4. 12. 2K

Derivative of function.

The derivative of a function $y = f(x)$ with respect to x is denoted by

$f'(x)$ or y' or $\dot{y}$ (Newton) or $\dot{f}(x)$

or (Leibnitz) $\dfrac{d\,f(x)}{dx}$ or $\dfrac{dy}{dx}$ or $Df(x)$

or Dy

Note:

Let $y = f(x)$ be the function.

The small increment in x is denoted by Δx or δx or h

The small increment in y is denoted by Δy or δy or K.

Derivative (Def^n)

The derivative of $y = f(x)$ w.r.to x $= f'(x)$ or $\dfrac{d\,f(x)}{dx}$ or $D\,f(x)$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$, provided the limit exists

and h is small increment in x.

Also $f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$.

The process of finding the derivative is called differentiation.

The derivative is also called as the differential co-efficient.

Note:

The process of finding the derivative using defⁿ is called 1st principle or delta method or using defⁿ or ab-initio

The calculus which deals with differnitiation is called differential calculus.

Derivative of a Constant

Let $f(x) = c$ where c is constant.

$\therefore f(x+h) = c$ where h is small increment in x

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{c - c}{h}$$

$$= \lim_{h\to 0} \frac{0}{h}$$

$$= \lim_{h\to 0} 0$$

$$= 0$$

$$\therefore \boxed{\frac{dc}{dx} = 0} \longrightarrow (1)$$

Ex

$$\frac{d3}{dx} = 0, \quad \frac{de}{dx} = 0$$

$$\frac{d\pi}{dx} = 0,$$

Derivative of x^n

Let $f(x) = x^n$.

$f(x+h) = (x+h)^n$ where h is small increment in x.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$$

Put $x+h = y$

$\Rightarrow h = y - x$

$h \to 0 \Rightarrow y \to x$

$$= \lim_{y\to x} \frac{y^n - x^n}{(y - x)}$$

$$= n \cdot x^{n-1} \quad \left(\because \lim_{x\to a} \frac{x^n - a^n}{x-a} = n \cdot a^{n-1}\right)$$

$$\therefore \boxed{\frac{d\,x^n}{dx} = \cancel{n x^{n-1}}\ n\,x^{n-1}} \longrightarrow (2)$$

Ex

$$\frac{d\,\{x^5\}}{dx} = 5 \cdot x^4$$

$$\frac{d\,x^e}{dx} = e\,x^{e-1}$$

$$\frac{d\,\frac{1}{x}}{dx} = \frac{d\,x^{-1}}{dx} = -1 \cdot x^{-1-1} = -1 \cdot x^{-2}$$

$$= \frac{-1}{x^2}$$

$$\frac{d\,\sqrt{x}}{dx} = \frac{d\,x^{\frac{1}{2}}}{dx} = \frac{1}{2} \cdot x^{-\frac{1}{2}}$$

$$= \frac{1}{2\sqrt{x}}$$

Derivative of Sinx ~~function~~

Let $f(x) = \sin x$

$f(x+h) = \sin(x+h)$ where h is small increment in x.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{2\cos\frac{2x+h}{2} \cdot \sin\frac{h}{2}}{h}$$

$$= \lim_{h \to 0} \frac{\cos\left(x+\frac{h}{2}\right) \cdot \left(\dfrac{\sin\frac{h}{2}}{\frac{h}{2}}\right)}{\cancel{\frac{h}{2}}}$$

$$= \lim_{\frac{h}{2} \to 0} \frac{\sin\frac{h}{2}}{\frac{h}{2}} \cdot \lim_{h \to 0} \cos\left(x+\frac{h}{2}\right)$$

$$= 1 \cdot \cos x$$

$$= \cos x$$

$$\therefore \boxed{\frac{d \sin x}{dx} = \cos x} \longrightarrow (3)$$

Derivative of Cosx

Let $f(x) = \cos x$

$f(x+h) = \cos(x+h)$ where h is small increment.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h}$$

$$= \lim_{h \to 0} \frac{-2 \sin\left(\frac{2x+h}{2}\right) \cdot \sin\frac{h}{2}}{h}$$

$$= \lim_{h \to 0} - \frac{\sin\left(\frac{2x+h}{2}\right) \cdot \sin\frac{h}{2}}{\frac{h}{2}}$$

$$= -\lim_{h\to 0} \sin\left(x+\frac{h}{2}\right) \cdot \lim_{h\to 0} \frac{\sin\frac{h}{2}}{\frac{h}{2}}$$

$$= -\sin x \,.\, 1$$

$$= -\sin x$$

$$\therefore \boxed{\frac{d\cos x}{dx} = -\sin x} \quad \text{—— (4)}$$

Derivative of tan x

Let $f(x) = \tan x$

$f(x+h) = \tan(x+h)$ where h is small increment in x.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\tan(x+h) - \tan(x)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x}}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x+h).\cos x - \cos(x+h).\sin x}{h \times \cos(x+h).\cos x}$$

$$= \lim_{h\to 0} \frac{\sin(x+h-x)}{h} \cdot \frac{1}{\cos(x+h).\cos x}$$

$$= \lim_{h\to 0} \frac{\sin h}{h} \cdot \lim_{h\to 0} \frac{1}{\cos(x+h).\cos x}$$

$$= 1 \cdot \frac{1}{\cos^2 x}$$

$$= \sec^2 x$$

$$\therefore \boxed{\frac{d}{dx} \tan x = \sec^2 x} \longrightarrow (5)$$

Derivative of $\cot x$ → $-\operatorname{cosec}^2 x$

Let $f(x) = \cot x$

$f(x+h) = \cot(x+h)$ where h is small increment in x.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\cot(x+h) - \cot x}{h}$$

$$= \lim_{h \to 0} \frac{\frac{\cos(x+h)}{\sin(x+h)} - \frac{\cos x}{\sin x}}{h}$$

$$= \lim_{h \to 0} \frac{\sin x \cdot \cos(x+h) - \cos x \cdot \sin(x+h)}{h \times \sin(x+h) \sin x}$$

$$= \lim_{h \to 0} \frac{\sin(x - x - h)}{h \sin(x+h) \cdot \sin x}$$

$$= \lim_{h \to 0} -\frac{\sin h}{h} \cdot \frac{1}{\sin(x+h) \cdot \sin x}$$

$$= -\lim_{h \to 0} \frac{\sin h}{h} \cdot \lim_{h \to 0} \frac{1}{\sin(x+h) \cdot \sin x}$$

$$= -1 \cdot \left(\frac{1}{\sin^2 x}\right) = -\operatorname{cosec}^2 x$$

$$\boxed{\therefore \frac{d}{dx} \cot x = -\operatorname{cosec}^2 x}$$

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Derivative of $\sec x$ → $(\sec x . \tan x)$

Let $f(x) = \sec x$

$f(x+h) = \sec(x+h)$, where h is small increment in x.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sec(x+h) - \sec x}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{\cos(x+h)} - \frac{1}{\cos x}}{h}$$

$$= \lim_{h \to 0} \frac{\cos x - \cos(x+h)}{h \times \cos(x+h) . \cos x}$$

$$= \lim_{h \to 0} \frac{-2 \sin\left(\frac{2x+h}{2}\right) . \sin\left(\frac{x-x-h}{2}\right)}{h \times \cos(x+h) . \cos x}$$

$$= \lim_{h \to 0} \frac{\sin\left(\frac{x}{1} + \frac{h}{2}\right)}{\cos(x+h) . \cos x} \frac{\sin \frac{h}{2}}{\frac{h}{2}}$$

$$= \lim_{h \to 0} \frac{\sin\left(x + \frac{h}{2}\right)}{\cos(x+h) . \cos x} \cdot \lim_{h \to 0} \frac{\sin \frac{h}{2}}{\frac{h}{2}}$$

$$= \frac{\sin x}{\cos x} . \frac{1}{\cos x} \cdot (1)$$

$= -\operatorname{cosec} x . \cot x,$

$$= \tan x . \sec x$$

$$\therefore \boxed{\frac{d(\sec x)}{dx} = \tan x . \sec x} \to 7$$

Derivative of cosecx

Let $f(x) = \text{cosec}x$

$f(x+h) = \text{cosec}(x+h)$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\text{cosec}(x+h) - \text{cosec}x}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{\sin(x+h)} - \frac{1}{\sin x}}{h}$$

$$= \lim_{h\to 0} \frac{\sin x - \sin(x+h)}{h \times \sin(x+h).\sin x}$$

$$= \lim_{h\to 0} \frac{2\cos\left(\frac{2x+h}{2}\right).\sin\left(\frac{-h}{2}\right)}{h.\sin(x+h).\sin x}$$

$$= -\lim_{h\to 0} \frac{\sin\frac{h}{2}}{\frac{h}{2}} . \lim_{h\to 0} \frac{\cos\left(x+\frac{h}{2}\right)}{\sin(x+h).\sin x}$$

$$= -(1). \frac{\cos x}{\sin^2 x}$$

$$= -\frac{\cos x}{\sin x}.\frac{1}{\sin x}$$

$$= -\cot x.\ \text{cosec}x$$

$$\therefore \boxed{\frac{d}{dx}(\text{cosec}x) = -\text{cosec}x.\cot x} \longrightarrow (8)$$

Derivative of e^x

Let $f(x) = e^x$

$f(x+h) = e^{x+h}$ where h is a small increment in x.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{e^{x+h} - e^x}{h}$$

$$= \lim_{h \to 0} \frac{e^x \cdot e^h - e^x}{h}$$

$$= \lim_{h \to 0} \frac{e^x (e^h - 1)}{h}$$

$$= e^x \cdot \lim_{h \to 0} \frac{e^h - 1}{h}$$

$$= e^x \cdot 1$$

$$= e^x \quad \left(\because \lim_{x \to 0} \frac{e^x - 1}{x} = 1 \right)$$

$$\therefore \boxed{\frac{d}{dx}(e^x) = e^x} \longrightarrow (9)$$

Derivative of a^x

Let $f(x) = a^x$

$f(x+h) = a^{x+h}$. where h is small increment in x

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{a^{x+h} - a^x}{h}$$

$$= \lim_{h \to 0} \frac{a^x . a^h - a^x}{h}$$

$$= \lim_{h \to 0} \frac{a^x (a^h - 1)}{h}$$

$$= a^x \lim_{h \to 0} \frac{a^h - 1}{h}$$

$$= a^x . \log a$$

$$\left(\because \lim_{x \to 0} \frac{a^x - 1}{x} = \log a \right)$$

$$\therefore \boxed{\frac{d}{dx}(a^x) = a^x . \log a} \rightarrow (10)$$

<u>Ex</u> $\frac{d}{dx}(2^x) = 2^x \log 2$

$\frac{d}{dx}(3^x) = 3^x . \log 3$

Derivative of $\log x$ or $\ln x$

Let $f(x) = \log x$

$f(x+h) = \log(x+h)$ where h is small increment in x.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\log(x+h) - \log x}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \log\left(\frac{x+h}{x}\right)$$

$$= \lim_{h \to 0} \frac{1}{h} \cdot \log\left(1 + \frac{h}{x}\right)$$

$$= \lim_{h \to 0} \log\left(1 + \frac{h}{x}\right)^{\frac{1}{h}}$$

Put $\frac{h}{x} = y$

$h \to 0 \Rightarrow y \to 0$

$h = xy$

$$= \lim_{y \to 0} \log(1+y)^{\frac{1}{xy}}$$

$$= \lim_{y \to 0} \left\{\log\left((1+y)^{\frac{1}{y}}\right)\right\}^{\frac{1}{x}}$$

$$= \lim_{y \to 0} \frac{1}{x} \cdot \log(1+y)^{\frac{1}{y}}$$

$$= \frac{1}{x} \cdot \lim_{y \to 0} \log(1+y)^{\frac{1}{y}}$$

$$= \frac{1}{x} \log\left(\lim_{y \to 0} (1+y)^{\frac{1}{y}}\right)$$

$$= \frac{1}{x} \cdot \log e \quad \left(\because \lim_{x \to 0} (1+x)^{\frac{1}{x}} = e\right)$$

$$= \frac{1}{x} \cdot 1$$

$$= \frac{1}{x}$$

$$\therefore \boxed{\frac{d(\log x)}{dx} = \frac{1}{x} \text{ or } \frac{d}{dx}\ln x = \frac{1}{x}} \rightarrow 11$$

Derivative of $\sin^{-1}x$

Imp

Let $y = f(x) = \sin^{-1}x$

$\Rightarrow x = \sin y$

$\therefore y + K = f(x+h) = \sin^{-1}(x+h)$

$\Rightarrow (x+h) = \sin(y+K)$

where h and K are small increments in x and y respectively.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sin^{-1}(x+h) - \sin^{-1}x}{h}$$

$$= \lim_{h \to 0} \frac{\sin^{-1}(x+h) - \sin^{-1}x}{(x+h) - x}$$

$$= \lim_{K \to 0} \frac{y+K - y}{\sin(y+K) - \sin y} \quad (\because h \to 0 \Rightarrow K \to 0)$$

$$= \lim_{K \to 0} \frac{K}{\sin(y+K) - \sin y}$$

$$= \lim_{K \to 0} \frac{K}{2 \cdot \cos\left(\frac{2y+K}{2}\right) \cdot \sin\left(\frac{K}{2}\right)} \quad = \lim_{K \to 0} \frac{\frac{K}{2}}{\cos\left(\frac{2y+K}{2}\right) \sin K/2}$$

$$= \lim_{K \to 0} \frac{1}{\cos\left(y + \frac{K}{2}\right)} \cdot \frac{1}{\frac{\sin\left(\frac{K}{2}\right)}{\frac{K}{2}}}$$

$$= \cancel{\lim_{K \to}} \; \frac{1}{\cos y} \cdot \frac{1}{1}$$

$$= \frac{1}{\cos y}$$

$$= \frac{1}{\sqrt{1 - \sin^2 y}}$$

$$= \frac{1}{\sqrt{1 - x^2}}$$

$$\boxed{\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}} \quad 12$$

3. $$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1 - x^2}}$$

4. $$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1 + x^2}$$

5. $$\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1 + x^2}$$

6. $$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x| \sqrt{x^2 - 1}}$$

7. $$\frac{d}{dx}(\operatorname{cosec}^{-1} x) = \frac{-1}{|x| \sqrt{x^2 - 1}}$$

18. $$\frac{d}{dx}\left(f(x)+g(x)\right) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

