point $(t, f(t))$

and Q be its neighbouring point $((t+\delta t), f(t+\delta t))$

on the curve $y = f(x)$

Let $A(t)$ be the area or region bounded by the curve $y=f(x)$

$x=a$, $x=t$ and X - axis.

Now area of the bigger rectangle

$= f(t+\delta t) . \delta t$

Area of the smaller rectangle

$= f(t) . \delta t$

Now $A(t+\delta t)$

= Area of the region bounded by the curve $y=f(x)$, $x=a$, $x=t+\delta t$ and x-axis

$\therefore$ Area of the region bounded by the curve $y=f(x)$, $x=t$, $x=t+\delta t$ and x-axis

$= A(t+\delta t) - A(t)$

$= \delta A$

$\therefore f(t).\delta t \leq \delta A \leq f(t+\delta t).\delta t$

$\Rightarrow f(t) \leq \frac{\delta A}{\delta t} \leq f(t+\delta t)$

But $\lim_{\delta t \to 0} f(t) = f(t)$

$\lim_{\delta t \to 0} f(t+\delta t) = f(t)$

$\therefore \lim_{\delta t \to 0} \frac{\delta A}{\delta t} = f(t)$

(by Sandwich theorem.)

$$\Rightarrow \quad \frac{dA}{dt} = f(t)$$

$$\Rightarrow \quad dA = f(t)\,.\,dt$$

$$\Rightarrow \quad A(t) = \int_a^t f(t)\,.\,dt$$

Taking $t = b$ we get the area of the region bounded by $y = f(x)$, $x = a$, $x = b$, and x-axis

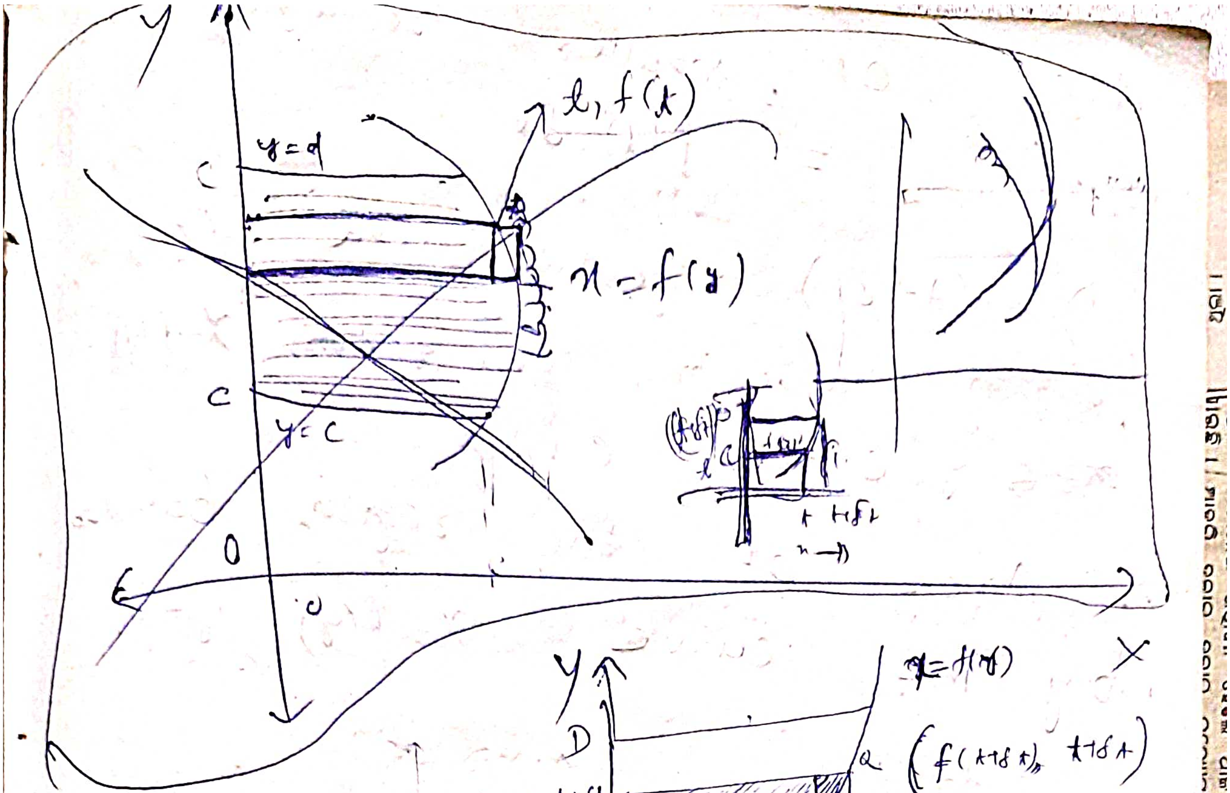
$$= \int_a^b f(t)\, dt$$

$$= \int_a^b f(x)\, dx \quad \text{(by property 1)}$$

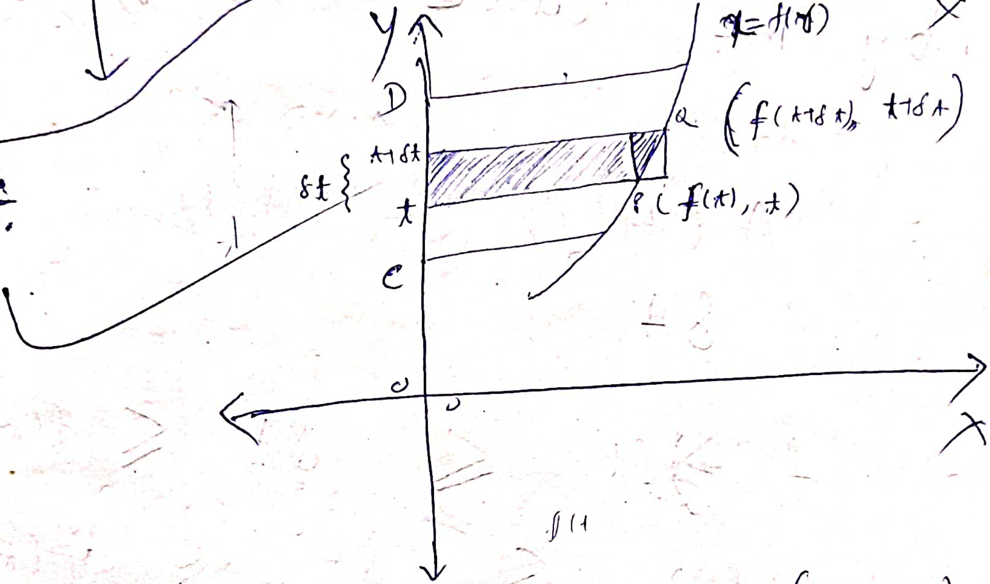
$$= \int_a^b y\, dx$$

Imp Notes :

1. The area of the region bounded by $x = f(y)$, $y = c$, $y = d$ and Y-axis $= \int_c^d x\, dy = \int_c^d f(y)\, dy$



Proof :



Let P be the point $(f(t), t)$ and Q be its neighbouring point $(f(t+\delta t), (t+\delta t))$

on the curve $x = f(y)$

Let $A(t)$ be area of region bounded by the curve $x = f(y)$, $y = c$, $y = t$ and Y-axis.

Now area of bigger rectangle

$= f(t+\delta t) . \delta t$

Area of smaller rectangle

$= f(t).\delta t$

Now $(A+\delta t)$ be the area of region bounded by curve ~~y=f(x)~~ $x=f(y)$ ~~axis~~ $y=c$ & $y=c+\delta t$

∴ Area of the region bounded by the curve $x=f(y)$, $y=t$; $y=t+\delta t$

$= A(t+\delta t) - A(t)$

δA

$=$

∴ $f(t).\delta t \leq \delta A \leq f(t+\delta t).\delta t$

$\Rightarrow f(t) \leq \frac{\delta A}{\delta t} \leq f(t+\delta t)$

But $\lim_{\delta t \to 0} f(t) = f(t)$

$\lim_{\delta t \to 0} f(t+\delta t) = f(t)$

$\lim_{\delta t \to 0} \frac{\delta A}{\delta t} = f(t)$ by Sandwich theorem.

$\Rightarrow \frac{dA}{dt} = f(t)$

$\Rightarrow dA = f(t)\, dt$

$\Rightarrow$ $A(t) = \int_c^t f(t)\,dt$

Taking $t = d$, we get the area of the region bounded by $x = f(y)$, $y = c$, $y = d$ and y-axis

$= \int_c^d f(t)\,dt$

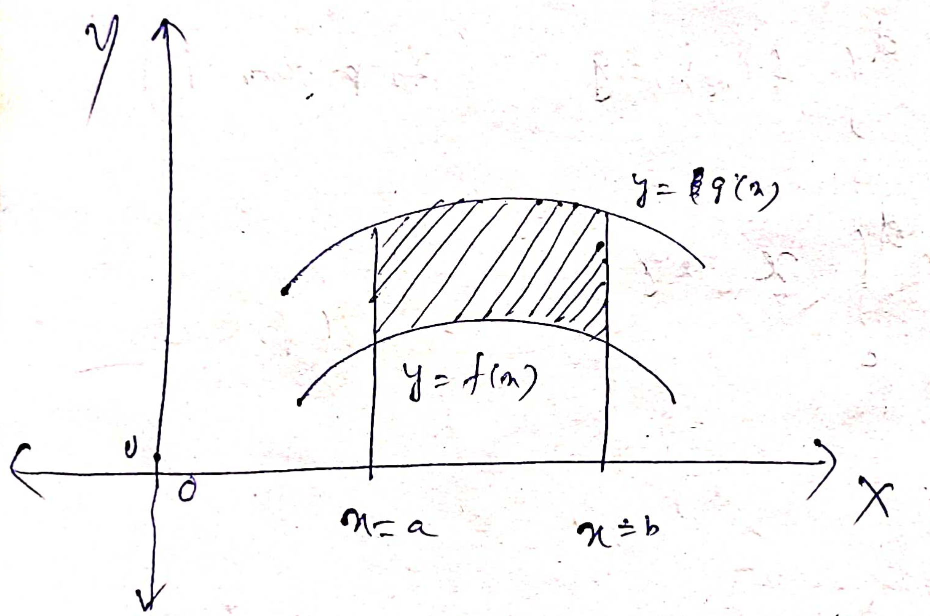
$= \int_c^d f(y).dy$ (By property 1)

$= \int_c^d x\,dy$

(Proved)

Note :- 2

The area of the region bounded by $y = f(x)$, $y = g(x)$, $x = a$ and $x = b$ is given by $\int_a^b (g(x) - f(x))\,dx$



3. The lower limit of integration should be taken as the smaller value of the independent variable while the greater value gives us the upper limit of integration.

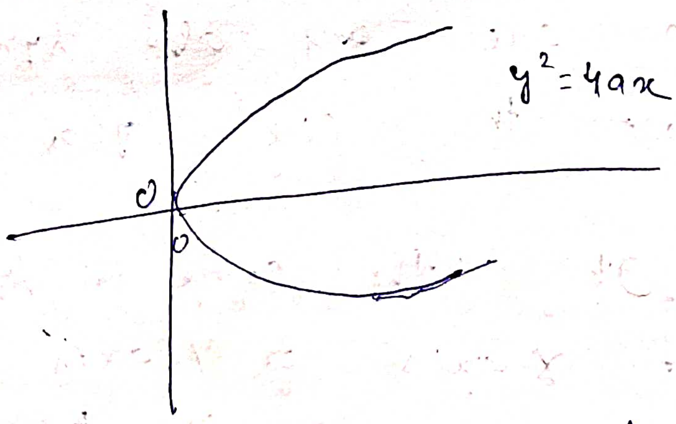
4. If we put $-x$ in place of x in the eqn of curve and the

eqn of curve remains unchanged, then the curve is symmetrical about y-axis

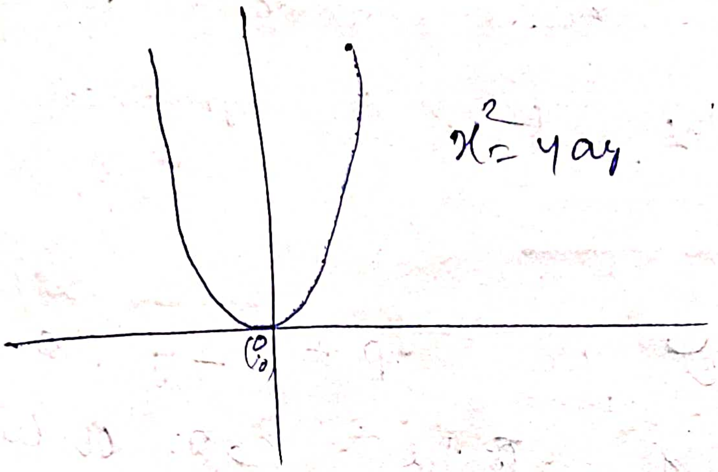
If we put $-y$ in place y in the eqn of curve and the eqn of curve remains unchanged then the curve is symmetrical about x-axis.

If we interchange x and y in the eqn of the curve and the eqn remains unchanged than the curve is symmetrical about the line $y = x$

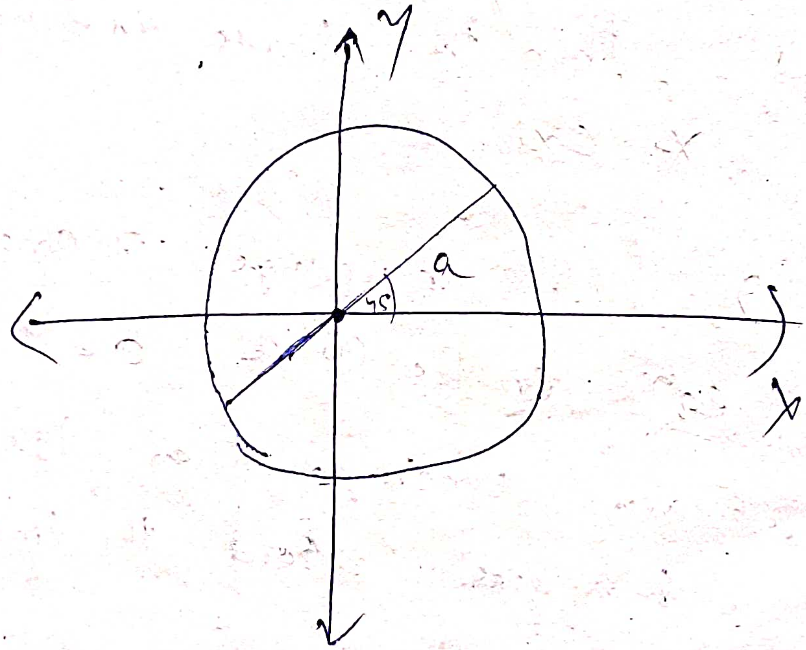
Ex:-



$y^2 = 4ax$ is symmetrical about x-axis



$x^2 = 4ay$ is symmetrical about y-axis



$x^2 + y^2 = a^2$ is symmetrical about both axes and also symmetrical about the line $y = x$

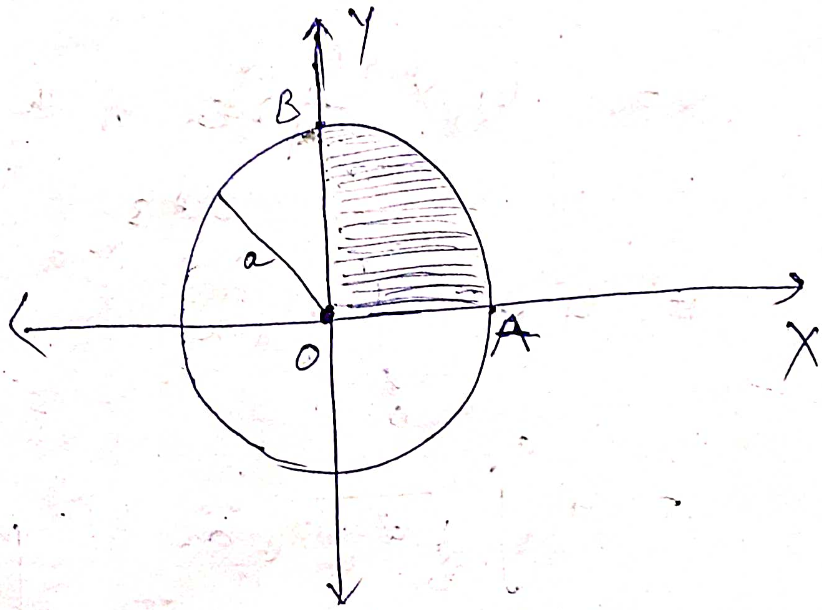
5. If the curve is symmetrical about x-axis or y-axis or both then we shall find the area of 1 symmetrical part and multiply it by the number of symmetrical parts to

get the whole area.

Problem

1) Find the area of a circle of radius 'a'.

Ans :-



Consider a circle with centre at (0,0) and radius 'a'.

Eqn of this circle is

$$x^2+y^2=a^2$$

It is symmetrical about both axes.

So consider the portion OAB in the first quadrant. For this portion

$$y^2=a^2-x^2$$

$$\Rightarrow y=\sqrt{a^2-x^2}$$

Now A is the point $(a, 0)$

For this position x varies from 0 to a.

$\therefore$ Area of the circle

$= 4 \times$ Area of the portion OAB

$= 4 \times \int_0^a y \, dx$

$= 4 \int_0^a \sqrt{a^2 - x^2} \, dx$

$= 4 \int_0^{\frac{\pi}{2}} a \cos\theta \cdot a \cos\theta \, d\theta$

$= 4a^2 \int_0^{\frac{\pi}{2}} \cos^2\theta \, d\theta$

Put

$x = a \sin\theta$

$dx = a \cos\theta \, d\theta$

$x = 0 \Rightarrow \theta = 0$

$x = a \Rightarrow \theta = \frac{\pi}{2}$

$\sqrt{a^2 - x^2} = a \cos\theta$

$= 4a^2 \frac{\Gamma\frac{0+1}{2} \cdot \Gamma\frac{2+1}{2}}{2 \cdot \Gamma\frac{2+2}{2}}$

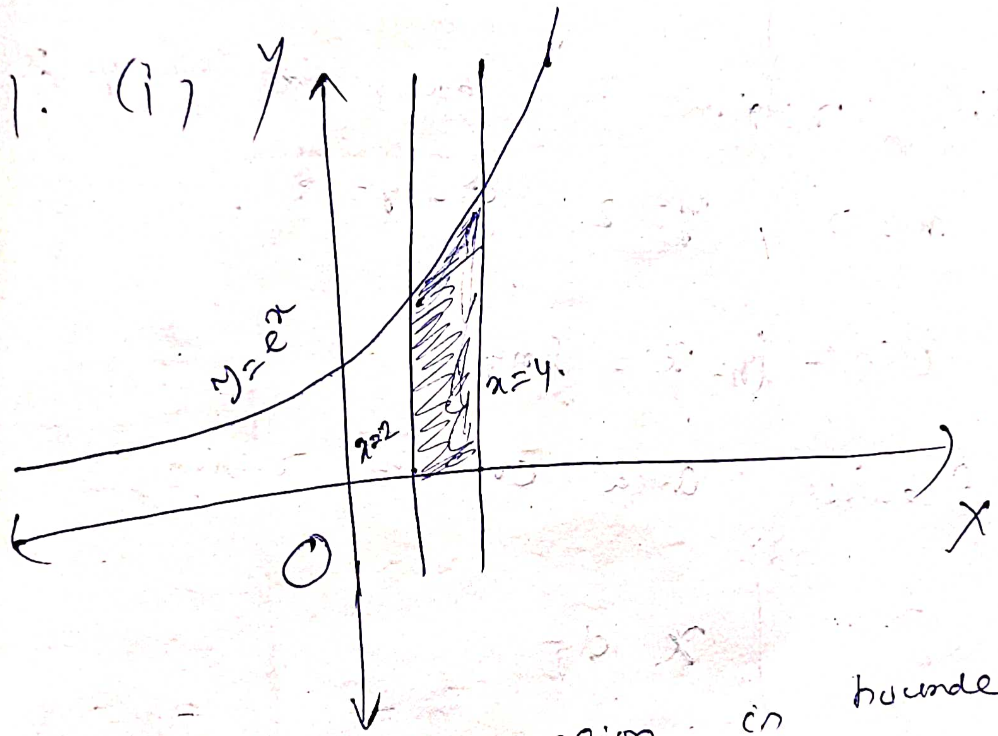
$= \dfrac{\frac{1}{2}a^2\sqrt{\pi} \cdot \frac{1}{2}\sqrt{\pi}}{1 \times \frac{1}{4}}$