Proof

Let $\phi(x) = f(x) + g(x)$

$\phi(x+h) = f(x+h) + g(x+h)$

where h is small increment in x.

$$\phi'(x) = \lim_{h \to 0} \frac{\phi(x+h) - \phi(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) + g(x+h) - f(x) - g(x)}{h}$$

$$= \lim_{h \to 0} \left\{\frac{f(x+h) - f(x)}{h}\right\} + \lim_{h \to 0}\left\{\frac{g(x+h) - g(x)}{h}\right\}$$

$$= \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

(Proved)

19. $$\frac{d}{dx}c\,f(x) = c \cdot \frac{d}{dx}f(x)$$

Proof

Let $\phi(x) = c \cdot f(x)$

$\phi(x+h) = c\,f(x+h)$

where 'h' is small increment in x

$$\phi'(x) = \lim_{h \to 0} \frac{\phi(x+h) - \phi(x)}{h}$$

$$= \lim_{h \to 0} \frac{c.f(x+h) - c.f(x)}{h}$$

$$= \lim_{h \to 0} c \frac{\{f(x+h) - f(x)\}}{h}$$

$$= c . \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= c . \frac{d}{dx} f(x)$$ (proved)

$$\frac{d}{dx}\{f(x) - g(x)\} = \frac{d}{dx} f(x) - \frac{d}{dx} g(x)$$

Proof Let $\cancel{\phi(x)} =$

$$\phi(x) = f(x) - g(x)$$

$$\phi(x+h) = f(x+h) - g(x+h)$$

where, h is small increment in x.

$$\phi'(x) = \lim_{h \to 0} \frac{\phi(x+h) - \phi(x)}{h}$$

$$= \lim_{h \to 0} \frac{\{f(x+h) - g(x+h)\} - \{f(x) - g(x)\}}{h}$$

$$= \lim_{h \to 0} \frac{\{f(x+h) - f(x)\} - \{g(x+h) - g(x)\}}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} - \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$= \frac{d}{dx} f(x) - \frac{d}{dx} g(x) \quad \text{(proved)}$$

21. $$\frac{d}{dx}\left(c_1 f_1(x) + c_2 f_2(x) \text{-----} + c_n f_n(x)\right)$$

$$= c_1 \frac{d}{dx} f_1(x) + c_2 \frac{d}{dx} f_2(x) + \text{-----} + c_n \frac{d}{dx} f_n(x)$$

Proof. Let $\phi(x) = c_1 f_1(x) + c_2 f_2(x) \text{----} + c_n f_n(x)$

$$\phi(x+h) = c_1 f_1(x+h) + c_2 f_2(x+h) + \text{---} + c_n f_n(x+h)$$

where h is small increment in x.

$$\phi' x = \lim_{h \to 0} \frac{\phi(x+h) - \phi(x)}{h}$$

$$= \lim_{h \to 0} \frac{\{c_1 f_1(x+h) + c_2 f_2(x+h) \text{----} + c_n f_n(x+h)\} - \{c_1 f_1(x) + c_2 f_2(x) + \text{---} + c_n f_n(x)\}}{h}$$

$$= \lim_{h \to 0} \frac{c_1 \{f_1(x+h) - f_1(x)\} + c_2 \{f_2(x+h) - f(x)\} + \text{-----} + c_n \{f_n(x+h) - f_n(x)\}}{h}$$

$$= \lim_{h \to 0} \frac{c_1 \{f_1(x+h) - f_1(x)\}}{h} + \lim_{h \to 0} \frac{c_2 \{f_2(x+h) - f_2(x)\}}{h} + \text{---} + \lim_{h \to 0} \frac{c_n \{f_n(x+h) - f_n(x)\}}{h}$$

$$= c_1 . \lim_{h \to 0} \frac{f_1(x+h) - f_1(x)}{h} + c_2 \lim_{h \to 0} \frac{f_2(x+h) - f_2(x)}{h} + \text{--} + c_n \lim_{h \to 0} \frac{f_n(x+h) - f_n(x)}{h}$$

$$= c_1 . \frac{d}{dx} f_1(x) + c_2 . \frac{d}{dx} f_2(x) + \text{-----} + c_n . \frac{d}{dx} f_n(x)$$

(proved)

22 Product Rule

$$\frac{d}{dx}\left(f(x) \cdot g(x)\right) = f(x) \cdot \frac{d}{dx}\{g(x)\} + g(x) \cdot \frac{d}{dx} f(x)$$

Proof :

Let $\phi(x) = f(x) \cdot g(x)$

$\therefore \phi(x+h) = f(x+h) \cdot g(x+h)$

where h is small increment in x.

$$\phi'(x) = \lim_{h \to 0} \frac{\phi(x+h) - \phi(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) \cdot g(x+h) - f(x) \cdot g(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) \cdot g(x+h) - f(x+h) \cdot g(x) + f(x+h) \cdot g(x) - f(x) \cdot g(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h)\{g(x+h) - g(x)\} + g(x)\{f(x+h) - f(x)\}}{h}$$

$$= \lim_{h \to 0} \left[\frac{f(x+h) \cdot \{g(x+h) - g(x)\}}{h} + \frac{g(x)\{f(x+h) - f(x)\}}{h}\right]$$

$$= \lim_{h\to 0} f(x+h) \cdot \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

$$g(x) \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= f(x) \cdot \frac{d}{dx} g(x) + gx \cdot \frac{d}{dx} f(x)$$

Quotient Rule $\rightarrow (23)$

$- f(x).gx$
$+ f(x).g(x)$

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x) \cdot \frac{d}{dx} f(x) - f(x) \cdot \frac{d}{dx} g(x)}{\{g(x)\}^2}$$

Proof

Let $\phi(x) = \frac{f(x)}{g(x)}$

$$\phi(x+h) = \frac{f(x+h)}{g(x+h)}$$

where h is small increment in x

$$\phi' x = \lim_{h\to 0} \frac{\phi(x+h) - \phi(x)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h).g(x) - f(x).g(x+h)}{h. g(x+h) \cdot g(x)}$$

$$= \lim_{h\to 0} \frac{f(x+h) \cdot g(x) - f(x).g(x) + f(x).g(x) - f(x).g(x+h)}{h \cdot g(x+h).g(x)}$$

$$= \lim_{h \to 0} \frac{g(x)\{f(x+h) - f(x)\} - f(x)\{g(x+h) - g(x)\}}{h \cdot g(x+h)g(x)}$$

$$= \frac{\lim_{h \to 0} \frac{g(x)\{f(x+h) - f(x)\}}{h} - \lim_{h \to 0} \frac{f(x)\{g(x+h) - g(x)\}}{h}}{\lim_{h \to 0} \frac{h \cdot g(x+h)(g(x))}{h}}$$

$$= \frac{g(x) \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} - f(x) \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}}{\lim_{h \to 0} g(x+h) \cdot g(x)}$$

$$= \frac{g(x) \cdot \frac{d}{dx}\{f(x)\} - f(x) \frac{d}{dx} g(x)}{\{g(x)\}^2} \quad \text{(Proved)}$$

24. $$\frac{d}{dx} \log_a x = \frac{1}{x} \cancel{\log_a e} \quad \frac{1}{x \cdot \log_e a}$$

or $$\frac{1}{x \cdot \ln a}$$

Proof

Let $f(x) = \log_a x$

$f(x+h) = \log_a (x+h)$ where h is small increment in x

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\log_a (x+h) - \log_a x}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \cdot \log_a \left(\frac{x+h}{x}\right)$$

$$= \lim_{h \to 0} \log_a \left(\frac{x+h}{x}\right)^{\frac{1}{h}} = \lim_{h \to 0} \log_a \left(1+\frac{h}{x}\right)^{\frac{1}{h}}$$

$$= \lim_{y \to 0} \log_a (1+y)^{\frac{1}{xy}}$$

Put $\frac{h}{x} = y$

$\Rightarrow h = xy$

$h \to 0 \Rightarrow y \to 0$

$$= \lim_{y \to 0} \log_a \left\{(1+y)^{\frac{1}{y}}\right\}^{\frac{1}{x}}$$

$$= \lim_{y \to 0} \frac{1}{x} \cdot \log_a (1+y)^{\frac{1}{y}}$$

$$= \frac{1}{x} \cdot \lim_{y \to 0} \log_a (1+y)^{\frac{1}{y}}$$

$$= \frac{1}{x} \log_a \lim_{y \to 0} (1+y)^{\frac{1}{y}}$$

$$= \frac{1}{x} \cdot \log_a e$$

$$= \frac{1}{x} \cdot \frac{1}{\log_e a} = \frac{1}{x \log_e a}$$

$$= \frac{1}{x \ln a}$$

$$\therefore \boxed{\frac{d}{dx} \log_a x = \frac{1}{x \log_e a} = \frac{1}{x \ln a}} \longrightarrow 24$$

Notes

(1) $\log_a b \cdot \log_b c = \log_a c$ —— (i)

(2) $\log_a b = \dfrac{1}{\log_b a}$

(3). $\log_a b = \dfrac{\log_e b}{\log_e a}$

For a

$\log_e a + \log_a b \therefore \log_e b$ from (i)

Inverse function

13. derivative of $\cos^{-1}x$

Let $y = f(x) = \cos^{-1}x$

$\Rightarrow x = \cos y$

$\therefore y+k = f(x+h) = \cos^{-1}(x+h)$

$\Rightarrow (x+h) = \cos(y+k)$

where h and k are small increments in x and y

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\cos^{-1}(x+h) - \cos^{-1}x}{h}$$

$$= \lim_{h\to 0} \frac{(y+k) - y}{(x+h) - x}$$

$$= \lim_{k\to 0} \frac{k}{\cos(y+k) - \cos y} \quad (\because h\to 0 \Rightarrow k\to 0$$

$$= \lim_{k\to 0} \frac{k}{-2\sin\left(y+\frac{k}{2}\right).\sin\left(\frac{k}{2}\right)}$$

$$= -\lim_{k\to 0} \frac{1}{\cancel{2}\sin\left(y+\frac{k}{2}\right)} \cdot \lim_{k\to 0} \frac{1}{\frac{\sin\left(\frac{k}{2}\right)}{\frac{k}{2}}}$$

$$= -\left(\frac{1}{\cancel{2}.\sin y} \cdot 1\right) = -\frac{1}{\sqrt{1-\cos^2 y}}$$

$$= -\frac{1}{\sqrt{1-x^2}}$$

$$\boxed{\therefore \frac{d}{dx}(\cot^{-1}x) = -\frac{1}{\sqrt{1-x^2}}} \rightarrow 13$$

14. <u>Derivative of $\tan^{-1}x$</u>

Let $y = f(x) = \tan^{-1}x$

$\Rightarrow x = \tan y$

$y+k = f(x+h) = \tan^{-1}(x+h)$

$\Rightarrow (x+h) = \cancel{\tan(y+k)}\ \tan(y+k)$

where h and k are small increments in x and y respectively.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\tan^{-1}(x+h) - \tan^{-1}x}{h}$$

$$= \lim_{h\to 0} \frac{\tan^{-1}(x+h) - \tan^{-1}x}{(x+h) - x}$$

$$= \lim_{h\to 0} \frac{\cancel{\tan^{-1}(x+h)}\ y+k - y}{\tan(y+k) - \tan y}$$

$$= \lim_{k\to 0} \frac{k}{\tan(y+k) - \tan y} \quad (\because h\to 0 \Rightarrow k\to 0)$$

$$= \lim_{K \to 0} \frac{K}{\frac{\sin(y+K)}{\cos(y+K)} - \frac{\sin y}{\cos y}}$$

$$= \lim_{K \to 0} \left(\frac{K}{\frac{\sin(y+K).\cos y - \cos(y+K).\sin y}{\cos(y+K).\cos y}} \right)$$

$$= \lim_{K \to 0} \frac{K.\cos(y+K).\cos y}{\sin K}$$

$$= \lim_{K \to 0} \frac{\cos(y+K).\cos y}{\frac{\sin K}{K}}$$

$$= \lim_{K \to 0} \frac{1}{\frac{\sin K}{K}} . \lim_{K \to 0} \cos(y+K).\cos y$$

$$= 1.\ \cos^2 y$$

$$= \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$$

$$\therefore \boxed{\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}} \text{ --- } 14$$

15. <u>Derivative of $\cot^{-1} x$</u>

Let $y = f(x) = \cot^{-1} x$

$\Rightarrow x = \cot y$

$y+K = f(x+h) = \cot^{-1}(x+h)$

$\Rightarrow x+h = \cot(y+K)$

where h and K are small increment in x and y respectively.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\cot^{-1}(x+h) - \cot^{-1}x}{h}$$

$$= \lim_{h\to 0} \frac{\cot^{-1}(x+h) - \cot^{-1}x}{(x+h) - x}$$

$$= \lim_{k\to 0} \frac{y+k-y}{\cot(y+k) - \cot y} \quad \left(\because h\to 0 \Rightarrow k\to 0 \right)$$

$$= \lim_{k\to 0} \frac{k}{\frac{\cos(y+k)}{\sin(y+k)} - \frac{\cos y}{\sin y}}$$

$$= \lim_{k\to 0} \frac{k}{\frac{\sin y.\cos(y+k) - \cos y.\sin(y+k)}{\sin(y+k).\sin y}}$$

$$= \lim_{k\to 0} \frac{k.\sin(y+k).\sin y}{\sin(-k)}$$

$$= -\lim_{k\to 0} \frac{\sin(y+k).\sin y}{\frac{\sin k}{k}}$$

$$= -\lim_{k\to 0} \frac{1}{\frac{\sin k}{k}} . \lim_{k\to 0} \sin(y+k).\sin y$$

$$= -(1.\sin^2 y) = \cancel{(1-\cos^2 y)}$$

$$= -\left(\frac{1}{\operatorname{cosec}^2 y}\right) = -\left(\frac{1}{1+\cot^2 y}\right) = -\frac{1}{1+x^2}$$

$$\boxed{\therefore \frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}} \longrightarrow 15$$

16. <u>Derivative of $\sec^{-1}x$</u>

Let $y = f(x) = \sec^{-1}x$

$\Rightarrow x = \sec y$

$y + K = f(x+h) = \sec^{-1}(x+h)$

$\Rightarrow x+h = \sec(y+K)$

where x and h are small increment in x and y respectively.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sec^{-1}(x+h) - \sec^{-1}x}{h}$$

$$= \lim_{h\to 0} \frac{\sec^{-1}(x+h) - \sec^{-1}x}{(x+h) - x}$$

$$= \lim_{K\to 0} \frac{y+K - y}{\sec(y+K) - \sec y} \quad \left(\because h \to 0 \Rightarrow K \to 0\right)$$

$$= \lim_{K\to 0} \frac{K}{\frac{1}{\cos(y+K)} - \frac{1}{\cos y}}$$

$$= \lim_{K\to 0} \frac{K \cdot \cos(y+K) \cdot \cos y}{\cos y - \cos(y+K)}$$

$$= \lim_{K\to 0} \frac{K \cdot \cos(y+K) \cdot \cos y}{-2\sin\left(y+\frac{K}{2}\right) \cdot \sin\left(-\frac{K}{2}\right)}$$

$$= \lim_{K\to 0} \frac{\frac{K}{2} \cdot \cos(y+K) \cdot \cos y}{\sin\left(y+\frac{K}{2}\right) \cdot \sin\frac{K}{2}}$$

$$= \lim_{K\to 0} \frac{\cos(y+K) \cdot \cos y}{\sin(y+K)} \cdot \lim_{K\to 0} \frac{\sin\frac{K}{2}}{\frac{K}{2}}$$

$$= \lim_{x \to 0} \frac{\cos^2 y}{\sin y} \quad \cdots (1)$$

$$= \frac{\cos^2 y}{\sqrt{1-\cos^2 y}} = \frac{\frac{1}{\sec^2 y}}{\sqrt{1-\frac{1}{\sec^2 y}}} = \frac{\frac{1}{x^2}}{\sqrt{1-\frac{1}{x^2}}}$$

$$= \frac{\frac{1}{x^2}}{\sqrt{\frac{x^2-1}{x^2}}} = \frac{1}{x^2} \times \frac{x}{\sqrt{x^2-1}}$$

$$= \frac{\cancel{x}\ 1}{x\sqrt{x^2-1}}$$

$$\boxed{\therefore \frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}} \rightarrow 16$$

17. <u>Derivative of $\operatorname{cosec}^{-1}x$</u>

Let $y = f(x) = \operatorname{cosec}^{-1}x$

$\Rightarrow x = \operatorname{cosec} y$

$y+k = f(x+h) = \operatorname{cosec}^{-1}(x+h)$

$\Rightarrow x+h = \operatorname{cosec}(y+k)$

where h and k are small increments in x and y.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\operatorname{cosec}^{-1}(x+h) - \operatorname{cosec}^{-1}x}{h}$$

$$= \lim_{h \to 0} \frac{\operatorname{cosec}^{-1}(x+h) - \operatorname{cosec}^{-1}x}{(x+h)-x}$$

$$= \lim_{k \to 0} \frac{y+k-y}{\operatorname{cosec}(y+k) - \operatorname{cosec} y} \quad (\because h \to 0 \Rightarrow k \to 0)$$

1. $$\lim_{x\to 0} \frac{\operatorname{cosec} x - \cot x}{x}$$

$$= \lim_{x\to 0} \frac{\frac{1}{\sin x} - \frac{\cos x}{\sin x}}{x} = \lim_{x\to 0} \frac{1-\cos x}{x \sin x}$$

$$= \lim_{x\to 0} \frac{(1-\cos x)(1+\cos x)}{x \sin x (1+\cos x)} = \lim_{x\to 0} \frac{1-\cos^2 x}{x \sin x (1+\cos x)}$$

$$= \lim_{x\to 0} \frac{\sin^2 x}{x \cdot \sin x (1+\cos)} = \lim_{x\to 0} \frac{\sin x}{x(1+\cos x)}$$

$$= \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{1}{(1+\cos x)} = \lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \frac{1}{(1+\cos x)}$$

$$= 1 \times \frac{1}{2} = \frac{1}{2} \text{ (ans)}$$

$$= \lim_{K\to 0} \frac{K}{\frac{1}{\sin(y+K)} - \frac{1}{\sin y}}$$

$$= \lim_{K\to 0} \frac{K \cdot \sin(y+K) \cdot \sin y}{\sin y - \sin(y+K)}$$

$$= \lim_{K\to 0} \frac{K \cdot \sin(y+K) \cdot \sin y}{2\cos\left(y+\frac{K}{2}\right) \cdot \sin\left(-\frac{K}{2}\right)} = -\lim_{K\to 0} \frac{\frac{K}{2} \cdot \sin(y+K) \cdot \sin y}{\cos\left(y+\frac{K}{2}\right) \cdot \sin\frac{K}{2}}$$

$$= -\lim_{K\to 0} \frac{\sin(y+K) \cdot \sin y}{\cos\left(y+\frac{K}{2}\right) \cdot \frac{\sin\frac{K}{2}}{\frac{K}{2}}}$$

$$= -\lim_{K\to 0} \frac{\sin(y+K) \cdot \sin y}{\cos\left(y+\frac{K}{2}\right)} \cdot \lim_{K\to 0} \frac{1}{\frac{\sin\frac{K}{2}}{\frac{K}{2}}}$$

$$= -\frac{\sin^2 y}{\cos y} \cdot (1) = -\frac{\sin^2 y}{\sqrt{1-\sin^2 y}} = -\frac{\frac{1}{\operatorname{cosec}^2 y}}{\sqrt{1-\frac{1}{\operatorname{cosec}^2 y}}}$$

$$= -\left(\frac{1}{\frac{\operatorname{cosec}^2 y}{\operatorname{cosec} y}} \times \frac{\operatorname{cosec} y}{\sqrt{\operatorname{cosec}^2 y - 1}}\right) = \frac{-1}{x\sqrt{x^2-1}}$$

$$\therefore \boxed{\frac{d}{dx}(\operatorname{cosec}^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}} \longrightarrow 17$$

Jay Jagannath

1. Find the derivative of

$$ax^3 + bx^2 + cx + d$$

Ans :

$$\frac{d}{dx}(ax^3 + bx^2 + cx + d)$$

$$= \frac{d}{dx}(ax^3) + \frac{d}{dx}(bx^2) + \frac{d}{dx}(cx) + \frac{d}{dx}(d)$$

$$= a\frac{d}{dx}(x^3) + b\frac{d}{dx}(x^2) + c\frac{d}{dx}(x) + 0$$

$$= a(3x^2) + b(2x) + c(1)$$

$$= 3ax^2 + 2bx + c$$

2. Find the derivative Of $x\sin x$

Ans

$$\frac{d}{dx}(x\sin x)$$

$$= x.\frac{d}{dx}(\sin x) + \sin x.\frac{d}{dx}(x)$$

$$= x\cos x + \sin x.(1)$$ (by Product Rule)

$$= x\cos x + \sin x$$

Find the derivative of $\frac{x^2}{\sin x}$

Ans

$$\frac{d}{dx}\left(\frac{x^2}{\sin x}\right)$$

$$= \frac{\sin x \frac{d}{dx}(x^2) - x^2 \frac{d}{dx}(\sin x)}{\sin^2 x}$$

(by quotient Rule)

$$= \frac{\sin x.(2x) - x^2 (\cos x)}{\sin^2 x}$$

$$= \frac{2x \sin x - x^2 \cos x}{\sin^2 x}$$

1. Find the derivative of $\frac{\tan x}{(x^2+1)\sec x}$

Ans

$$\frac{d}{dx}\left(\frac{\tan x}{(x^2+1)\sec x}\right)$$

$$= \frac{(x^2+1)\sec x . \frac{d}{dx}(\tan x) - \tan x . \frac{d}{dx}(x^2+1)\sec x}{\{(x^2+1)^2 \sec^2 x\}}$$ by quotient Rule.

$$= \frac{(x^2+1)\sec x . (\sec^2 x) - \tan x\left((x^2+1)\frac{d}{dx}(\sec x) + \sec x \frac{d}{dx}(x^2+1)\right)}{(x^2+1)^2 \sec^2 x}$$

by product Rule.

$$= \frac{(x^2+1)\sec^3 x - \tan x\left((x^2+1)\sec x.\tan x + \sec x.\ 2x\right)}{(x^2+1)^2 \sec^2 x}$$

$$= \frac{(x^2+1)\sec^2 x - \tan^2 x\{(x^2+1) - 2x\tan x}{(x^2+1)^2 \sec x}$$

$$= \frac{(x^2+1)(\sec^2 x - \tan^2 x) - 2x\tan x}{(x^2+1)^2 \sec x}$$

$$= \frac{(x^2+1) - (2x\tan x)}{(x^2+1)^2 \sec x}$$

5. Find the derivative of $(x^2+1)(\sin x+\cos x)(\tan x+\cot x)$

Ans: $\frac{d}{dx}\left\{(x^2+1)(\sin x+\cos x)(\tan x+\cot x)\right\}$

$= (x^2+1)(\sin x+\cos x)\frac{d}{dx}(\tan x+\cot x) +$
$(x^2+1)(\tan x+\cot x)\frac{d}{dx}(\sin x+\cos x) +$
$(\sin x+\cos x).(\tan x+\cot x)\frac{d}{dx}(x^2+1)$

(by product Rule)

$(x^2+1)(\sin x+\cos x).(\sec^2 x-\csc^2 x)$ +
$(x^2+1)(\tan x+\cot x)(\cos x-\sin x)$ +
$(\sin x+\cos x)(\tan x+\cot x)\cancel{(x^2+1)}(2x)$.

(Ans)

11-(d) exceptn. - 35
11-(a) - 1, 2.

1st principle

Find the derivative of $\frac{\tan x}{e^x}$ by first principle.

Ans

Let $f(x)=\frac{\tan x}{e^x}$

$f(x+h)=\frac{\tan(x+h)}{e^{x+h}}$ where h is small increment in x

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\frac{\tan(x+h)}{e^{x+h}}-\frac{\tan x}{e^x}}{h}$$

$$=\lim_{h\to 0}\frac{\tan(x+h)-e^h.\tan x}{h.e^{x+h}}$$

$$= \lim_{h \to 0} \frac{\tan(x+h) - \tan x + \tan x - e^h . \tan x}{h . e^{x+h}}$$

$$= \lim_{h \to 0} \frac{\left(\tan(x+h) - \tan x\right) - \tan x\left(e^h - 1\right)}{h . e^{x+h}}$$

$$= \lim_{h \to 0} \frac{\frac{\tan(x+h) - \tan x}{h} - \frac{\tan x\,(e^h - 1)}{h}}{e^{x+h}}$$

$$= \lim_{h \to 0} \frac{\frac{1}{h} \cdot \left(\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x}\right) - \frac{\tan x\,(e^h - 1)}{h}}{e^{x+h}}$$

$$= \lim_{h \to 0} \frac{\frac{\sin(x+h-x)}{h . \cos(x+h) . \cos x} - \frac{\tan x\,(e^h - 1)}{h}}{e^{x+h}}$$

$$= \lim_{h \to 0} \frac{\frac{\sin h}{h} . \frac{1}{\cos(x+h) . \cos x} - \frac{\tan x\,(e^h - 1)}{h}}{e^{x+h}}$$

$$= \frac{1 . \frac{1}{\cos^2 x} - \tan x}{e^x} = \frac{\sec^2 x - \tan x}{e^x}$$

Ans

$$\frac{d}{dx}\left(\frac{\ln x}{x^2}\right)$$

$$= \frac{x^2 \cdot \frac{d}{dx}\ln x - \ln x \cdot \frac{d}{dx}x^2}{x^4}$$

$$= \frac{x^2 \cdot \frac{1}{x} - \ln x \cdot 2x}{x^4}$$

$$= \frac{x - 2x\ln x}{x^4}$$

$$= \frac{\cancel{x}(1-2\ln x)}{\cancel{x^4}\, x^3}$$

Imp

Chain Rule

The Chain rule is applied for function Of a function OR composite function.