$= \pi a^2$ (Ans)

— 0 —

Exercise - 11(K)

1. (i)



Here the region is bounded by $y = e^x$, $y = 0$, $x = 2$ and $x = 4$

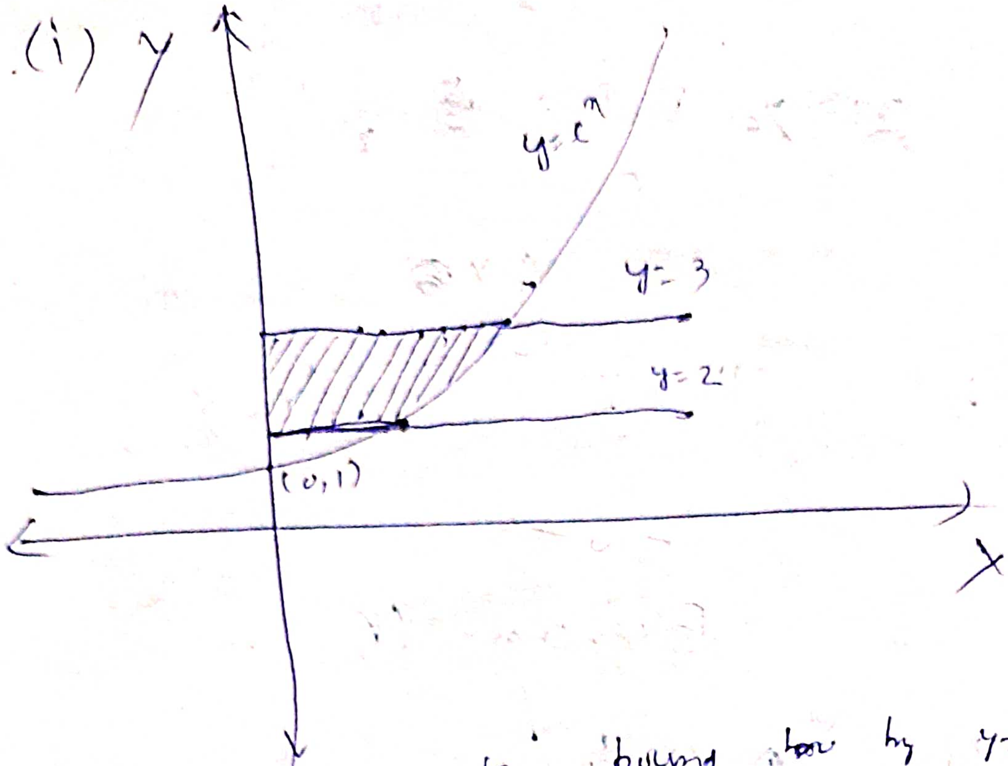
Area of the region

$= \int_2^4 y \, dx$

$= \int_2^4 e^x \, dx = \left[e^x \right]_2^4$

$= (e^4 - e^2)$ (Ans)

2.(i)



The region is bound by $y = e^x$

$x = 0$, $y = 2$ and $y = 3$.

Here $f(y) = x = \ln y$

Required area of the region

$$= \int_2^3 x \, dy$$

$$= \int_2^3 \ln y \, dy$$

$$= \int_2^3 \ln y \cdot 1 \, dy$$

$$= \left[\ln y \cdot y - \int \frac{1}{y} \cdot y \, dy \right]_2^3$$

$$= \left[y \ln y - y \right]_2^3$$

$$= (3\ln 3 - 3) - (2\ln 2 - 2)$$

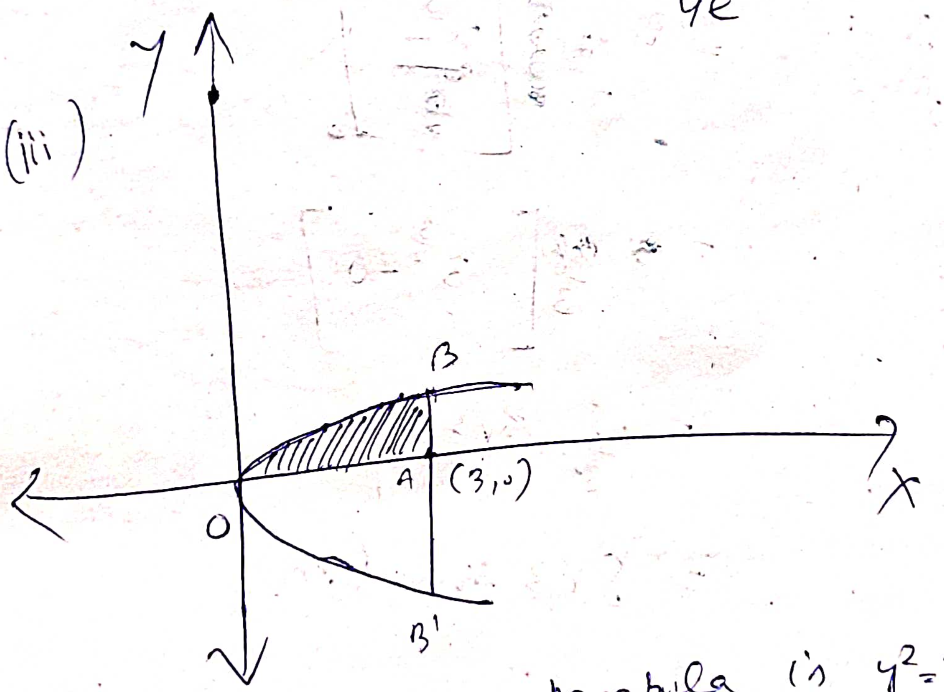
$$= 3\ln 3 - 2\ln 2 - 1$$

$$= \ln 27 - \ln 4 - 1$$

$$= \ln \frac{27}{4} - 1 = \ln \frac{27}{4} - \ln e$$

$$= \ln \frac{27}{4e}$$

3. (iii)



The eqn of parabola is $y^2 = 4x$. It is right handed parabola. It is symmetrical about x-axis.

Let BB' be the double ordinate passing through $A(3,0)$.

$\therefore$ Area bounded by parabola and double ordinate $= 2 \times$ Area of OAB

$$= 2 \times \int_0^3 y \, dx$$

($\because$ For the portion OAB x varies from 0 to 3)

$$= 2 \times \int_0^3 2\sqrt{x} \, dx$$

$$= 4 \int_0^3 (x)^{\frac{1}{2}} dx = 4 \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}} \right]_0^3$$

$$= \frac{4 \times 2}{3} \left[x^{\frac{3}{2}} \right]_0^3$$

$$= \frac{8}{3} \left[3^{\frac{3}{2}} - 0 \right]$$

$$= \frac{8}{3} 3\sqrt{3}$$

$$= 8\sqrt{3} \text{ sq units. (Ans) 2}$$

3. (ii) The eqn of circle is

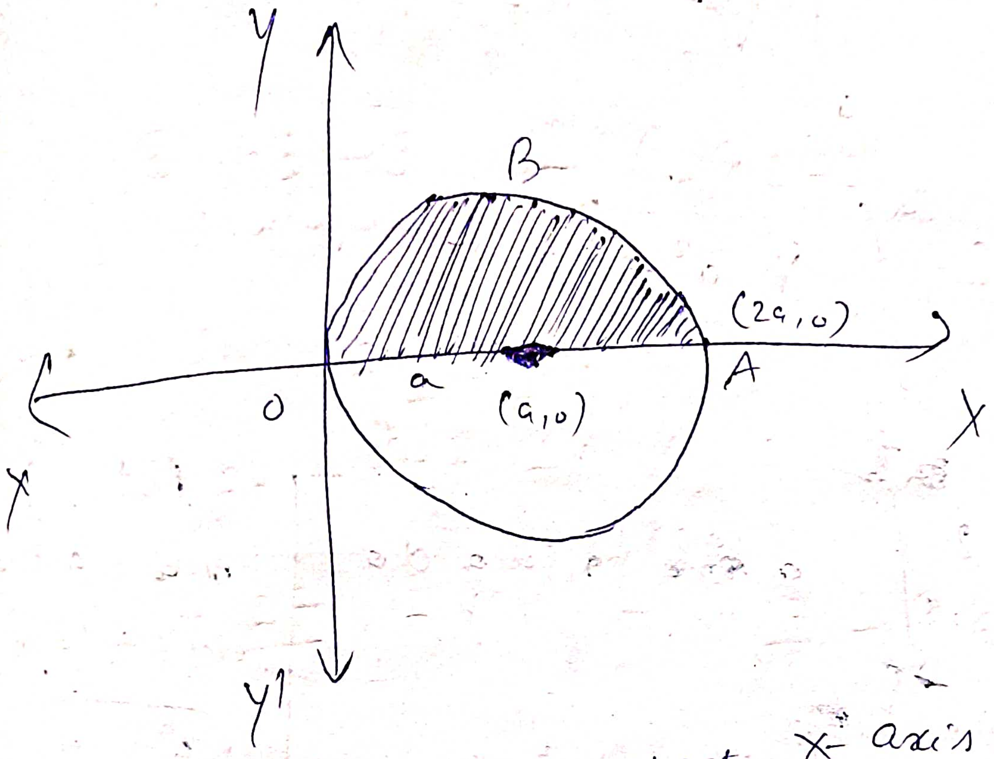
$$x^2 + y^2 = 2ax$$

$$\Rightarrow x^2 - 2ax + y^2 = 0$$

$$\Rightarrow x^2 - 2ax + a^2 + y^2 = a^2$$

$$\Rightarrow (x-a)^2 + (y-0)^2 = a^2$$

Centre is $(a, 0)$ and radius is a.



It is symmetrical about x-axis, and not symmetrical about y-axis.

Hence consider the upper half OAB. For this portion x varies from 0 to $2a$.

Here $y^2 = 2ax - x^2$

$\Rightarrow y = \sqrt{2ax - x^2}$

Required Area $= 2 \times$ Area of OAB

$$= 2 \times \int_0^{2a} y \, dx$$

$$= 2 \times \int_0^{2a} \sqrt{2ax - x^2} \, dx$$

$$= 2 \int_0^{2a} \sqrt{-(x^2 + a^2 - 2ax - a^2)}\, dx$$

$$= 2 \int_0^{2a} \sqrt{a^2 - (x-a)^2}\, dx$$

$$= 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a \cos\theta \cdot a \cos\theta \, d\theta$$

Put

$x - a = a \sin\theta$

$dx = a \cos\theta \, d\theta$

$\sqrt{a^2 - (x-a)^2} = a \cos\theta$

$x = 0 \Rightarrow \theta = -\frac{\pi}{2}$

$x = 2a \Rightarrow \theta = \frac{\pi}{2}$

$$= 2a^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta \, d\theta$$

$$= 4a^2 \int_0^{\frac{\pi}{2}} \cos^2\theta \, d\theta$$

(by property 5)

$$= 4a^2 \left\{ \frac{\Gamma\frac{1}{2} \cdot \Gamma\frac{3}{2}}{2 \cdot \Gamma 2} \right\}$$

$$= 4a^2 \left\{ \frac{\sqrt{\pi} \cdot \frac{1}{2} \cdot \sqrt{\pi}}{2 \cdot 1} \right\}$$

$$= \pi a^2$$

(Ans)

4.(i) The eqn of circle is

Another process in problem sheet

$$x^2 + y^2 = 4$$

Its Centre is $(0,0)$, radius $= 2$

Here $y = \pm\sqrt{4 - x^2}$

The eqn of line is

$$x + \sqrt{3}y = 2$$

$$\Rightarrow y = \frac{2 - x}{\sqrt{3}}$$

The intersection of the circle and the straight line by solving both eqns. $x^2 + y^2 = 4$

and $y = \frac{2-x}{\sqrt{3}}$

$$\therefore x^2 + \left(\frac{2-x}{\sqrt{3}}\right)^2 = 4$$

$$\Rightarrow x^2 + \frac{(2-x)^2}{3} = 4$$

$$\Rightarrow \frac{3x^2 + (4 + x^2 - 4x)}{3} = 4$$

$$\Rightarrow 4x^2 - 4x + 4 = 12$$

$$\Rightarrow x^2 - x + 1 = 3$$

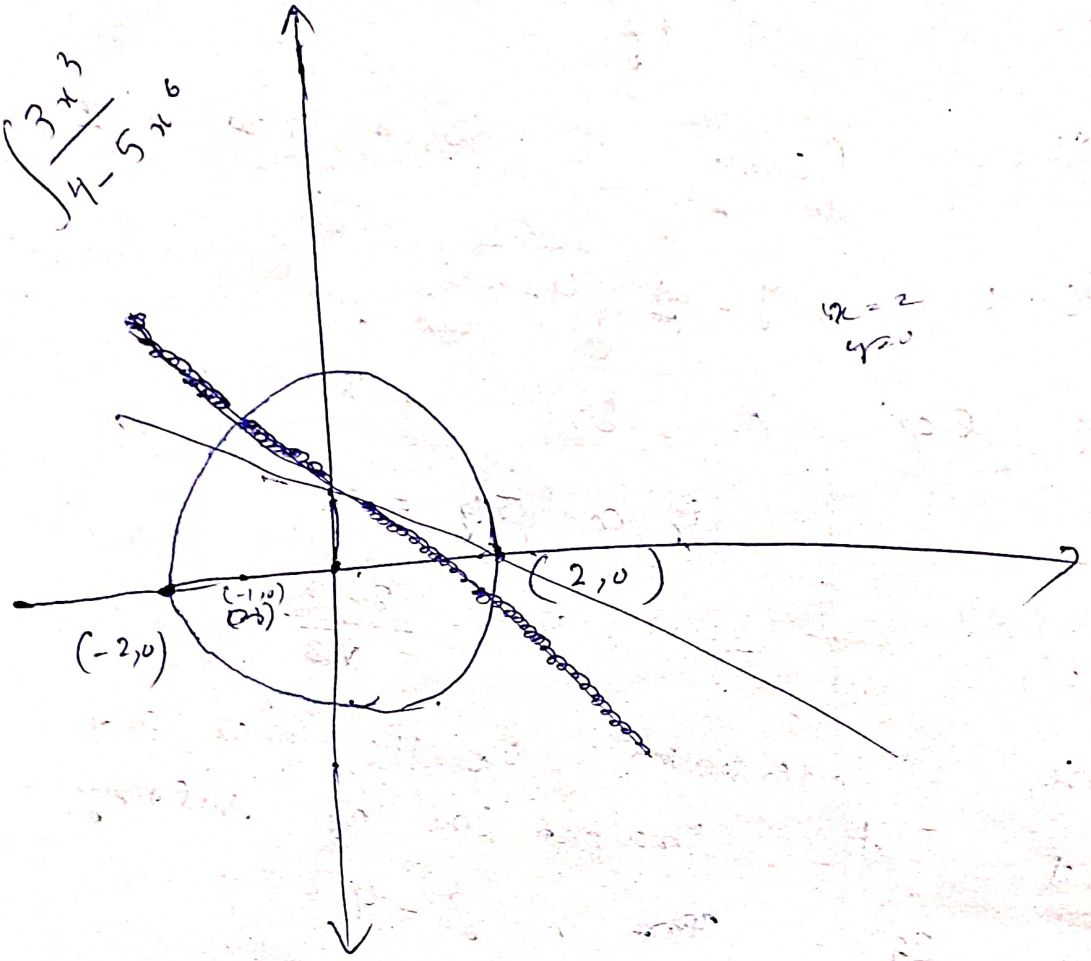
$$\Rightarrow x^2 - x - 2 = 0 \quad \Rightarrow x^2 - 2x + x - 2 = 0$$

$$\Rightarrow x(x-2) + 1(x-2) = 0$$

$$\Rightarrow x = -1, 2 \longleftarrow \Rightarrow (x-2)(x+1) = 0$$

∴ x varies from -1 to 2

$\int \frac{3x^3}{4-5x^6}$



∴ Area between straight line and circle lying above the line

$$= \int_{-1}^{2} \left[\sqrt{4-x^2} - \frac{2-x}{\sqrt{3}} \right] dx$$

$$= \int_{-1}^{2} \sqrt{4-x^2}\, dx - \frac{1}{\sqrt{3}} \int_{-1}^{2} (2-x)\, dx$$

$$= \int_{-1}^{2} \sqrt{(2)^2 - x^2}\, dx \quad - \frac{1}{\sqrt{3}} \left[2x - \frac{x^2}{2} \right]_{-1}^{2}$$

$$= \left[\frac{4}{2}\sin^{-1}\frac{x}{2} + \frac{x}{2}\sqrt{4-x^2}\right]_{-1}^{2} - \frac{1}{\sqrt{3}\times 2}\left[4x - x^2\right]_{-1}^{2}$$

$$= \left\{\left(2\sin^{-1}1 + \sqrt{4-4}\right) - \left(\frac{4}{2}\sin^{-1}\frac{-1}{2} - \frac{1}{2}\sqrt{4-1}\right)\right\} -$$

$$\frac{1}{2\sqrt{3}}\left\{\left(4\right) - \left(-4-1\right)\right\}$$

$$= \left\{\pi + \frac{\pi}{3} + \frac{\sqrt{3}}{2}\right\} - \left\{\frac{9}{2\sqrt{3}}\right\}$$

$$= \frac{4\pi}{3} - \sqrt{3}$$

Total area of the circle

$$= 4\pi$$

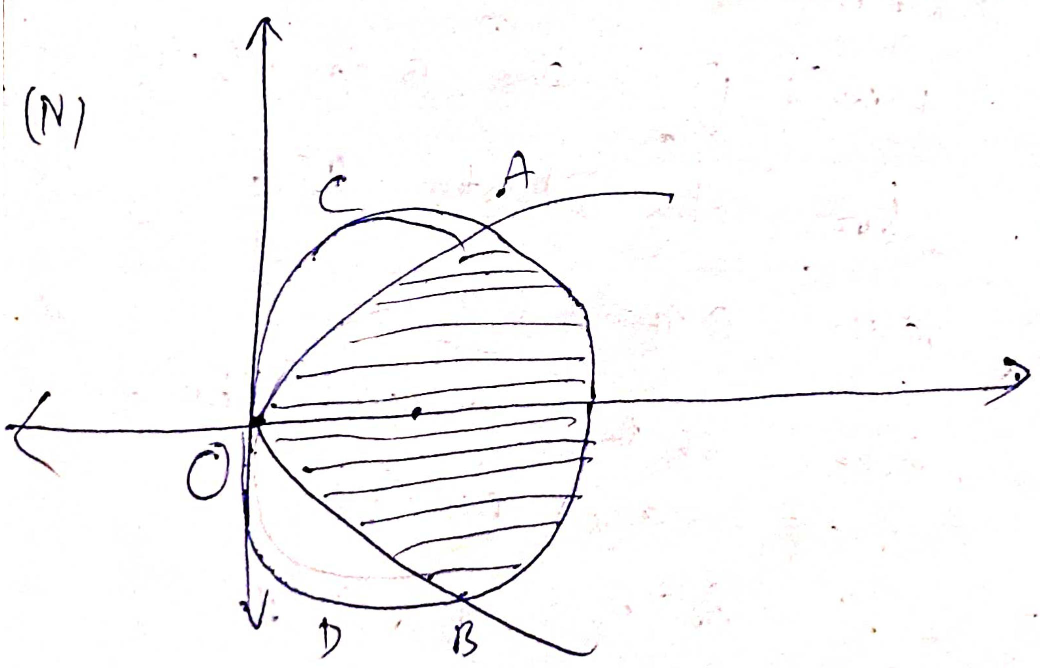
Area between straight line and circle lying below the line

$$= 4\pi - \left(\frac{4\pi}{3} - \sqrt{3}\right)$$

$$= \frac{12\pi - 4\pi + 3\sqrt{3}}{3}$$

$$= \frac{8\pi}{3} + \sqrt{3}$$

(N)



For finding the points of intersection of circle and parabola i.e A and B, we have to solve both eq^{n}s $x^2+y^2=2x$ and $y^2=x$.