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot \frac{d}{dx} g(x)$$

$$= \frac{d}{dg(x)} f(g(x)) \cdot \frac{d}{dx} g(x)$$

Ex

$$\frac{d}{dx} (\tan x^2)$$

$$= \sec^2 x^2 \cdot \frac{d}{dx} x^2 \quad \text{by Chain rule}$$

$$= \sec^2 x^2 \cdot (2x)$$

$$= 2x \sec^2 x^2$$

2. $$\frac{d}{dx} \sin(x^3 + 5x + 6)$$

$$= \cos (x^3 + 5x + 6) \cdot \frac{d}{dx} (x^3 + 5x + 6) \quad \text{by chain Rule}$$

$$= \cos (x^3 + 5x + 6) \cdot (3x^2 + 5 + 0)$$

$$= (3x^2 + 5) \cos (x^3 + 5x + 6)$$

$\frac{d}{dx} \sec(x^3+x^2)$

$= \sec(x^3+x^2) \cdot \tan(x^3+x^2) \cdot \frac{d}{dx}(x^3+x^2)$ by chain rule

$= \sec(x^3+x^2) \cdot \tan(x^3+x^2) \cdot (3x^2+2x)$

$= (3x^2+2x) \sec(x^3+x^2) \cdot \tan(x^3+x^2)$

4. $\frac{d}{dx}(x^4+x^2+5x+8)^{200}$

$= 200 \cdot (x^4+x^2+5x+8)^{199} \cdot \frac{d}{dx}(x^4+x^2+5x+8)$ by chain rule

$= 200(x^4+x^2+5x+8)^{199} \cdot (4x^3+2x+5)$

5. $\frac{d}{dx} e^{x^2+5x+3}$

$= e^{x^2+5x+3} \cdot \frac{d}{dx}(x^2+5x+3)$ by chain rule

$= e^{x^2+5x+3} (2x+5)$

$= (2x+5) e^{x^2+5x+3}$

6. $\frac{d}{dx} a^{x^2+5x+3}$

$= a^{x^2+5x+3} . \log a \cdot \frac{d}{dx}(x^2+5x+3)$

by chain rule

$= a^{x^2+5x+3} . \log a \; (2x+5)$

$= \cancel{2}$

7. $\frac{d}{dx} \log (x^2+5x+3)$

$= \frac{1}{x^2+5x+3} \cdot \frac{d}{dx}(x^2+5x+3)$

by chain rule

$= \frac{2x+5}{x^2+5x+3}$

8. $\frac{d}{dx} \sin^{-1}(x^2+5x+3)$

$= \frac{1}{\sqrt{1-(x^2+5x+3)^2}} \cdot \frac{d}{dx}(x^2+5x+3)$

$= \frac{2x+5}{\sqrt{1-(x^2+5x+3)^2}}$

8. $\frac{d}{dx}(\tan^2 x)$

$= \frac{d}{dx}(\tan x)^2$

$= 2\tan x \cdot \frac{d}{dx}(\tan x)$

$= 2\tan x \cdot \sec^2 x$

$= 2\tan x\,(1+\tan^2 x)$

---

9. $\frac{d}{dx}(x-1)^2$

$= 2(x-1) \cdot \frac{d}{dx}(x-1)$

$= 2(x-1)\,(1)$

$= 2(x-1)$

10. $\frac{d}{dx}(x^2-x+2)^2$

$= 2\cdot(x^2-x+2)\cdot\frac{d}{dx}(x^2-x+2)$

$= 2(x^2-x+2)(2x-1)$ — Ans

1.6 $\frac{d}{dx}\left(\frac{x-1}{x+1}\right)^2$

$= 2\left(\frac{x-1}{x+1}\right) \cdot \frac{d}{dx}\left(\frac{x-1}{x+1}\right)$

(by chain rule)

$= 2\left(\frac{x-1}{x+1}\right)\left\{\frac{(x+1)\frac{d}{dx}(x-1) - (x-1)\frac{d}{dx}(x+1)}{(x+1)^2}\right\}$

$= 2\left(\frac{x-1}{x+1}\right)\left\{\frac{(x+1)(1) - (x-1)(1)}{(x+1)^2}\right\}$

$= 2\left(\frac{x-1}{x+1}\right)\frac{2}{(x+1)^2}$

$= 4\,\frac{(x-1)}{(x+1)^3}$ —— Ans

---

18. $\frac{d}{dx}\left(x^3 \sin x \, e^{4\ln x}\right)$

$= \frac{d}{dx}\left(x^3 \sin x \, e^{\ln x^4}\right)$

$= \frac{d}{dx}\left(x^3 \cdot \sin x \, x^4\right)$

$= \frac{d}{dx}\left(x^7 \cdot \sin x\right)$

$= x^7 \frac{d}{dx}\sin x + \sin x \cdot \frac{d}{dx}(x^7)$

$= x^7 \cos x + 7x^6 \sin x$ by product Rule

-(e)

5. $\frac{d}{dt} e^{\sin t}$

$= e^{\sin t} \cdot \frac{d}{dt}(\sin t)$

$= e^{\sin t} \cdot \cos t$

6. $\frac{d}{dx}\sqrt{ax^2+bx+c}$

$= \frac{d}{dx}(ax^2+bx+c)^{\frac{1}{2}}$

$= \frac{1}{2} \cdot (ax^2+bx+c)^{-\frac{1}{2}} \cdot \frac{d}{dx}(ax^2+bx+c)$

$= \frac{1}{2} \cdot (ax^2+bx+c)^{-\frac{1}{2}} (2ax+b)$

$= \frac{2ax+b}{2\sqrt{ax^2+bx+c}}$

---

1. $\frac{d}{dx} \sin(e^x \log x)$

$= \cos(e^x \log x) \cdot \frac{d}{dx}(e^x \log x)$ by chain rule

$= \cos(e^x \cdot \log x)\{e^x \cdot \frac{1}{x} + \log x \cdot e^x\}$

$= \cos(e^x \log x) \cdot e^x (\frac{1}{x} + \log x)$

2. $\frac{d}{dx}\left(\sec \log\sqrt{a^2+x^2}\right)$

$$= \frac{d}{dx}\left\{\sec\left(\frac{1}{2}\log(a^2+x^2)\right)\right\}$$

$$= \sec\left(\frac{1}{2}\log(a^2+x^2)\right) \cdot \tan\left\{\frac{1}{2}\log(a^2+x^2)\right\}$$

$$\cdot \frac{d}{dx}\frac{1}{2}\log(a^2+x^2)$$ by chain rule

$$= \sec\left(\log\sqrt{a^2+x^2}\right)\tan\left(\log\sqrt{a^2+x^2}\right)$$

$$\cdot \frac{d}{dx}\frac{1}{2}\log(a^2+x^2)$$

$$= \left\{\frac{\sec\left(\log\sqrt{a^2+x^2}\right) \cdot \tan\left(\log\left(\sqrt{a^2+x^2}\right)\right)}{2}\right\}$$

$$\times \frac{d}{dx}\log(a^2+x^2)$$

$$= \frac{\sec\left(\log\sqrt{a^2+x^2}\right) \cdot \tan\left(\log\sqrt{a^2+x^2}\right)}{2} \times$$

$$\frac{1}{a^2+x^2} \cdot \frac{d}{dx}(a^2+x^2)$$ by chain rule again.

$$= \left(\frac{\sec\left(\log\sqrt{a^2+x^2}\right) \cdot \tan\log\sqrt{a^2+x^2}}{\cancel{2}(a^2+x^2)}\right)(0+\cancel{2}x)$$

$$= \frac{x\sec\left(\log\sqrt{a^2+x^2}\right) \cdot \tan\left(\log\sqrt{a^2+x^2}\right)}{a^2+x^2}$$

$U_r(x)$

$$y = \frac{x^{-3}\tan^{-1}x}{x^2+3e^x} - 5^x\tan x - 7\sin x$$

$$\frac{d}{dx}\left(\frac{x^{-3}\tan^{-1}x}{x^2+3e^x} - 5^x\tan x - 7\sin x\right)$$

$$= \frac{d}{dx}\left(\frac{x^{-3}\tan^{-1}x}{x^2+3e^x}\right) - \frac{d}{dx}(5^x\tan x) - \frac{d}{dx}7\sin x$$

$$= \frac{x^2+3e^x\frac{d}{dx}(x^{-3}.\tan^{-1}x) - (x^{-3}.\tan^{-1}x).\frac{d}{dx}(x^2.3e^x)}{x^4\ 9e^{2x}}$$

$$- \{\cancel{\ }\ \{5^x(\sec^2x) + \tan x.\ 5^x\log 5\}$$

$$- 7\cos x$$

$$= \frac{x^2\ 3e^x\ \{x^{-3}.\left(\frac{1}{1+x^2}\right) + \tan^{-1}x(-3.x^{-4}) - (x^{-3}.\tan^{-1}x).(2x - 3e^x)}{x^4\ .9e^{2x}}$$

$$- 5^x\sec^2x + \tan x\quad 5^x\log 5 - 7\cos x$$

$$= \frac{x^3.3e^x.\left(\frac{1}{x^3+x^5}\right) - \frac{3x}{x^4}(\tan^{-1}x) - \frac{\tan^{-1}x}{x^3}(2x-3e^x)}{x^4\ 9e^{2x}}$$

$$- 5^x\sec^2x + \tan x\ 5^x\log 5 - 7\cos x$$

(b) $\log\left(\sin^{-1}\ln\cos^{-1}(x+y)\right)$

Note :

Sometimes we first simplify the function as far as practicable then we take the derivative. These things are generally applicable in trigonometric or inverse trigonometric functions.

Ex : Find the derivative of

$\sin^{-1}\sqrt{1-x^2}$

Ans

$$\frac{d}{dx}\sin^{-1}\sqrt{1-x^2}$$

$$= \cancel{\frac{1}{\sqrt{1-(1-x^2)^2}}} = \frac{1}{\sqrt{1-(\sqrt{1-x^2})^2}}\cdot\frac{d}{dx}\sqrt{1-x^2}$$

$$= \frac{1}{\sqrt{1-1+x^2}}\cdot\frac{d}{dx}(1-x^2)^{\frac{1}{2}}$$

$$= \frac{1}{\sqrt{x^2}}\cdot\frac{1}{2}\cdot(1-x^2)^{\frac{1}{2}-1}\cdot\frac{d}{dx}(1-x^2)$$

$$= \frac{1}{\cancel{x}}\cdot\frac{1}{\cancel{2}}(1-x^2)^{\frac{-1}{2}}\,\cancel{(-2x)}(-1)$$

$$= \frac{-1}{\sqrt{1-x^2}} \quad \text{(ans)}$$

Alternative method

$\cancel{\frac{d}{dx} \sin}$ Now $\sin^{-1} \sqrt{1-x^2}$

Put $x = \cos\theta$, $\theta = \cos^{-1} x$

$= \sin^{-1} \sqrt{1-\cos^2\theta}$

$= \sin^{-1} \sin\theta$

$= \theta$

$= \cos^{-1} x$

$\therefore \frac{d}{dx} \sin^{-1} \sqrt{1-x^2}$

$= \frac{d}{dx} \cos^{-1} x$

$= -\frac{1}{\sqrt{1-x^2}}$ (ans)

Note

a = constant, x = variable

If $a^2 - x^2$ then put $x = a\sin\theta$

or $x = a\cos\theta$

If $a^2 + x^2$ then put

$x = a\tan\theta$

OR $x = a\cot\theta$

If $x^2 - a^2$ then put

$x = a\sec\theta$

OR $x = a\operatorname{cosec}\theta$

Ex

Find the derivative of

$$\sqrt{\frac{1-\sin x}{1+\sin x}}$$

Ans

$$\frac{d}{dx}\sqrt{\frac{1-\sin x}{1+\sin x}}$$

$$= \frac{d}{dx}\sqrt{\frac{(1-\sin x)^2}{1-\sin^2 x}}$$

$$= \frac{d}{dx}\sqrt{\frac{(1-\sin x)^2}{\cos^2 x}}$$

$$= \frac{d}{dx}\left(\frac{1-\sin x}{\cos x}\right)$$

$$= \frac{d}{dx}(\sec x - \tan x)$$

$$= \sec x \cdot \tan x - \sec^2 x$$

$$= \sec x(\tan x - \sec x)$$

Ex Find the derivative of

$$\tan^{-1}\left[\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right]$$

~~= $\tan^{-1}$~~

$$\frac{d}{dx}\left[\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right]$$

Put

$x^2 = \cos 2\theta$

$\Rightarrow 2\theta = \cos^{-1}x^2$

$\Rightarrow \theta = \frac{1}{2}\cos^{-1}x^2$

$$\frac{d}{dx}\tan^{-1}\left[\frac{\sqrt{1+\cos2\theta}+\sqrt{1-\cos2\theta}}{\sqrt{1+\cos2\theta}-\sqrt{1-\cos2\theta}}\right]$$

$$=\frac{d}{dx}\tan^{-1}\left[\frac{\sqrt{2\cos^2\theta}+\sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta}-\sqrt{2\sin^2\theta}}\right]$$

$$=\frac{d}{dx}\tan^{-1}\left[\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right]$$

$$=\frac{d}{dx}\tan^{-1}\left[\frac{1+\tan\theta}{1-\tan\theta}\right]$$

$$=\frac{d}{dx}\tan^{-1}\left[\tan\left(\frac{\pi}{4}+\theta\right)\right]$$

$$=\frac{d}{dx}\left(\frac{\pi}{4}+\theta\right)$$

$$=\frac{d}{dx}\left(\frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2\right)$$

$$=0+\frac{1}{2}\frac{d}{dx}\cos^{-1}x^2$$

$$=\frac{1}{2}\left\{\frac{(-1)}{\sqrt{1-x^4}}\right\}\cdot\frac{d}{dx}(x^2)$$

$$=\frac{1}{\cancel{2}}\cdot\frac{-1}{\sqrt{1-x^4}}\cdots(\cancel{2}x)$$

$$=\frac{-x}{\sqrt{1-x^4}}\quad\text{(Ans.)}$$

11 - F, G

(11- G)

Ans. (1) ~~x = sin a~~ 1. Put x = sin a

1. $\frac{d}{dx} \sin^{-1} 2x\sqrt{1-x^2}$

Put
$x = \sin a$
$a = \sin^{-1} x$

$= \frac{d}{dx} \sin^{-1} 2\sin\theta . \cos a$

$= \frac{d}{dx} \sin^{-1} \sin 2\theta$

$= \frac{d}{dx} 2a$

$= \cancel{2 \frac{d}{dx} \theta} = \frac{d}{dx} 2\sin^{-1} x$

$= 2 . \frac{d}{dx} \sin^{-1} x$

$= 2 . \frac{1}{\sqrt{1-x^2}}$

$= \frac{2}{\sqrt{1-x^2}}$ → Ans

5. $\frac{d}{dx} \tan^{-1} \left(\frac{\sqrt{x}+\sqrt{a}}{1-\sqrt{x}.\sqrt{a}} \right)$