$\therefore x^2+x=2x$

$\Rightarrow x^2-x=0$

$\Rightarrow x(x-1)=0$

$\Rightarrow x=0$ and $x=1$

If $x=0$, $y=0$ $\therefore (0,0)$ is a point of intersection.

$x=1$, $y^2=1$ $\Rightarrow y=\pm 1$

$\therefore (1,1)$ and $(1,-1)$ are points of intersection.

A is $(1,1)$ and B is $(1,-1)$

$\therefore$ Here for the portion OCA

x varies from $(0,1)$

$\therefore$ Area of the region bounded by circle and parabola :

$= 2 \times$ Area of OCA

$$= 2 \times \int_0^1 \left(\sqrt{2x - x^2} - \sqrt{x} \right) dx$$

$$\left(\because \text{ for Circle } y = \sqrt{2x - x^2} \quad \text{for parabola } y = \sqrt{x} \right)$$

$$= 2 \times \left[\int_0^1 \sqrt{2x - x^2}\, dx - \int_0^1 \sqrt{x}\, dx \right]$$

$$= 2 \times \left[\int_0^1 \sqrt{1 - (x-1)^2} - \left[\frac{x^{3/2}}{3/2} \right]_0^1 \right]$$

$$= 2 \times \left[\int_{-\pi/2}^{0} \cos\theta \cdot \cos\theta \, d\theta - \frac{2}{3} \right]$$

$$= 2 \times \left[\int_{-\pi/2}^{0} \frac{1 + \cos 2\theta}{2} d\theta - \frac{2}{3} \right]$$

Put
$x - 1 = \sin\theta$
$dx = \cos\theta \, d\theta$
$\sqrt{1 - (x-1)^2} = \cos\theta$
$x = 0 \Rightarrow \theta = -\frac{\pi}{2}$
$x = 1 \Rightarrow \theta = 0$

$$= 2 \left[\frac{1}{2} \cdot \int_{-\pi/2}^{0} 1 + \cos 2\theta \, d\theta - \frac{2}{3} \right]$$

$$= 2 \left[\frac{1}{2} \left[\left(\theta + \frac{\sin 2\theta}{2} \right) \right]_{-\pi/2}^{0} - \frac{2}{3} \right]$$

$$= 2\left[\frac{1}{2}\cdot\frac{\pi}{2} - \frac{2}{3}\right]$$

$$= 2\left[\frac{\pi}{4} - \frac{2}{3}\right]$$

$$= \frac{\pi}{2} - \frac{4}{3}.$$

Area of the circle $= \pi(1)^2$

$= \pi$

Area common to parabola and circle

$$= \pi - \left(\frac{\pi}{2} - \frac{4}{3}\right)$$

Ans

$$= \pi - \frac{\pi}{2} + \frac{4}{3} = \frac{4}{3} + \frac{\pi}{2} \text{ sq unit}$$

$$= \frac{6\pi - 3\pi + 8}{6}$$

$$= \frac{3\pi + 8}{6} \text{ sq unit}$$

Differential eqⁿ

A differential eqⁿ is an eqⁿ involving derivatives. Differential eqⁿs are mainly classified into two types.

(i) Ordinary differential eqⁿ.
(ii) Partial differential eqⁿ.

The eqⁿ which contains the ordinary derivatives is called ordinary differential eqⁿ.

The eqⁿ which contains partial derivatives is called partial differential eqⁿ.

We will discuss only ordinary differential eqⁿ.

Order and degree of differential eqⁿ

The highest order of the derivative occured in the eqⁿ is called order of differential eqⁿ.

The highest power of the highest order of derivative provided it is free of radicals and fractional powers is called degree of differential eq^n.

Ex :- $$\left(\frac{d^2y}{dx^2}\right)^1 + \left(\frac{dy}{dx}\right)^5 + 3y = 0$$

Order = 2 , degree = 1

$$\frac{d^2r}{d\theta^2} = \sqrt{r + \left(\frac{dr}{d\theta}\right)^2}$$

order first degree then one day 2

$$\Rightarrow \left(\frac{d^2r}{d\theta^2}\right)^2 = r + \left(\frac{dr}{d\theta}\right)^2$$

order = 2 , degree = 2

General Sol^n of differential eq^n

A function which satisfies the differential eq^n is called a sol^n of differential eq^n.

General sol^n is that sol^n ~~is that sol^n~~ which contains as many arbitrary constants as the order of differential eq^n.

Particular Solⁿ can be obtained from general solⁿ by assigning suitable values to the arbitrary constants.

Another type solⁿ → Singular solⁿ.

Formation of differential eqⁿs.

Differential eqⁿs can be formed from the given function by finding derivatives of the given function.

For finding differential eqⁿs we have to eliminate the arbitrary constants in the given function.

Ex: Form the differential eqⁿ whose solⁿ is $y = mx + c$

Solⁿ:

$$y = mx + c$$

$$\Rightarrow \frac{dy}{dx} = m$$

$$\Rightarrow \frac{d^2y}{dx^2} = 0$$

which is the required ~~general~~ differential eqⁿ.

2. Form the differential eq^n of all straight lines passing through the origin.

Ans → The eq^n of all straight lines passing through the origin is $y = mx$

$$\Rightarrow \frac{dy}{dx} = m$$

Now $y = mx$

~~⇒ m = y/x~~ $\Rightarrow y = \frac{dy}{dx} x$

$\Rightarrow \frac{dy}{dx} = \frac{y}{x}$ which is the required diff. eq^n.

Q→ Form the differential eq^n whose general sol^n is $y = ae^{5x} + be^{3x}$

Ans: $y = ae^{5x} + be^{3x}$ —— (i)

$\Rightarrow \frac{dy}{dx} = 5ae^{5x} + 3be^{3x}$ —— (ii)

$\frac{d^2y}{dx^2} = 25ae^{5x} + 9be^{3x}$ —— (iii)

Multiplying eq^n (i) by 3 and subtracting

eq^n (2), we get

$$\frac{dy}{dx} - 3y = 2a e^{5x}$$

$$\Rightarrow a = \frac{\frac{dy}{dx} - 3y}{2e^{5x}}$$

Multiplying eq^n (ii) by 3 and

subtracting from (iii), we get

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} = 10 a e^{5x}$$

$$\Rightarrow a = \frac{\frac{d^2y}{dx^2} - 3\frac{dy}{dx}}{10 e^{5x}}$$

$$\therefore \frac{\frac{d^2y}{dx^2} - 3\frac{dy}{dx}}{\cancel{10}^{5} \cancel{e^{5x}}} = \frac{\frac{dy}{dx} - 3y}{\cancel{2} \cancel{e^{5x}}}$$

$$\Rightarrow \frac{d^2y}{dx^2} - 3\frac{dy}{dx} = 5\frac{dy}{dx} - 15y$$

$$\Rightarrow \frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 15y = 0$$

Which is the required diff. eq^n.

1st order and 1st degree differential eq^n

There are 4 types of 1st order and 1st degree differential eq^{n}s.

(i) Variables seperables.

(ii) Linear,

(iii) Homogeneous,

*(iv) Exact.

Variables separable

In this type of eq^{n}s, the variables (dependent independent) i.e y and x can be written seperately and hence can be integrated. Hence we get the general sol^n.

Note: Sometimes the eq^{n}s are not variable separable, but under substitution the eq^{n}s are reduced to variables separable.

Problem

1. Solve $\frac{dy}{dx} = x a^{x^2}$

Ans :- $\frac{dy}{dx} = x a^{x^2}$

$\Rightarrow dy = x a^{x^2} dx$

Integrating both sides, we get

$\int dy = \int x a^{x^2} dx$

$\Rightarrow y = \int a^t \frac{dt}{2}$

$\Rightarrow y = \frac{1}{2} \int a^t dt$

$\Rightarrow y = \frac{1}{2} \frac{a^t}{\ln a} + C$

$\Rightarrow y = \frac{1}{2} \frac{a^{x^2}}{\ln a} + C$

Put $x^2 = t$
$\Rightarrow 2x\,dx = dt$
$\Rightarrow x\,dx = \frac{dt}{2}$

which is the required general soln

2. Solve : $\frac{dx}{dy} = \frac{e^y - e^{-y}}{e^y + e^{-y}}$

Sol^n : $\dfrac{dx}{dy} = \dfrac{e^y - e^{-y}}{e^y + e^{-y}}$

$\Rightarrow dx = \dfrac{e^y - e^{-y}}{e^y + e^{-y}} dy$

~~Integrating~~ $\Rightarrow \int dx = \int \dfrac{e^y - e^{-y}}{e^y + e^{-y}} dy$

$\Rightarrow x = \ln(e^y + e^{-y}) + C$

which is required general sol^n.

{ Putting $e^y + e^{-y} = t$

$\therefore$ Integral $= \int \frac{dt}{t}$

$= \ln t$

$= \ln(e^y + e^{-y})$

3. Solve $(1+x^2)\tan^{-1}y \dfrac{dy}{dx} = (1+y^2)(\tan^{-1}x)$

Sol^n $(1+x^2)\tan^{-1}y \dfrac{dy}{dx} = (1+y^2)\tan^{-1}x$

$\Rightarrow \dfrac{\tan^{-1}y}{1+y^2} dy = \dfrac{\tan^{-1}x}{1+x^2} dx$

$\Rightarrow \int \dfrac{\tan^{-1}y}{1+y^2} dy = \int \dfrac{\tan^{-1}x}{1+x^2} \cdot dx$

$$\Rightarrow \int t\,dt = \int u\,du$$

$$\Rightarrow \frac{t^2}{2} = \frac{u^2}{2} + \frac{C}{2}$$

$$\Rightarrow t^2 = u^2 + C$$

$$\Rightarrow (\tan^{-1} y)^2 = (\tan^{-1} x)^2 + C$$

which is required general sol^n.