Put
$\sqrt{x} = \tan\theta$
$\sqrt{a} = \tan\beta$
$\theta = \tan^{-1}\sqrt{x}$
$\beta = \tan^{-1}\sqrt{a}$

$= \frac{d}{dx} \tan^{-1} \left(\frac{\tan a + \tan\beta}{1 - \tan a . \tan\beta} \right)$

$= \frac{d}{dx} \tan^{-1} \tan(\theta + \beta)$

$= \frac{d}{dx}(\theta + \beta)$

$= \frac{d}{dx}\left(\tan^{-1}\sqrt{x} + \tan^{-1}\sqrt{a}\right)$

$= \frac{d}{dx}\tan^{-1}\sqrt{x} + 0$

$= \cancel{\frac{1}{\sqrt{1+x}}} = \frac{1}{1+(\sqrt{x})^2} \cdot \frac{d}{dx}(\sqrt{x})$

$= \frac{1}{1+x} \cdot \frac{1}{2} \cdot x^{-\frac{1}{2}}$

$= \frac{1}{(1+x)} \frac{1}{(2\sqrt{x})}$

4. $t^2 = \cos 2\theta$

$\frac{d}{dt}\left[\frac{1+t^2}{1-t^2} - 1\right]^{\frac{1}{2}}$

$= \frac{d}{dt}\left[\frac{1+\cos 2\theta}{1-\cos 2\theta} - 1\right]^{\frac{1}{2}}$

$= \frac{d}{dt}\left[\frac{2\cos^2\theta}{2\sin^2\theta} - 1\right]^{\frac{1}{2}}$

$= \frac{d}{dt}\left(\cot^2\theta - 1\right)^{\frac{1}{2}}$

put
$t^2 = \cos 2\theta$
$2\theta = \cos^{-1} t^2$
$\Rightarrow \theta = \frac{1}{2}\cos^{-1} t^2$

$$= \frac{d}{dt} \quad 4 \cdot \frac{d}{dt}\left[\left(\frac{1+t^2}{1-t^2}\right) - 1\right]^{\frac{1}{2}}.$$

$$= \frac{d}{dt}\left(\frac{\cancel{1}+t^2 - \cancel{1}+t^2}{1-t^2}\right)^{\frac{1}{2}}$$

$$= \frac{d}{dt}\left(\frac{2t^2}{1-t^2}\right)^{\frac{1}{2}}$$

$$= \frac{d}{dt} \frac{\sqrt{2}\,t}{\sqrt{1-t^2}}$$

$$= \sqrt{1-t^2} \cdot \frac{d}{dt}\sqrt{2}\,t - \sqrt{2}\,t \cdot \frac{d}{dt}$$

Derivative of $f(x)$ with respect to $g(x)$ $= \frac{d}{dg(x)} f(x) = \frac{d\, f(x)}{d\, g(x)}$

$$= \frac{\frac{d\, f(x)}{dx}}{\frac{d\, g(x)}{dx}} \quad \text{—— (26)}$$

Ex Find the derivative of

~~$\tan(1+x)^2$ with respect to $(1+x)^2$~~ ~~$1+x^2$~~

$\tan(1+x)^2$ w.r. to $1+x^2$

Ans Derivative of $\tan(1+x)^2$ w.r.to $1+x^2$

$$= \frac{d \tan(1+x)^2}{d\, (1+x^2)}$$

$$= \frac{\frac{d}{dx} \tan(1+x)^2}{\frac{d}{dx}(1+x^2)}$$

$$= \frac{\cancel{2} \cdot sec^2(1+x)^2 \cdot \frac{d}{dx}(1+x)^2}{0+2x}$$

$$= \frac{sec^2(1+x)^2 \cdot \cancel{2}(1+x) \cdot \frac{d}{dx}(1+x)}{\cancel{2}x}$$

$$= \frac{sec^2(1+x)^2 \; (1+x) \; (1)}{(x)}$$

$$= \frac{(1+x) \; sec^2(1+x)^2}{x} \quad \text{(Ans)}$$

Ex (2)

Find the derivative of $\sin x$ with r.t $\tan x$

Ans

Derivative of $\sin x$ w.r. to $\tan x$.

$$= \frac{d \sin x}{d \tan x}$$

$$= \frac{\frac{d}{dx}\sin x}{\frac{d}{dx}\tan x} = \frac{\cos x}{sec^2 x} = \frac{\cos x}{\frac{1}{\cos^2 x}} = \cos^3 x$$

(Ans)

Exercise → 12(k) 1 (j)

5. Derivative of $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ w.r.t $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

$$= \frac{d \sin^{-1}\left(\frac{2x}{1+x^2}\right)}{d \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)}$$

$$= \frac{\frac{d}{dx} \sin^{-1}\left(\frac{2x}{1+x^2}\right)}{\frac{d}{dx} \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)}$$

$$= \frac{\frac{d}{dx} 2\tan^{-1}x}{\frac{d}{dx} 2\tan^{-1}x}$$

$\left(\because 2\tan^{-1}x = \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) \right)$

$$= 1$$

Derivative Of parametric equations

$x = f(t)$ and $y = g(t)$ are called parametric eq^{n}s and t is called parameter.

$$\boxed{\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{g'(t)}{f'(t)}} \rightarrow 27$$

Ex

Find $\frac{dy}{dx}$ when $x = at^2$ and $y = 2at$

Solⁿ

$x = at^2$

$\Rightarrow \frac{dx}{dt} = \frac{d}{dt}(at^2) = a\frac{d}{dt}(t^2) = a(2t) = 2at$

$y = 2at$

$\Rightarrow \frac{dy}{dt} = \frac{d}{dt}(2at) = 2a\frac{d}{dt}(t) = 2a\cdot(1) = 2a$

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2a}{2at} = \frac{1}{t}$

$\left.\frac{dy}{dx}\right]_{t=\frac{1}{2}} = \frac{1}{1/2} = 2$ (Ans)

Find $\frac{dy}{dx}$ when $x = a\log(\sin\theta - \frac{1}{3}\sin 3\theta)$

$y = a\log(\cos\theta + \frac{1}{3}\cos 3\theta)$

Sol^n

$x = a\log\left(\sin\theta - \frac{1}{3}\sin 3\theta\right)$

$= a\log\left(\sin\theta - \frac{1}{3}(3\sin\theta - 4\sin^3\theta)\right)$

$= a\log\left(\sin\theta - \sin\theta + \frac{4}{3}\sin^3\theta\right)$

$= a\log\left(\frac{4}{3}\sin^3\theta\right)$

$= a\left(\log\frac{4}{3} + \log\sin^3\theta\right)$

$= a\left\{\log\frac{4}{3} + \log(\sin\theta)^3\right\}$

$= a\left\{\log\frac{4}{3} + 3\log\sin\theta\right\}$

$\frac{dx}{d\theta} = a\left[\frac{d}{d\theta}\left(\log\frac{4}{3}\right) + \frac{d}{d\theta}3\log\sin\theta\right]$

$= a\left[0 + 3 \times \frac{d}{d\theta}\log\sin\theta\right]$

$= 3a\left[\frac{1}{\sin\theta} \cdot \frac{d}{d\theta}\sin\theta\right]$

$= 3a\left[\frac{1}{\sin\theta} \cdot \cos\theta\right]$

$= 3a\cot\theta$

$$y = a \log \left(\cos\theta + \frac{1}{3} \cos 3\theta \right)$$

$$= a \log \left\{ \cos\theta + \frac{1}{3} \left(4\cos^3\theta - 3\cos\theta \right) \right\}$$

$$= a \log \left(\cancel{\cos\theta} + \frac{4}{3} \cos^3\theta - \cancel{\cos\theta} \right)$$

$$= a \log \left(\frac{4}{3} \cos^3\theta \right)$$

$$= a \left(\log \frac{4}{3} + \log \cos^3\theta \right)$$

$$= a \left(\log \frac{4}{3} + \log (\cos\theta)^3 \right)$$

$$= a \left(\log \frac{4}{3} + 3 \log \cos\theta \right)$$

$$\frac{dy}{d\theta} = a \left[\frac{d}{d\theta} \log \frac{4}{3} + \frac{d}{d\theta} 3 \log \cos\theta \right]$$

$$= a \left[0 + 3 \cdot \frac{d}{d\theta} \log \cos\theta \right]$$

$$= 3a \left(\frac{1}{\cos\theta} \cdot \frac{d}{d\theta} \cos\theta \right)$$

$$= 3a \left(\frac{1}{\cos\theta} \cdot (-\sin\theta) \right)$$

$$= -3a \tan\theta$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-3a\tan\theta}{3a\cot\theta} = -\tan^2\theta$$

(Ans)

Find the derivative of $\sin x^2$ by first principle.

Let $f(x) = \sin x^2$

$f(x+h) = \sin(x+h)^2$ where h is small increment in x.

$f'(x)$

$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h \to 0} \frac{\sin(x+h)^2 - \sin x^2}{h}$

$\boxed{= \lim_{h \to 0} \cancel{\sin(x+h+x)} \cdot \sin(}$

$= \lim_{h \to 0} \frac{\{\sin(x+h)^2 - \sin x^2\}\{(x+h)^2 - x^2\}}{h\{(x+h)^2 - x^2\}}$

$= \lim_{h \to 0} \frac{2\cos\frac{(x+h)^2+x^2}{2} \cdot \sin\frac{(x+h)^2 - x^2}{2} \cdot (h^2+2hx)}{h(h^2+2hx)}$

$= \lim_{h \to 0} \frac{2\cos\left(\frac{(x+h)^2+x^2}{2}\right)(h^2+2hx) \cdot \sin\frac{h^2+2hx}{2}}{h\left(\frac{h^2+2hx}{2}\right)}$

$= \lim_{h \to 0} \cos\frac{(x+h)^2+x^2}{2} \cdot \frac{(h^2+2hx)}{h} \cdot \frac{\sin\frac{h^2+2hx}{2}}{\frac{h^2+2hx}{2}}$

$$= \lim_{h\to 0} \cos\left(\frac{(x+h)^2 + x^2}{2}\right).(h+2x).\ \frac{\sin \frac{h^2+2hx}{2}}{\frac{h^2+2hx}{2}}$$

$$= \cos\left(\frac{x^2+x^2}{2}\right).(2x).(1)$$

$$= \cos x^2 .(2x).$$

$$= 2x \cos x^2$$

Proof Of Chain rule by 1st principle

$$\frac{d}{dx} f(g(x)) = \frac{d}{dg(x)} f(g(x)) \cdot \frac{d}{dx} g(x)$$

Proof

~~Let f(x) =~~

$$\text{Let } \phi(x) = f(g(x))$$

$$\phi(x+h) = f(g(x+h))$$

where h is small increment in g

$$\phi'(x) = \lim_{h\to 0} \frac{\phi(x+h) - \phi(x)}{h}$$

$$= \lim_{h\to 0} \frac{\{f(g(x+h)) - f(g(x))\}\{g(x+h) - g(x)\}}{h\{g(x+h) - g(x)\}}$$

$$= \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \cdot \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \cdot \frac{d}{dx} g(x)$$

~~Put~~
~~$g(x) = y$~~
~~$g(x+h) = y+h$~~
~~$h\to 0 \therefore y \to g(x)$~~

$\lim_{y \to 0} \dfrac{f(g(x)+y) - f(g(x))}{y} \cdot \dfrac{dg(x)}{dx}$

put
$g(x+h) - g(x) = y$
$\Rightarrow g(x+h) = y + g(x)$
$h \to 0 \Rightarrow y \to 0$

$= \dfrac{d}{dx} f(g(x)) \cdot \dfrac{d}{dx} g(x)$

(proved)

$\dfrac{d}{dx} x^y = \cancel{\dfrac{d}{dx} x^5} = 5x^4 = y x^{y-1}$

$\dfrac{d}{dx} y^x = \cancel{\dfrac{d}{dx} a^x} = y^x \cdot \ln y$

$\dfrac{d}{dx} x^y = \dfrac{-f_x}{f_y} = -\left(\dfrac{y \cdot x^{y-1}}{x^y \ln x} \right)$

$\dfrac{d}{dx} y^x = \dfrac{-f_x}{f_y} = -\left(\dfrac{y^x \ln y}{x \cdot y^{x-1}} \right)$

$$\lim_{n \to \infty} \left(1 + \frac{1}{e} + \frac{1}{e^2} + \frac{1}{e^3} \text{------} \frac{1}{e^n}\right)$$

$$\lim_{n \to \infty} \frac{1\left(1 - \frac{1}{e^{n+1}}\right)}{1 - \frac{1}{e}} = \frac{1-0}{1-\frac{1}{e}}$$

$$= \frac{1}{\frac{e-1}{e}} = \frac{e}{e-1}$$

---

(28) → Derivative of implicit function

(1) $\frac{d}{dx} \sin y = \cos y \cdot \frac{dy}{dx}$

(2) $\frac{d}{dx} \log y = \frac{1}{\cancel{\log} y} \cdot \frac{dy}{dx}$

(3) $\frac{d}{dx} \tan^{-1} y = \frac{1}{1+y^2} \cdot \frac{dy}{dx}$

(4) $\frac{d}{dx} y^{50} = 50 \cdot y^{49} \cdot \frac{dy}{dx}$

(5) $\frac{d}{dx} e^y = e^y \cdot \frac{dy}{dx}$

(6) $\frac{d}{dx} a^y = a^y \log a \cdot \frac{dy}{dx}$

7. $\frac{d}{dx} \tan y = \sec^2 y \cdot \frac{dy}{dx}$

(12-i)