Put $\tan^{-1} y = t$ in L.H.S and $\tan^{-1} x = u$ in R.H.S

$\frac{dt}{dy} = \frac{1}{1+y^2}$

$dt = \frac{dy}{1+y^2}$

$du = \frac{dx}{1+x^2}$

4. Solve $\frac{dy}{dx} = e^{x-y}$

Sol^n: $\frac{dy}{dx} = e^{x-y} = \frac{e^x}{e^y}$

$$\Rightarrow e^x\,dx = e^y\,dy$$

$$\Rightarrow \int e^y\,dy = \int e^x\,dx$$

$$\Rightarrow e^y = e^x + C$$

which is required general sol^n.

OR

$$\Rightarrow \ln e^y = \ln e^x + \ln c$$

$$\Rightarrow y = x + \ln c$$

~~$\Rightarrow y = x + K$~~

$$\Rightarrow y - x = K$$

(where $K = \ln c$)

5. Solve

$$(x+y)\,dx + dy = 0$$

Soln

$$(x+y)\,dx + dy = 0$$

Put
$x+y = u$
$\Rightarrow \frac{du}{dx} = 1 + \frac{dy}{dx}$
$\Rightarrow du = dx + dy$
$\Rightarrow dy = du - dx$

$$\Rightarrow u\,dx + du - dx = 0$$

$$\Rightarrow (u-1)\,dx = -du$$

$$\Rightarrow dx = \frac{-du}{u-1}$$

$$\Rightarrow \int dx = -\int \frac{du}{u-1}$$

$$\Rightarrow x = -\ln(u-1) + \ln c$$

$$\Rightarrow x = -\left(\ln \frac{u-1}{c}\right)$$

$$\Rightarrow -x = \ln \frac{u-1}{c}$$

$$\Rightarrow e^{-x} = \frac{u-1}{c}$$

$$\Rightarrow u-1 = c\,e^{-x}$$

$$\Rightarrow x+y-1 = c\,e^{-x}$$

which is required general soln.

13-a) 1 — 5

Linear differential eqn of first order

The first order differential eqn linear in y is of the form

$$\frac{dy}{dx} + P(x).y = Q(x)$$

Here we will find a function which is multiplied throughout the eqn to make the eqn integrable. ~~is called~~ and this function is called integrating factor or I.F. or μ

In this case $I.F = e^{\int P dx}$

Method of Solution :-

Multiplying integrating factor throughout the given eqn, we get

$$e^{\int P dx} \frac{dy}{dx} + e^{\int P dx} P(x) y = e^{\int P dx} . Q(x)$$

$$\Rightarrow \frac{d}{dx}\left(y . e^{\int P dx}\right) = e^{\int P dx} Q(x)$$

$$\Rightarrow d\left\{y\, e^{\int P\,dx}\right\} = e^{\int P\,dx}\cdot Q(x)\cdot dx$$

Integrating both the side, we get

$$\Rightarrow y\, e^{\int P\,dx} = \int e^{\int P\,dx}\cdot Q(x)\,dx + C$$

$$\Rightarrow y = e^{-\int P\,dx}\left[\int e^{\int P\,dx}\, Q(x)\,dx + C\right]$$

Which is required general soln.

Problem → Solve $(x+y)\,dx + dy = 0$

Soln

$$(x+y)\,dx + dy = 0$$

$$\Rightarrow x+y + \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} + y = -x \quad \text{———} \ (1)$$

It is linear and it is of the form

$$\frac{dy}{dx} + P(x)\,y = Q(x) \quad \text{where}$$

$$P = 1, \quad Q = -x$$

$\therefore$ I.F $= e^{\int P dx} = e^{\int 1 dx} = e^{x}$

Multiplying e^x both the sides of the eqn (i), we get

$\Rightarrow e^x \frac{dy}{dx} + e^x y = -x e^x$

$\Rightarrow \frac{d}{dx}(y e^x) = -x e^x$

$\Rightarrow d(y e^x) = -x e^x dx$

$\Rightarrow \int d(y e^x) = -\int x e^x dx$

$\Rightarrow y e^x = -\left[x e^x - \int e^x\right] + c$

$\Rightarrow y = -x + 1 + c e^{-x}$

Linear in x

The first order differential eqn linear in x is of the form

$$\frac{dx}{dy} + P(y) x = Q(y)$$

Here $IF = e^{\int P(y)\,dy}$

My trick : to remember the form of linear eqn, (as it begins) x then y then x then y / y then x then y then x

Method of soln

Multiplying integrating factor throughout the given eqn, we get

$$e^{\int P\,dy} \frac{dx}{dy} + e^{\int P\,dy} \cdot P(y)\,x = e^{\int P\,dy}\, Q(y)$$

$$\Rightarrow \frac{d}{dy}\left(x \cdot e^{\int P\,dy} \right) = Q(y)\, e^{\int P\,dy}$$

$$\Rightarrow d\left(x \cdot e^{\int P\,dy} \right) = Q\, e^{\int P\,dy}\, dy$$

Integrating,

$$x\, e^{\int P\,dy} = \int Q\, e^{\int P\,dy}\, dy + C$$

$$\Rightarrow x = e^{-\int P\,dy} \left[\int Q\, e^{\int P\,dy}\, dy + C \right]$$

Solve $(1+y^2)\,dx = (\tan^{-1}y - x)\,dy$

Sol^n

$$(1+y^2)\,dx = (\tan^{-1}y - x)\,dy$$

$$\Rightarrow (1+y^2)\frac{dx}{dy} = \tan^{-1}y - x$$

$$\Rightarrow \frac{dx}{dy} = \frac{\tan^{-1}y}{1+y^2} - \frac{x}{1+y^2}$$

$$\Rightarrow \frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1}(y)}{1+y^2} \quad -(1)$$

It is linear in x and it is of the form

$$\frac{dx}{dy} + P.x = Q(y)$$

where $P = \frac{1}{1+y^2}$

$$Q = \frac{\tan^{-1}y}{1+y^2}$$

Here $IF = e^{\int P(y)\,dy} = e^{\int \frac{1}{1+y^2}dy}$

$$= e^{\tan^{-1}y}$$

Multiplying $e^{\tan^{-1}y}$ through out the eq^n (1),

we get

$$e^{\tan^{-1}y}\frac{dx}{dy} + e^{\tan^{-1}y}\frac{x}{1+y^2} = e^{\tan^{-1}y}\cdot\frac{\tan^{-1}y}{1+y^2}$$

$$\Rightarrow \frac{d}{dy}\left(x \cdot e^{\tan^{-1}y}\right) = e^{\tan^{-1}y} \frac{\tan^{-1}y}{1+y^2}$$

$$\Rightarrow d\left(x \cdot e^{\tan^{-1}y}\right) = e^{\tan^{-1}y} \frac{\tan^{-1}y}{1+y^2} dy$$

$$\Rightarrow \int d\left(x\, e^{\tan^{-1}y}\right) = \int e^{\tan^{-1}y} \frac{\tan^{-1}y}{1+y^2} dy$$

$$\Rightarrow x\, e^{\tan^{-1}y} = \int t e^{t} \cdot dt$$

$$= t e^{t} - e^{t} + c$$

Put $\tan^{-1}y = t$

$\Rightarrow dt = \frac{dy}{1+y^2}$

$$\Rightarrow x \cdot e^{\tan^{-1}y} = \tan^{-1}y \cdot e^{\tan^{-1}y} - e^{\tan^{-1}y} + C$$

$$\Rightarrow x = \tan^{-1}y - 1 + C e^{-\tan^{-1}y}$$

which is required general sol^n. (Ans)

Eq^{n}s reducible to linear form

Bernoulli's eq^n

The eq^n of the form $\frac{dy}{dx} + P(x)\, y = Q(x)\, y^n$ where

$n \neq 1$ and $n \neq 0$, is called Bernoulli's eq^n.

Methods of sol^n

$$\frac{dy}{dx} + P(x)\, y = Q(x)\, y^n$$

$$\Rightarrow y^{-n} \frac{dy}{dx} + y^{1-n} P(x) = Q(x) \quad \text{— (1)}$$

(Dividing by y^n)

~~Put~~ ~~$y^{1-n} = Z$~~ ~~dz =~~

Put $y^{1-n} = Z$

$$\Rightarrow \frac{dZ}{dx} = (1-n)\, y^{-n} \cdot \frac{dy}{dx}$$

$$\Rightarrow \frac{1}{1-n} \frac{dz}{dx} = y^{-n} \frac{dy}{dx}$$

$\therefore$ Eq^n (1) becomes.

$$\frac{1}{1-n} \frac{dz}{dx} + Z\, P(x) = Q(x)$$

$$\Rightarrow \frac{dz}{dx} + (1-n) Z\, P(x) = (1-n)\, Q(x)$$

which is linear in Z

and can be solved.

Rule ① Firstly divide by y^n

② Put $y^{1-n} = Z$ and the eqn becomes linear.

Problem:

Solve $x\frac{dy}{dx} + y = x^4 y^3$

Soln:

$$x\frac{dy}{dx} + y = x^4 y^3$$

$$\Rightarrow \frac{dy}{dx} + \frac{y}{x} = x^3 y^3 \quad \left(\text{Dividing by } x\right)$$

which is Bernoulli's eqn of the form $\frac{dy}{dx} + P(x)\, y = Q(x) \cdot y^3$

where $P(x) = \frac{1}{x}$, $Q(x) = x^3$

Dividing by y^3 throughout, we get

$$y^{-3}\frac{dy}{dx} + \frac{y^{-2}}{x} = x^3$$

Put $y^{-2} = Z$

$\Rightarrow \frac{dZ}{dx} = -2y^{-3}\frac{dy}{dx}$

$\Rightarrow y^{-3}\frac{dy}{dx} = -\frac{1}{2}\frac{dZ}{dx}$

$$\Rightarrow -\frac{1}{2}\frac{dZ}{dx} + \frac{Z}{x} = x^3$$

$$\Rightarrow \frac{dZ}{dx} - \frac{2Z}{x} = -2x^3$$

Which is linear.

$$I.F = e^{-\int \frac{2}{x} dx} = e^{-2\ln x} = e^{\ln x^{-2}} = x^{-2} = \frac{1}{x^2}$$

~~Multiplying by $\frac{1}{x^2}$ both the sides, we get~~

General Sol^n is $Z \cdot \frac{1}{x^2} = \int \frac{1}{x^2} (-2x^3)\, dx + C$

$$\Rightarrow \frac{Z}{x^2} = -x^2 + C$$

$$\Rightarrow Z = Cx^2 - x^4$$

$$\Rightarrow \frac{1}{y^2} = Cx^2 - x^4$$

Homogeneous eqⁿs

The differential eqⁿ $Mdx + Ndy = 0$ —(i)

is called homogeneous if M and N are homogeneous functions of the same degree.

i.e the sum of the powers of x and y in each term remains same.

Ex: $(x^2+xy)\,dx + y^2 dy = 0$

is homogeneous differential eqⁿ.

Method of solⁿ:

In eqⁿ (i) put $y = vx$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

The eqⁿ (i) is reduced to variables separable and it can be solved by integration.

Solve $(x^2-2y^2)\,dx + xy\,dy = 0$

Solⁿ → $(x^2-2y^2)\,dx + xy\,dy = 0$

$$\Rightarrow (x^2-2y^2)\,dx = -xy\,dy$$

$$\Rightarrow x^2-2y^2 = -xy\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-x^2+2y^2}{xy} \quad \text{——— (i)}$$

It is homogeneous. Put $y = vx$

$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore$ Eqn (i) becomes

$v + x\frac{dv}{dx} = \frac{2v^2x^2 - x^2}{x \cdot vx}$

$= \frac{2v^2-1}{v}$

$\Rightarrow x\frac{dv}{dx} = \frac{2v^2-1}{v} - v = \frac{v^2-1}{v}$

$\Rightarrow \frac{v}{v^2-1}dv = \frac{dx}{x}$

$\Rightarrow \int \frac{v}{v^2-1}dv = \int \frac{1}{x}dx$

$\Rightarrow \int \frac{dt}{2t} = \ln x + \ln c$

Put $v^2-1=t$

$\Rightarrow dt = 2v\,dv$

$\Rightarrow v\,dv = \frac{dt}{2}$

$\Rightarrow \frac{1}{2}\ln|t| = \ln x + \ln c$

$\Rightarrow \frac{1}{2}\ln(v^2-1) = \ln xc$

$\Rightarrow \ln(v^2-1) = 2\ln cx = \ln c^2x^2$

$\Rightarrow v^2-1 = c^2x^2$

$\Rightarrow \left(\frac{y^2}{x^2}-1\right) = c^2x^2$

$\Rightarrow \frac{y^2-x^2}{x^2} = c^2x^2$

$\Rightarrow y^2-x^2 = c^2x^4 \Rightarrow y^2 = x^2+c^2x^4$

It is the required general soln.

Reducible to homogeneous and Special forms of homogeneous eqⁿs

Case-I → Suppose the eqⁿ is of the form

$$\frac{dy}{dx} = \frac{ax+by+c}{a'x+b'y+c'} \quad \text{---} (i)$$

where $\frac{a}{a'} \neq \frac{b}{b'}$.

It is not homogeneous

Put, $x = X+h$

and $y = Y+k$

$\therefore \frac{dy}{dx} = \frac{dY}{dX}$

$\therefore$ The given eqⁿ (i), becomes

$$\frac{dY}{dX} = \frac{a(X+h)+b(Y+k)+c}{a'(X+h)+b'(Y+k)+c'}$$

$$\Rightarrow \frac{dY}{dX} = \frac{aX+bY+(ah+bk+c)}{a'X+b'Y+(a'h+b'k+c')}$$

Choose h and k such that $ah+bk+c = 0$

and $a'h+b'k+c' = 0$

Thus the eqⁿ reduces to $\frac{dY}{dX} = \frac{aX+bY}{a'X+b'Y}$

which is homogeneous & Put

$y = vx$ and solve

Case-II If $\frac{a}{a'} = \frac{b}{b'} = \frac{1}{K}$ in

eqn (i), then $a' = aK$, $b' = bK$

$$\therefore \frac{dy}{dx} = \frac{ax+by+c}{aKx+bKy+c'}$$

$$= \frac{ax+by+c}{K(ax+by)+c'} \quad \text{—(ii)}$$

Put $Z = ax+by$

$$\Rightarrow \frac{dZ}{dx} = a + b\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{b}\left(\frac{dz}{dx} - a\right)$$

The eqn (ii) becomes

$$\frac{1}{b}\left(\frac{dz}{dx} - a\right) = \frac{z+c}{Kz+c'}$$

which is variable separable and

can be solved.

Solve : $\frac{dy}{dx} = \frac{x-y+1}{x+y-3}$

Sol^n $\frac{dy}{dx} = \frac{x-y+1}{x+y-3}$ ——— (i)

Put $x = X+h$, $y = Y+K$

$\therefore$ $dx = dX$, $dy = dY$

$\therefore$ $\frac{dy}{dx} = \frac{dY}{dX}$

$\therefore$ Eq^n (i), becomes

$$\frac{dY}{dX} = \frac{X+h-Y-K+1}{X+h+Y+K-3}$$

$$\Rightarrow \frac{dY}{dX} = \frac{(X-Y)+(h-K+1)}{(X+Y)+(h+K-3)} \quad \text{——— (ii)}$$

Choose h, K such that $h-K+1=0$

$h+K-3=0$

$\Rightarrow h=1,\ K=2$

$\therefore$ Eq^n (ii), becomes $\frac{dY}{dX} = \frac{X-Y}{X+Y}$ ——— (iii) which is homogeneous. Put $y = vx$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Eqn (iii) becomes

$$v + x\frac{dv}{dx} = \frac{x - vx}{x + vx} = \frac{1-v}{1+v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1-v}{1+v} - v = \frac{1-2v-v^2}{1+v}$$

$$\Rightarrow \frac{dx}{x} = \frac{1+v}{1-2v-v^2}\,dv$$

$$\Rightarrow \int \frac{dx}{x} = \int \frac{1+v}{1-2v-v^2}\,dv$$

$$\Rightarrow \ln x = -\frac{1}{2}\int \frac{dt}{t} + \frac{1}{2}\ln c$$

Put $1-2v-v^2 = t$

$\Rightarrow dt = (-2-2v)\,dv$

$\Rightarrow -\frac{dt}{2} = (1+v)\,dv$

$$\Rightarrow \ln(x) = -\frac{1}{2}\ln t + \frac{1}{2}\ln c$$

$$\Rightarrow \ln x = \frac{1}{2}\ln\sqrt{\frac{c}{t}}$$

$$\Rightarrow x = \frac{\sqrt{c}}{\sqrt{t}} = \frac{\sqrt{c}}{\sqrt{1-2v-v^2}}$$

$$\Rightarrow x^2 = \frac{c}{1-2v-v^2}$$

$$\Rightarrow x^2 = \frac{c}{1-\frac{2y}{x}-\frac{y^2}{x^2}}$$

$$\Rightarrow x^2 = \frac{c\,x^2}{x^2-2yx-y^2}$$

$$\Rightarrow x^2 - 2xy - y^2 = c$$

$$\Rightarrow (x-h)^2 - 2(x-h)(y-k) - (y-k)^2 = c \qquad (\because X = x-h,\ Y = y-k)$$

$$\Rightarrow (x-1)^2 - 2(x-1)(y-2) - (y-2)^2 = c$$

~~which is homogeneous. Put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$~~

~~Eqn (ii) becomes $(v+x)\frac{dv}{dx} = \frac{x - vx}{x + vx} = \frac{1-v}{1+v}$~~

$\Rightarrow x^2 - 2xy + 2x - y^2 + 6y = C + 7 = C_1$

which is the required general soln.

Solve: $(2x - 4y + 5)dy + (x - 2y + 3)dx = 0$

Soln: $(2x - 4y + 5)dy + (x - 2y + 3)dx = 0$

$\Rightarrow \frac{dy}{dx} = -\frac{(x-2y+3)}{(2x-4y+5)} = -\left\{\frac{(x-2y)+3}{2(x-2y)+5}\right\}$ —— (i)

Put $Z = x - 2y$

$\Rightarrow \frac{dZ}{dx} = 1 - 2\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(1 - \frac{dZ}{dx}\right)$

$\therefore$ Eqn (i) becomes $\frac{1}{2}\left(1 - \frac{dZ}{dx}\right) = -\left\{\frac{Z+3}{2Z+5}\right\}$

Put $\Rightarrow 1 - \frac{dZ}{dx} = \frac{-(2Z+6)}{(2Z+5)}$

$\Rightarrow \frac{dZ}{dx} = 1 + \frac{2Z+6}{2Z+5} = \frac{4Z+11}{2Z+5}$

~~Put Z = vx~~

$\Rightarrow dx = \frac{2Z+5}{4Z+11}dZ$

$\Rightarrow \int dx = \int \frac{1}{2}\left(1 - \frac{1}{4Z+11}\right)dZ$

$\Rightarrow \int dx = \frac{1}{2}\int\left(1 - \frac{1}{4Z+11}\right)dZ$

$\Rightarrow x = \frac{1}{2}\left[Z - \frac{1}{4}\ln(4Z+11)\right] + C_1$

$\Rightarrow x = \frac{Z}{2} - \frac{1}{8}\ln(4Z+11) + C_1$

$\Rightarrow 8x = 4Z - \ln(4Z+11) + 8C_1$

$\Rightarrow 8x = 4(x-2y) - \ln(4x - 8y + 11) + C$ (where $c = 8C_1$)

$4x + 8y + \ln(4x - 8y + 11) = C$ which is req'd general soln.