Find $\dfrac{dy}{dx}$ when $xy^2+x^2y+1=0$

(i) Ans

$$xy^2+x^2y+1=0$$

$$\Rightarrow \frac{d}{dx}(xy^2+x^2y+1)=\frac{d(0)}{dx}=0$$

$$\Rightarrow \frac{d}{dx}xy^2+\frac{d}{dx}x^2y+\frac{d}{dx}(1)=0$$

$$\Rightarrow \left(x\cdot\frac{d}{dx}y^2+y^2\cdot\frac{d(x)}{dx}\right)+\left(x^2\cdot\frac{dy}{dx}+y\cdot\frac{d}{dx}x^2\right)+0=0$$

$$\Rightarrow \left(x\cdot 2y\cdot\frac{dy}{dx}+y^2\right)+\left(x^2\cdot\frac{dy}{dx}+y\cdot 2x\right)=0$$

$$\Rightarrow 2xy\frac{dy}{dx}+x^2\cdot\frac{dy}{dx}=-(y^2+2xy)$$

$$\Rightarrow \frac{dy}{dx}(2xy+x^2)=-(y^2+2xy)$$

$$\Rightarrow \frac{dy}{dx}=-\frac{(y^2+2xy)}{(x^2+2xy)}$$

$$=\frac{-y(y+2x)}{x(2y+2x)}$$ (Ans)

5. Find $\dfrac{dy}{dx}$ when $y=\tan(xy)$

Soln

$$y=\tan(xy)$$

$$\Rightarrow \frac{dy}{dx}=\frac{d}{dx}\tan(xy)$$

$$\Rightarrow \frac{dy}{dx} = \sec^2 xy \cdot \frac{d}{dx}(xy)$$

$$\Rightarrow \frac{dy}{dx} = \sec^2 xy \left(x \cdot \frac{dy}{dx} + y \frac{dx}{dx} \right)$$

$$\Rightarrow \frac{dy}{dx} = \sec^2 xy \left(x \frac{dy}{dx} + y \right)$$

$$\Rightarrow \frac{dy}{dx} = \sec^2 xy \cdot \left(x \cdot \frac{dy}{dx} \right) + \sec^2 xy \cdot y$$

$$\Rightarrow \frac{dy}{dx} - x \cdot \sec^2 xy \cdot \frac{dy}{dx} = y \sec^2 xy$$

$$\Rightarrow \frac{dy}{dx}\left(1 - x \sec^2 xy \right) = y \sec^2 xy$$

$$\Rightarrow \frac{dy}{dx} = \frac{y \sec^2 xy}{1 - x \sec^2 xy} \quad \text{(Ans)}$$

Logarithmic Dibberentiations

It is appliable when the power contains variables or the function contains product, quotiens, and ditherent powers.

Ex: (1) Find the derivative of

x^x

Ans: Let $y = x^x$

$\Rightarrow \cancel{\log_{e} y = x}$

$\Rightarrow \log y = x \log x^x = x \log x$

$\Rightarrow \frac{d}{dx} \log y = \frac{d}{dx} x \log x$

$\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = x \frac{d}{dx} \log x + \log x . \frac{dx}{dx}$

$\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = (\cancel{x} \times \frac{1}{\cancel{x}}) + \log x (1)$

$= 1 + \log x$

$\Rightarrow \frac{dy}{dx} = y(1 + \log x)$

$\Rightarrow \frac{dy}{dx} = x^x (1 + \log x)$

(Ans)

(2) Find the derivative of $\boxed{x^x + x^{\frac{1}{x}}}$

Ans

~~Let $y = x^x + x^{\frac{1}{x}}$~~

~~$\Rightarrow \log y = \log(x^x + x^{\frac{1}{x}})$~~

~~$\Rightarrow \log y = \log x^x + \ell$~~

Let $y = x^x + x^{\frac{1}{x}}$

$= u + v$ (say)

where $u = x^x$, $v = x^{\frac{1}{x}}$

$\therefore \dfrac{dy}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx}$ ——(i)

Now $u = x^x$

Do it

$\dfrac{du}{dx} = x^x(1 + \log x)$

Now $v = x^{\frac{1}{x}}$

$\Rightarrow \log v = \log x^{\frac{1}{x}} = \frac{1}{x} \cdot \log x$

$\Rightarrow \dfrac{d}{dx}\log v = \dfrac{d}{dx}\left(\frac{1}{x} \cdot \log x\right)$

$\Rightarrow \dfrac{1}{v} \cdot \dfrac{dv}{dx} = \dfrac{1}{x} \cdot \dfrac{d}{dx}\log x + \log x \cdot \dfrac{d}{dx}\left(\frac{1}{x}\right)$

$\Rightarrow \dfrac{1}{v} \cdot \dfrac{dv}{dx} = \left(\dfrac{1}{x} \times \dfrac{1}{x}\right) + \log x \dfrac{d}{dx}(x^{-1})$

$= \dfrac{1}{x^2} + \log x \cdot (-1 \cdot x^{-2})$

$$\frac{1}{v}\cdot\frac{dv}{dx} = \frac{1}{x^2} - \log x \cdot x^{-2}$$

$$= \frac{1}{x^2} - \frac{\log x}{x^2}$$

$$= \frac{1}{x^2}(1-\log x)$$

$$\therefore \frac{dv}{dx} = \frac{v}{x^2}(1-\log x)$$

$$= \frac{x^{\frac{1}{x}}}{x^2}(1-\log x)$$

$$= x^{\frac{1}{x}-2}(1-\log x)$$

$$= \cancel{\frac{1}{x}}$$

From (i)

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

$$= x^x(1+\log x) + x^{\frac{1}{x}-2}(1-\log x)$$

(Ans)

Ans

Let $y = x^{\sin x}$

$\Rightarrow \log y = \log x^{\sin x} = \sin x \log x$

$\Rightarrow \frac{d}{dx} \log y = \frac{d}{dx} \sin x . \log x$

$= \sin x \frac{d}{dx} \log x + \log x . \frac{d}{dx} \sin x$

$\Rightarrow \frac{1}{y} . \frac{dy}{dx} = \sin x . \frac{1}{x} + \log x . \cos x$

$\Rightarrow \frac{dy}{dx} = y \left(\frac{\sin x}{x} + \log x \cos x \right)$

$= x^{\sin x} \left(\frac{\sin x}{x} + \log x \cos x \right)$

4. Find $\frac{dy}{dx}$ when $y^x = x^y$

Ans:

$y^x = x^y$

$\Rightarrow \log y^x = \log x^y$

$\Rightarrow x \log y = y \log x$

$$\Rightarrow \frac{d}{dx}(x \cdot \log y) = \frac{d}{dx}(y \cdot \log x)$$

$$\Rightarrow x \cdot \frac{d}{dx}\log y + \log y \cdot \frac{dx}{dx} = y \cdot \frac{d}{dx}\log x + \log x \cdot \frac{dy}{dx}$$

$$\Rightarrow \frac{x}{y}\frac{dy}{dx} + \log y = \frac{y}{x} + \log x \cdot \frac{dy}{dx}$$

$$\cancel{\Rightarrow \frac{dy}{dx} = \frac{\frac{x}{y} + \log y - \frac{y}{x}}{\log x}}$$

$$\Rightarrow \frac{x}{y}\frac{dy}{dx} - \log x \frac{dy}{dx} = \frac{y}{x} - \log y$$

$$\Rightarrow \frac{dy}{dx}\left(\frac{x}{y} - \log x\right) = \frac{y}{x} - \log y$$

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{y}{x} - \log y}{x/y - \log x}$$

$\boxed{11-b}$

Ex :- Find $\frac{dy}{dx}$ when $y = x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}$ ad inf

Ans

$$y = x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} = x^y$$

$$\Rightarrow \log y = \log x^y = y \log x$$

$$\Rightarrow \frac{d}{dx}\log y = \frac{d}{dx}(y \log x)$$

$$\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = y \cdot \frac{d}{dx}\log x + \log x \cdot \frac{dy}{dx}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = y \cdot \frac{1}{x} + \log x \cdot \frac{dy}{dx}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} - \log x \frac{dy}{dx} = \frac{y}{x}$$

$$\Rightarrow \frac{dy}{dx}\left(\frac{1}{y} - \log x\right) = \frac{y}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x\left(\frac{1}{y} - \log x\right)} = \frac{y^2}{x(1 - y\log x)}$$

Find $\frac{dy}{dx}$ when

$$y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \cdots}}}$$

Soln :

$$y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \cdots}}}$$

$$= \sqrt{\sin x + y}$$

$$\Rightarrow y^2 = \sin x + y$$

$$\Rightarrow \frac{d}{dx} y^2 = \frac{d}{dx}\sin x + \frac{dy}{dx}$$

$$\Rightarrow 2y \cdot \frac{dy}{dx} = \cos x + \frac{dy}{dx}$$

$$\Rightarrow 2y\frac{dy}{dx} - \frac{dy}{dx} = \cos x$$

$$\Rightarrow \frac{dy}{dx}(2y-1) = \cos x$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos x}{(2y-1)}$$

(Ans)

Ex:- Find the derivate at the indicated point.

$\sqrt{x} + \sqrt{y} = 3$ at $(1, 4)$

Ans: $\sqrt{x} + \sqrt{y} = 3$

$\Rightarrow \frac{d}{dx}\sqrt{x} + \frac{d}{dx}\sqrt{y} = \frac{d(3)}{dx} = 0$

$\Rightarrow \frac{1}{\cancel{2}} \cdot x^{-\frac{1}{2}} + \frac{1}{\cancel{2}} \cdot y^{-\frac{1}{2}} \cdot \frac{dy}{dx} = 0$

$\Rightarrow \frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}} \cdot \frac{dy}{dx} = 0$

$\Rightarrow \frac{1}{\sqrt{y}} \frac{dy}{dx} = -\frac{1}{\sqrt{x}}$

$\Rightarrow \frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}$

$\left.\frac{dy}{dx}\right]_{(1,4)} = -\frac{\sqrt{4}}{\sqrt{1}} = -\frac{2}{1} = -2$ (Ans)

Ex: Find the derivative of

$\log_{\cos x} \sin x$

Ans: $\frac{d}{dx} \log_{\cos x} \sin x = \frac{d}{dx}\left(\frac{\log \sin x}{\log \cos x}\right)$

$=$

$$\Rightarrow \frac{d}{dx} \log_{\cos x} \sin x = \frac{\log \cos x \cdot \frac{d}{dx} \log \sin x - \log \sin x \cdot \frac{d}{dx} \log \cos x}{(\log \cos x)^2}$$

$$= \frac{\log \cos x \cdot \frac{\cos x}{\sin x} - \log \sin x \cdot \frac{1}{\cos x}(-\sin x)}{(\log \cos x)^2}$$

$$= \frac{\cot x \log \cos x + \tan x \log \sin x}{(\log \cos x)^2}$$

(Ans)

Ex 11. Find the derivative of $|x|$ when $x > 0$ and $x < 0$

Ans For $x > 0$

$|x| = x$

$$\therefore \frac{d}{dx}|x| = \frac{dx}{dx} = 1$$

for $x < 0$

$|x| = -x$

$$\frac{d}{dx}|x| = \frac{d}{dx}(-x) = -1$$

Hyperbolic function

$\sinh x = \frac{1}{2}(e^x - e^{-x})$

$\cosh x = \frac{1}{2}(e^x + e^{-x})$

$\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

$\coth x = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

$\text{sech}\, x = \dfrac{1}{\cosh x} = \dfrac{1}{\frac{1}{2}\cdot(e^x + e^{-x})}$

$\text{cosech}\, x = \dfrac{1}{\sinh x} = \dfrac{1}{\frac{1}{2}(e^x - e^{-x})}$

(i) $\cosh^2 x - \sinh^2 x = 1$

Proof : L.H.S

$\cosh^2 x - \sinh^2 x$

$= \dfrac{(e^x + e^{-x})^2}{4} - \dfrac{(e^x - e^{-x})^2}{4}$

$$\frac{1}{4}\left[e^{2x}+e^{-2x}+2 - e^{2x} - e^{-2x}+2\right]$$

$$= \frac{1}{4} \times 4.$$

$$= 1 \quad = R.H.S \quad \text{(Proved)}$$

(ii) $\tanh^2 x + \operatorname{sech}^2 x = 1$

(iii) $\coth^2 x - \operatorname{cosech}^2 x = 1$

(iv) $\sinh(x+y) = \sinh x . \cosh y + \cosh x . \sinh y$

(v) $\cosh(x+y) = \cosh x . \cosh y + \sinh x . \sinh y$

(vi) $\tanh(x+y) = \dfrac{\tanh x + \tanh y}{1 + \tanh x . \tanh y}$

(vii) $\sinh(-x) = -\sinh x$; $\cosh(-x) = \cosh x$

(viii) $\cosh^2 x = 1 + \sinh^2 x$

(ix) $\sinh^2 x = \cancel{\cosh^2 x} + \cosh^2 x - 1$

(x) $\cosh^2 x + \sinh^2 x = \cosh 2x$

(xi) $\sinh 2x = 2 . \sinh x . \cosh x$.