2nd Order and 1st degree differential eq^n

Solve: $\frac{d^2y}{dx^2} = 0$

Sol^n $\frac{d^2y}{dx^2} = 0$

$\Rightarrow \frac{d}{dx}\left(\frac{dy}{dx}\right) = 0$ (My notes $\Rightarrow d\left(\frac{dy}{dx}\right) = 0 \cdot dx$ $\Rightarrow \int d\frac{dy}{dx} = \int 0\, dx = \text{constant}$)

$\Rightarrow \frac{dy}{dx} = c_1$ (when c_1 is a constant).

$\Rightarrow dy = c_1 dx$

$\Rightarrow \int dy = \int c_1 dx = c_1 \int dx$

$\Rightarrow y = c_1 x + c_2$ where c_2 is another constant

which is required general sol^n.

Solve: $\cos^2 x \cdot \frac{d^2y}{dx^2} = 1$

Sol^n: $\cos^2 x \frac{d^2y}{dx^2} = 1$

$\Rightarrow \frac{d^2y}{dx^2} = \sec^2 x$ $\Rightarrow \frac{d}{dx}\left(\frac{dy}{dx}\right) = \sec^2 x$

$\Rightarrow d\left(\frac{dy}{dx}\right) = \sec^2 x\, dx$

$\Rightarrow \int d\left(\frac{dy}{dx}\right) = \int \sec^2 x\, dx = \tan x$

$\Rightarrow \frac{dy}{dx} = \tan x + c_1$

$\Rightarrow dy = (\tan x + c_1)\, dx$

$\Rightarrow \int dy = \int (\tan x + c_1)\, dx$

$\Rightarrow y = \ln \sec x + c_1 x + c_2$

which is required general soln.

Initial Value problem (I.V.P)

The differential eqn which has some initial conditions is called initial value problem. Provided ~~for~~ ~~one~~ ~~value~~ in the initial conditions only one value of independent ~~value~~ variable is given.

Q → Solve : $\frac{dy}{dx} = 2+x$, $y(0) = 3$

Soln :

$$\frac{dy}{dx} = 2+x$$

$$\Rightarrow dy = (2+x)\, dx$$

$$\Rightarrow \int dy = \int (2+x)\, dx$$

$$\Rightarrow y = 2x + \frac{x^2}{2} + c.$$

Also for $x = 0$, $y = 3$

$$\therefore 3 = 0 + 0 + c$$

$$\Rightarrow c = 3$$

Hence the required particular soln is

$$y = 2x + \frac{x^2}{2} + 3$$

Solve: $\frac{d^2y}{dx^2} = 3x^2 - x + 1$, $y(0) = 0$

and $\frac{dy}{dx} = 1$ at $x = 0$

Soln:

$$\frac{d^2y}{dx^2} = 3x^2 - x + 1$$

$$\Rightarrow \frac{d}{dx}\left(\frac{dy}{dx}\right) = 3x^2 - x + 1$$

$$\Rightarrow d\left(\frac{dy}{dx}\right) = (3x^2 - x + 1)\, dx$$

$$\Rightarrow \int d\left(\frac{dy}{dx}\right) = \int (3x^2 - x + 1)\, dx$$

$$\Rightarrow \frac{dy}{dx} = \frac{3x^3}{3} - \frac{x^2}{2} + x + C_1$$

~~Also at $x = 0$, $y = 0$~~

Given that $\frac{dy}{dx} = 1$ at $x = 0$

$\therefore 1 = C_1 \quad \therefore C_1 = 1$

$$\therefore \frac{dy}{dx} = x^3 - \frac{x^2}{2} + x + 1$$

$$\Rightarrow dy = \left(x^3 - \frac{x^2}{2} + x + 1\right) dx$$

Integrating

$$\Rightarrow y = \frac{x^4}{4} - \frac{x^3}{6} + \frac{x^2}{2} + x + C_2$$

Boundary value Problem. → In this problem 2 values of x is given. So 2 conditions.

Given that $x=0$ and $y=0$

$\therefore C_2 = 0$

Hence the required particular sol^n is

$$y = \frac{x^4}{4} - \frac{x^3}{6} + \frac{x^2}{2} + x$$

$$\int_0^{\frac{\pi}{4}} \sqrt{\cot\theta}\, d\theta$$

Put $\cot\theta = x^2$

$-\operatorname{cosec}^2\theta\, d\theta = 2x\,dx$

$d\theta = -\frac{2x\,dx}{\operatorname{cosec}^2\theta} = \frac{-2x\,dx}{1+x^4}$

$\theta = \frac{\pi}{4} \Rightarrow x = 1$

$x = 0 \Rightarrow x = \infty$

$$= \int_{\infty}^{1} \frac{x \cdot (-2x)}{1+x^4}\, dx$$

$$= \int_1^{\infty} \frac{2x^2}{1+x^4}\, dx$$

$$= \int_1^{\infty} \left[\frac{x^2+1}{1+x^4} + \frac{x^2-1}{1+x^4}\right] dx$$

$$= \int_1^{\infty} \frac{x^2+1}{1+x^4}\, dx + \int_1^{\infty} \frac{x^2-1}{1+x^4}\, dx$$

$$= \left[\frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x^2-1}{\sqrt{2}\,x}\right) + \frac{1}{2\sqrt{2}} \ln\left|\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right|\right]_1^{\infty}$$

$$= \left[\frac{1}{\sqrt{2}} \cdot \frac{\pi}{2} + \frac{1}{2\sqrt{2}} \cdot 0\right] - \left[0 + \frac{1}{2\sqrt{2}} \ln \frac{\sqrt{2}-1}{\sqrt{2}+1}\right]$$

($\because \lim_{x\to\infty} \tan^{-1}\left(\frac{x^2-1}{\sqrt{2}x}\right) = \frac{\pi}{2}$ (by using L'Hospital's rule)

$\lim_{x\to\infty} \ln\left|\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right| = 0$ by using L'Hospital's rule)

$$= \frac{\pi}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} \ln(\sqrt{2}-1)^2 = \frac{\pi\sqrt{2}}{4} - \frac{1}{\sqrt{2}} \ln(\sqrt{2}-1)$$

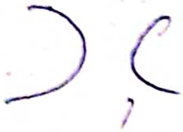
(Ans)

Condition: $y = mx + c$ touches

Circle → $c^2 = r^2(1+m^2)$

$x^2 + y^2 = r^2$

parabola $x^2 = 4ay$, $c = -am^2$



$y^2 = 4ax$, $c = \frac{a}{m}$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $c^2 = a^2m^2 + b^2$

$\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1$, $c^2 = b^2m^2 + a^2$

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, $c^2 = a^2m^2 - b^2$

$\frac{x^2}{b^2} - \frac{y^2}{a^2} = 1$, ~~$c^2 = b^2m^2 - a^2$~~

$c^2 = a^2 - b^2m^2$

(3) $\int \frac{dx}{a^2-x^2} = \int \frac{1}{(a+x)(a-x)} dx$

Let $\frac{1}{(a+x)(a-x)} = \frac{A}{(a+x)} + \frac{B}{(a-x)}$

$= \frac{A(a-x)+B(a+x)}{(a+x)(a-x)}$

$\Rightarrow 1 = A(a-x) + B(a+x)$

Putting $x = a$

$1 = 2aB$

$\Rightarrow B = \frac{1}{2a}$

Putting $x = -a$

$A = \frac{1}{2a}$

$\therefore \frac{1}{(a+x)(a-x)} = \frac{1}{2a(a+x)} + \frac{1}{2a(a-x)}$

$\therefore \int \frac{dx}{(a+x)(a-x)} = \frac{1}{2a}\int \frac{dx}{a+x} + \frac{1}{2a}\int \frac{dx}{a-x}$

$= \frac{1}{2a} \ln|a+x| - \frac{1}{2a} \ln|a-x|$

$= \frac{1}{2a} \ln\left|\frac{a+x}{a-x}\right| + C$ (Ans)

Problem n decents

Lemis cates or Bojnoff

Using partial fraction → ② and ③ of 9 integrals

② $\int \frac{1}{x^2-a^2}\,dx = \int \frac{1}{(x+a)(x-a)}$

Let $\frac{1}{(x+a)(x-a)} = \frac{A}{x+a} + \frac{B}{x-a}$

$= \frac{A(x-a)+B(x+a)}{(x+a)(x-a)}$

$\therefore\ 1 = A(x-a)+B(x+a)$

Putting $x = a$, we get

$B(a+a) = 1$

$\Rightarrow 2aB = 1$

$\Rightarrow B = \frac{1}{2a}$

Putting $x = -a$, we get

$1 = -A(2a)$

$\Rightarrow A = -\frac{1}{2a}$

$\therefore \frac{1}{(x+a)(x-a)} = \frac{-1}{2a(x+a)} + \frac{1}{2a(x-a)}$

Now $\int \frac{1}{x^2-a^2}\,dx$

$= \int \frac{1}{2a(x-a)}\,dx - \int \frac{1}{2a(x+a)}\,dx$

$= \frac{1}{2a}\int \frac{1}{x-a}\,dx - \frac{1}{2a}\int \frac{1}{x+a}\,dx$

$= \frac{1}{2a}\ln|x-a| - \frac{1}{2a}\ln|x+a| = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$

$$\int \sqrt{a^2+x^2}\, dx$$

Put

$x = a \sinh\theta$

$dx = a \cosh\theta \, d\theta$

$\sqrt{a^2+x^2}$

$= \sqrt{a^2 + a^2 \sinh^2\theta}$

$= a\cosh\theta$

$$= \int a\cosh\theta \cdot a\cosh\theta \, d\theta$$

$$= \int a^2 \cosh^2\theta \, d\theta$$

$$= a^2 \int \cosh^2\theta \, d\theta$$

$\cosh^2\theta + \sinh^2\theta = \cosh 2\theta$

$\downarrow$

$1 + \cosh^2$

$2\cosh^2\theta + 1 = \cosh x$

$$= a^2 \int \frac{\cosh 2\theta - 1}{2} \, d\theta$$

$$= \frac{a^2}{2} \left[\int \cosh 2\theta \, d\theta - \int 1 \, d\theta \right]$$

$$= \frac{a^2}{2} \left[\frac{\sinh 2\theta}{2} - \theta \right]$$