(xii) $\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right), x \in R$

Proof: Let $y = \sinh^{-1} x$

$$\Rightarrow x = \sinh y$$

$$\Rightarrow x = \frac{e^{y} - e^{-y}}{2}$$

$$\Rightarrow 2x = e^{y} - e^{-y} = e^{y} - \frac{1}{e^{y}}$$

$$\Rightarrow 2x e^{y} = e^{2y} - 1$$

$$\Rightarrow e^{2y} - 2x e^{y} - 1 = 0$$

$$\Rightarrow e^{y} = \frac{2x \pm \sqrt{4x^2 + 4}}{2}$$

$$= x \pm \sqrt{x^2 + 1}$$

The principal value is defined by taking the +ve sign of $\sqrt{x^2+1}$

$$\therefore e^{y} = x + \sqrt{x^2+1}$$

$$\Rightarrow y = \ln\left(x + \sqrt{x^2+1}\right)$$

$$\Rightarrow \sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$$

(xiii) $\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right), \quad x \geq 1$

(xiv) $\tanh^{-1} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right), \quad |x| < 1$

(xv) $\coth^{-1} x = \frac{1}{2} \ln\left(\frac{x+1}{x-1}\right), \quad |x| > 1$

$$\operatorname{sech}^{-1} x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right), \quad 0 < x < 1.$$

xvi

$$\operatorname{csch}^{-1} x = \ln\left(\frac{1 \pm \sqrt{1+x^2}}{x}\right), \quad x \neq 0$$

(a) $\dfrac{d}{dx} \sinh x = \cosh x$

Proof

$$\frac{d}{dx} \sinh x$$

$$= \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right)$$

$$= \frac{1}{2}\left\{\frac{d}{dx}(e^x) - \frac{d}{dx} e^{-x}\right\}$$

$$= \frac{1}{2}\left\{e^x - e^{-x} \cdot \frac{d}{dx}(-x)\right\}$$

$$= \frac{1}{2}\left\{e^x + e^{-x}\right\}$$

$$= \cosh x$$

(b) $\dfrac{d}{dx} \cosh x = \sinh x$

(c) $\dfrac{d}{dx} \tanh x = \operatorname{sech}^2 x$

(d) $\dfrac{d}{dx} \coth x = -\operatorname{csch}^2 x$

(e) $\frac{d}{dx}\operatorname{sech} x = -\operatorname{sech} x \cdot \tanh x$

(f) $\frac{d}{dx}\operatorname{cosech} x = -\operatorname{cosech} x \cdot \coth x$

(g) $\frac{d}{dx}\sinh^{-1} x = \frac{1}{\sqrt{x^2+1}}$

Proof Let $y = \sinh^{-1} x$

$\Rightarrow x = \sinh y$

$\cancel{\Rightarrow x = e^{y}}$

Diff $\Rightarrow \frac{dx}{dy} = \frac{d}{dy}\sinh y = \cosh y$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1+\sinh^2 y}} = \frac{1}{\sqrt{1+x^2}}$

(h) $\frac{d}{dx}\cosh^{-1} x = \frac{1}{\sqrt{x^2-1}}, \; x > 1$

(i) $\frac{d}{dx}\tanh^{-1} x = \frac{1}{1-x^2} \quad |x| < 1$

(j) $\frac{d}{dx}\coth^{-1} x = \frac{-1}{x^2-1}, \quad |x| > 1$

(E) $\frac{d}{dx} \operatorname{sech}^{-1} x = \frac{1}{x\sqrt{1-x^2}}, \quad 0 < x < 1$

(x) $\frac{d}{dx} \operatorname{cosech}^{-1} x = \frac{-1}{|x|\sqrt{1+x^2}}, \quad x \neq 0$

Alternative method of $\frac{d}{dx} \sin^{-1} x$

(a) ~~Ans~~

Let $y = \sin^{-1} x$

$\Rightarrow x = \sin y$

$\Rightarrow \frac{dx}{dy} = \frac{d}{dy} \sin y$

$\Rightarrow \frac{dx}{dy} = \cos y =$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

$\Rightarrow \frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$

Exercise -11-(a)

Learned

(1)

$$\frac{d}{dx} \sinh 2x$$

$$= \frac{d}{dx} \cosh 2x \cdot \frac{d}{dx} 2x$$ by chain rule

$$= 2 \cosh 2x$$

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

4. $$\frac{d}{dx} \sin \cosh 5x$$

$$= \cos \cosh 5x \cdot \frac{d}{dx} \cosh 5x$$

$$= \cos \cosh 5x \cdot \sinh 5x \cdot \frac{d}{dx}(5x)$$

$$= 5 \cos \cosh 5x \cdot \sinh 5x.$$

Differentiability and Continuity

A function f is differentiable at $x = a$ if $f'(a) = \phi$

$$= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists.

Putting $(a+h) = x$ $\therefore h = x - a$

also $h \to 0 \Rightarrow x \to a$

$$\boxed{f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}}$$

Note :

1. $\lim_{x \to a+} \frac{f(x) - f(a)}{x - a}$ is called right hand derivative at $x = a$.

$\lim_{x \to a-} \frac{f(x) - f(a)}{x - a}$ is called left hand derivative at $x = a$.

If R.H.D = L.H.D then we say f is differentiable at $x = a$.

i.e $f'(a)$ exists.

Imp Theorem :

A function which is differentiable at $x = a$ is continuous at ~~at~~ that point and the converse is not true.

Proof : Let $f(x)$ be differentiable at $x = a$ i.e

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x-a} \quad \text{exists} \longrightarrow (i)$$

Consider $\lim_{x \to a} \left(f(x) - f(a)\right)$

$$= \lim_{x \to a} \left(\frac{f(x) - f(a)}{x-a}\right)(x-a)$$

$$= \lim_{x \to a} \frac{f(x) - f(a)}{x-a} \cdot \lim_{x \to a} (x-a)$$

$$= f'(a) \cdot 0 \quad \text{from eq}^n \ (i)$$

$$= 0$$

$$\therefore \lim_{x \to a} \left(f(x) - f(a)\right) = 0$$

$$\Rightarrow \lim_{x \to a} f(x) - \lim_{x \to a} f(a) = 0$$

$$\Rightarrow \lim_{x \to a} f(x) - f(a) = 0$$

$$\Rightarrow \lim_{x \to a} f(x) = f(a)$$

$\Rightarrow$ f is contineous at $x = a$

The converse is not true

it can be shown by a counter example

Counter example

Let $f(x) = |x|$ at $x=0$

To test the continuity at $x=0$

(i) $f(0) = |0| = 0$ which exists.

(ii) Consider $\lim_{x \to 0} f(x)$

$= \lim_{x \to 0} |x|$

R.H.L $= \lim_{x \to 0^+} |x|$

$= \lim_{x \to 0^+} x$ $\left(\because x \to 0^+ \right.$ x is $+ve$ $\therefore |x| = x \left. \right)$

$= 0$

L.H.L $\lim_{x \to 0^-} |x|$

$= \lim_{x \to 0^-} -x$ $\left(\because x \to 0^- \right.$ x is $-ve$ $\therefore |x| = -x \left. \right)$

$= -0$

$= 0$

R.H.L $=$ L.H.L $= 0$

$\therefore \lim_{x \to 0} f(x)$ exists and $\lim_{x \to 0} f(x) = 0$

(iii) $\lim_{x \to 0} f(x) = 0 = f(0)$

$\therefore$ f is continuous at $x = 0$

To test differentiability at $x = 0$

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{x \to 0} \frac{|x| - 0}{x}$$

$$= \lim_{x \to 0} \frac{|x|}{x}$$

R.H.D $= \lim_{x \to 0+} \frac{|x|}{x}$

$= \lim_{x \to 0+} \frac{x}{x}$ $\left(\because x \to 0+ \right.$ x is +ve and $\left. |x| = x \right)$

$= \lim_{x \to 0+} 1$

$= 1$

L.H.D $\lim_{x \to 0-} \frac{|x|}{x}$

$= \lim_{x \to 0-} \frac{-x}{x}$ $\left(\because x \to 0- \right.$ x is -ve and $\left. |x| = -x \right)$

$= \lim_{x \to 0-} -1$

$= -1$

L.H.D $\neq$ R.H.D

$\therefore$ $f'(0)$ does not exist.

$\therefore$ f is not differentiable at $x=0$

(Proved)

Note

1. $|2x+1|$ is not differentiable at

$x = -\frac{1}{2}$

Question

Give an example of a function which is not differentiable at two points

Ans :- $f(x) = |x| + |x-1|$ is not differentiable at $x = 0$ and at $x = 1$

---

1. Discuss the differentiability of the following function at $x=0$

$$f(x) = \begin{cases} 3+2x & \text{when } x \leq 0 \\ 3-2x & \text{when } x > 0 \end{cases}$$

Ex 3

$$f(x) = \begin{cases} 3+2x & \text{when } x \leq 0 \\ 3-2x & \text{when } x > 0 \end{cases}$$

To test the differentiability at $x=0$

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x-0}$$

$$= \lim_{x\to 0} \frac{f(x) - 3}{x-0} \quad \left(\because \text{At } x=0 \;\; f(x) = 3+2x \;\; \text{i.e } f(0) = 3 \right)$$

R.H.D

$$\lim_{x\to 0+} \frac{f(x) - 3}{x}$$

$$= \lim_{x\to 0+} \frac{\cancel{3} - 2x - \cancel{3}}{x} \quad \left(\because x\to 0+ \Rightarrow x > 0 \;\; \therefore f(x) = 3-2x \right)$$

$$= \lim_{x\to 0+} -2$$

$$= -2$$

L.H.D

$$\lim_{x\to 0-} \frac{f(x) - 3}{x}$$

$$= \lim_{x\to 0-} \frac{\cancel{3}+2x - \cancel{3}}{x} \quad \left(\because x\to 0- \Rightarrow x < 0 \;\; \therefore f(x) = 3+2x \right)$$

$$= \lim_{x\to 0-} 2$$

$$= 2$$

R.H.D $\neq$ L.H.D

$\therefore$ $f'(0)$ does not exists.

$\therefore$ f is not differentiable at $x=0$.

Ex 2. Discuss the differentiability of the following function at $x=0$

$$f(x) = \begin{cases} x^2 \cdot \sin\frac{1}{x} & \text{when } x \neq 0 \\ 0 & \text{when } x = 0 \end{cases}$$

Ans: $$f(x) = \begin{cases} x^2 \cdot \sin\frac{1}{x} & \text{when } x \neq 0 \\ 0 & \text{when } x = 0 \end{cases}$$

To test the differentiability at $x=0$

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{x \to 0} \frac{x^2 \cdot \sin\frac{1}{x} - 0}{x}$$

$\left[\because f(x) = 0 \text{ when } x = 0 \text{ i.e } f(0) = 0 \right]$

$$= \lim_{x \to 0} x \sin\frac{1}{x}$$

We know

$$0 \leq \left|\sin\frac{1}{x}\right| \leq 1$$

$$\Rightarrow \cancel{0 \cdot \frac{}{x}|x| < \frac{}{x}}$$

$$\Rightarrow 0 \cdot |x| \leq |x| \cdot \left|\sin\frac{1}{x}\right| \leq |x| \cdot 1$$

$$\Rightarrow 0 \leq \left|x \sin\frac{1}{x}\right| \leq |x|$$

$$\lim_{x \to 0} 0 = 0$$

and $\lim_{x \to 0} |x| = 0$

$\therefore \lim_{x \to 0} \left|x \sin\frac{1}{x}\right| = 0$ by sandwich theorem.

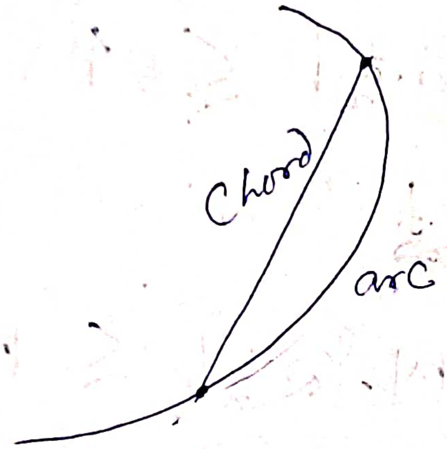
$$\lim_{x \to 0} x \sin\frac{1}{x} = 0$$

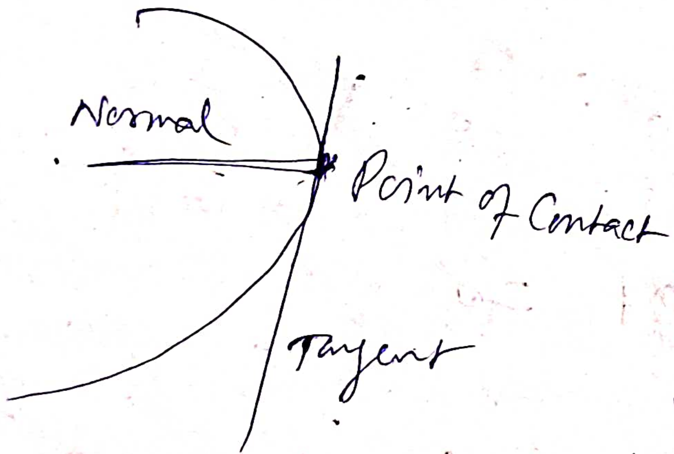
$\therefore \Rightarrow f'(0) = 0$

$\Rightarrow f$ is differentiable at $x = 0$.

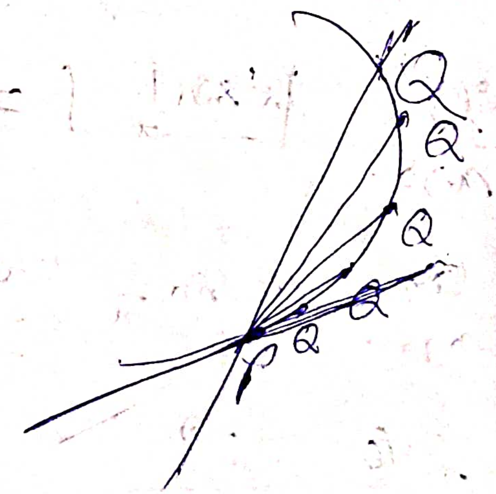
11-1 — Teca

Geometrical ~~inter pidation~~ interpretation of derivative

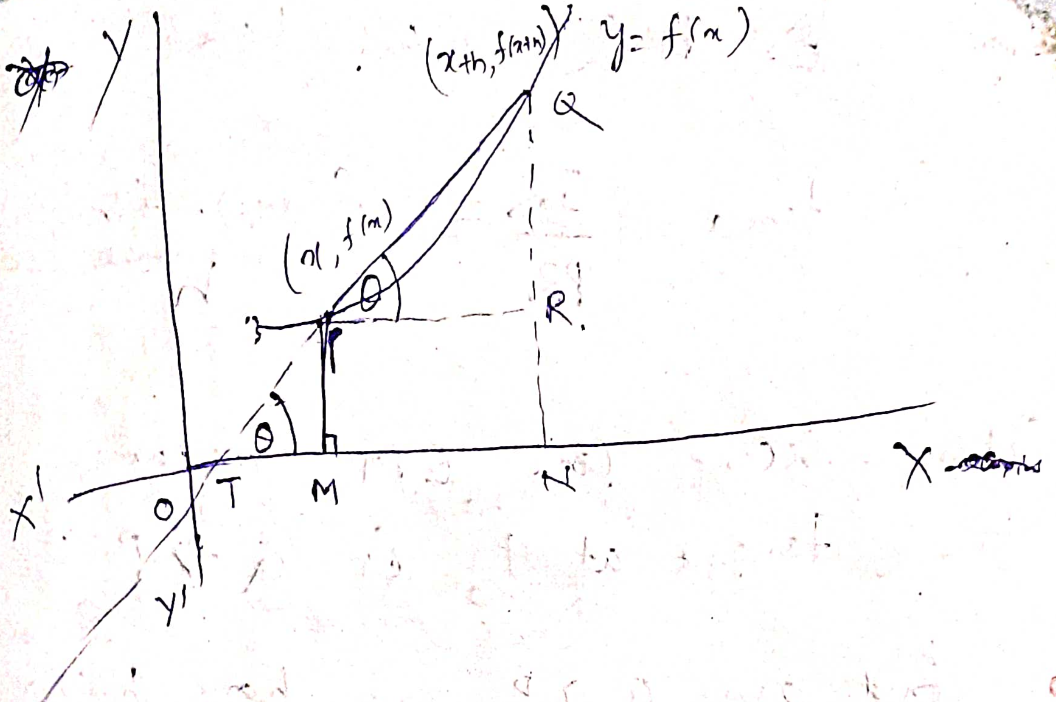




$Q \to P$



As $Q \to P$, the secant line PQ becomes tangent at P



Let $y = f(x)$ be the curve

Let $P(x, f(x))$ and $Q(x+h, f(x+h))$ be two neighbouring point.

Join QP and let it meets X-axis at T. Let $\angle QTX = \theta$

$\therefore$ Inclination of the secant $= QP$
$= \theta$

Draw $\perp$ $\overline{PM}$ and $\overline{QN}$ on x-axis and $\perp$ $\overline{PR}$ on $\overline{QN}$. Now $\angle QPR = \theta$

also $PM = f(x)$

$QN = f(x+h)$

$QR = QN - RN = QN - PM$
$= f(x+h) - f(x)$

$PR = MN = ON - OM$
$= x+h - x = h$

From $\triangle$ PRQ we have

$$\tan\theta = \frac{QR}{PR} = \frac{f(x+h)-f(x)}{h}$$

The secant line QPT becomes the tangent at P if $Q \to P$

But as $Q \to P$ we have, $h \to 0$

Hence the secant line QPT becomes the tangent at P if $h \to 0$

Now slope of secant line QPT

$$= \tan\theta = \frac{f(x+h)-f(x)}{h}$$

Slope of tangent line at P

$$\cancel{= \lim_{h\to 0}}$$

$$= \lim_{h\to 0} \left[\text{Slope of secant line QPT} \right]$$

$$= \lim_{h\to 0} \left(\frac{f(x+h)-f(x)}{h} \right)$$

$$= f'(x) = \left.\frac{dy}{dx}\right]_{(x,y)}$$

∴ The slope of the tangent at any point of the curve is the derivative at that point.

Problem

1. Find the slope of the tangent to the curve $y = x + \frac{1}{x}$ at $x = \alpha$

Ans: The eqn of curve is

$$y = x + \frac{1}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(x + \frac{1}{x}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{dx}{dx} + \frac{d\frac{1}{x}}{dx}$$

$$= 1 - \frac{1}{x^2}$$

Slope of the tangent at $x = \alpha$ is

$$\left.\frac{dy}{dx}\right]_{x=\alpha} = 1 - \frac{1}{\alpha^2}$$

4.(i) The eqn of curve

$$y = mx + c$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(mx + c) = m.1 + 0 = m$$

Slope of tangent at $x = 3$ is

$$\left.\frac{dy}{dx}\right]_{x=3} = m$$

Fll.n

5. (ii) $f(x) = \begin{cases} 1-2x, & x \leq \frac{1}{2} \\ x - \frac{1}{2}, & x > \frac{1}{2} \end{cases}$ at $x=\frac{1}{2}$

To test the differentiable at $x \frac{1}{2}$

$$f'\left(\frac{1}{2}\right) = \lim_{x \to \frac{1}{2}} \frac{f(x) - f\left(\frac{1}{2}\right)}{x - \frac{1}{2}}$$

R.L.D

$$\lim_{x \to \frac{1}{2}^+} \frac{f(x) + 0}{x - \frac{1}{2}} \quad \left(\because \text{At } x = \frac{1}{2},\ f(x) = 1-2x,\ f\left(\frac{1}{2}\right) = 0 \right)$$

$$= \lim_{x \to \frac{1}{2}^+} \frac{-x + \frac{1}{2}}{x - \frac{1}{2}}$$

$$= \lim_{x \to \frac{1}{2}^+} - \left(\frac{x - \frac{1}{2}}{x - \frac{1}{2}} \right)$$

$$= \lim_{x \to \frac{1}{2}^+} -1$$

$$= -1$$

R.L.D

$$= \lim_{x \to \frac{1}{2}^+} \frac{f(x) - x - \frac{1}{2}}{x - \frac{1}{2}}$$

Limit $= \lim_{x \to \frac{1}{2}^+} 1 = 1$

L.H.D. $\lim_{x \to \frac{1}{2}^-} \dfrac{f(x) - f(\frac{1}{2})}{x - \frac{1}{2}}$

$= \lim_{x \to \frac{1}{2}^-} \dfrac{1 - 2x - 0}{x - \frac{1}{2}}$ $\left(\because \text{At } x = \frac{1}{2} \quad f(x) = 1-2x \quad f(\frac{1}{2}) = 0 \right)$

$= \lim_{x \to \frac{1}{2}^-} \dfrac{1 - 2x}{x - \frac{1}{2}}$

$= \lim_{x \to \frac{1}{2}^-} \dfrac{2(1 - 2x)}{(2x - 1)} = \lim_{x \to \frac{1}{2}^-} 2 \dfrac{\cancel{(1-2x)}}{-\cancel{(2x-1)}}$

$= \lim_{x \to \frac{1}{2}^-} (-2)$

$= -2$

L.H.D $\neq$ R.H.D

Again

5.(ii) $f(x) = \begin{cases} 1 - 2x, & x \le \frac{1}{2} \\ x - \frac{1}{2}, & x > \frac{1}{2} \end{cases}$

To test the differentiability at $x = \frac{1}{2}$

$f'(\frac{1}{2}) = \lim_{x \to \frac{1}{2}} \dfrac{f(x) - f(\frac{1}{2})}{x - \frac{1}{2}}$

$= \lim_{x \to \frac{1}{2}} \dfrac{f(x) - 0}{x - \frac{1}{2}}$ $\left(\because \text{At } x = \frac{1}{2} \quad f(x) = 1-2x \quad \text{i.e } f(\frac{1}{2}) = 0 \right)$

$$= \lim_{x \to \frac{1}{2}} \frac{f(x)}{x - \frac{1}{2}}$$

R.H.D

$$\lim_{x \to \frac{1}{2}+} \frac{x - \frac{1}{2}}{x - \frac{1}{2}} \qquad \left(\because x \to \tfrac{1}{2}+ \quad \therefore x > \tfrac{1}{2} \quad \therefore f(x) = x - \tfrac{1}{2} \right)$$

$$= \lim_{x \to \frac{1}{2}+} 1 = 1$$

L.H.D

$$\lim_{x \to \frac{1}{2}-} \frac{1 - 2x}{x - \frac{1}{2}}$$

$$= \lim_{x \to \frac{1}{2}-} \frac{2(1-2x)}{(2x-1)} \qquad \left(\because x \to \tfrac{1}{2}- \quad \therefore x < \tfrac{1}{2}- \quad \therefore f(x) = 1 - 2x \right)$$

$$= \lim_{x \to \frac{1}{2}} \frac{-2\cancel{(2x-1)}}{\cancel{(2x-1)}}$$

$$= -2$$

$\therefore$ L.H.D $\neq$ R.H.D

$\therefore f'(0)$ does not exists.

$\therefore f$ is not differentiable at $x = \frac{1}{2}$

1. First principle $\dfrac{\tan x}{e^x}$

Let $f(x) = \dfrac{\tan x}{e^x}$

$f(x+h) = \dfrac{\tan(x+h)}{e^{x+h}}$ where h is small increment in x.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\dfrac{\tan(x+h)}{e^{x+h}} - \dfrac{\tan x}{e^x}}{h}$$

$$= \lim_{h\to 0} \frac{e^x \tan(x+h) - e^{x+h}\cdot \tan x}{h \times e^{x+h}\cdot e^x}$$

$$= \lim_{h\to 0} \frac{e^x \tan(x+h) - e^x \tan x + e^x \tan x - e^{x+h}\cdot \tan x}{h\, e^x \cdot e^{x+h}}$$

$$= \lim_{h\to 0} \frac{e^x \{\tan(x+h) - \tan x\} - \tan x\,(e^{x+h} - e^x)}{h \cdot e^x \cdot e^{x+h}}$$

$$= \lim_{h\to 0} \frac{e^x \dfrac{\sin(x+h)}{\cos(x+h)} - \dfrac{\sin x}{\cos x} - \tan x\, e^x (e^h - 1)}{h \cdot e^x \cdot e^{x+h}}$$

$$\lim_{h\to 0} \frac{\cancel{e^x}\left(\dfrac{\sin h}{\cos(x+h).\cos x}\right)}{h.\cancel{e^x}.e^{x+h}} - \frac{\tan x.\cancel{e^x}(e^h-1)}{h.\cancel{e^x}.e^{x+h}}$$

$$= \lim_{h\to 0} \frac{\sin h}{h}.\frac{1}{\cos(x+h).\cos x}.\frac{1}{e^{x+h}} - \tan x \lim_{h\to 0} \frac{e^h-1}{h}.\frac{1}{e^{x+h}}$$

$$= 1\times\frac{1}{\cos^2 x}\times\frac{1}{e^x} - \tan x\,(1).\frac{1}{e^x}$$

$$= \frac{\sec^2 x}{e^x} - \frac{\tan x}{e^x}$$

$$= \frac{\sec^2 x - \tan x}{e^x}$$

Eqn of tangent and normal

Suppose the point of contact (x_1, y_1) and slope of tangent m.

Eqn of tangent $y - y_1 = m(x - x_1)$

Slope normal $= \dfrac{-1}{\text{slope of tangent}} = -\dfrac{1}{m}$

Eqn of normal is $= (y - y_1) = -\dfrac{1}{m}(x - x_1)$

(1) The eqn of parabola is

$$y^2 = 4ax$$

$$\Rightarrow \frac{d}{dx} y^2 = 4a \frac{d}{dx} x$$

$$\Rightarrow 2y \cdot \frac{dy}{dx} = 4a$$

$$\Rightarrow \frac{dy}{dx} = \frac{4a}{2y} = \frac{2a}{y}$$

Let the point of contact be (x_1, y_1)

~~satisfy~~ $\because (x_1, y_1)$ is on the curve

$$\therefore (y_1)^2 = 4ax_1$$

Slope of the tangent at $(x_1, y_1) = \left.\frac{dy}{dx}\right]_{(x_1, y_1)}$

$$= \frac{2a}{y_1}$$

Eqn of the tangent at (x_1, y_1) is

$$y - y_1 = \frac{2a}{y_1}(x - x_1)$$

$$\Rightarrow yy_1 - y_1^2 = 2ax - 2ax_1$$

$$\Rightarrow yy_1 - 4ax_1 = 2ax - 2ax_1$$

$$\Rightarrow yy_1 = 2ax + 2ax_1 = \cancel{2a(x+x_1)}$$

$$\Rightarrow y = \frac{2a}{y_1}(x + x_1) = \frac{2ax}{y_1} + \frac{2ax_1}{y_1}$$

$$\Rightarrow y = \frac{2a}{y_1}(x) + \frac{2ax_1}{y_1}$$

But $y = mx + c$ is tangent given (given)

$$\therefore m = \frac{2a}{y_1}, \quad c = \frac{2ax_1}{y_1}$$

Now $$m \times c = \frac{2a}{y_1} \times \frac{2ax_1}{y_1} = \frac{4a^2x_1}{y_1^2}$$

$$= \frac{\cancel{4}a^{\cancel{2}}\cancel{x_1}}{\cancel{4a}\cancel{x_1}} = a$$

$$\Rightarrow c = \frac{a}{m}$$ (Proved)

Indeterminant forms

$\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \times \infty$, $\infty - \infty$, 1^{∞}, 0^0, ∞^0

L' Hospital's Rule

When $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form occure in the limit then we defferentiate the numerator and denominator separately till the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ will be (দূর) abolished.

Problem

1. Find $\lim_{x \to 0} \frac{a^x - b^x}{x}$

Ans

$$\lim_{x \to 0} \frac{a^x - b^x}{x} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0} \frac{a^x \ln a - b^x \ln b}{1} \quad \left(\text{by } L'\text{-Hospital's rule}\right)$$

$$= \ln a - \ln b$$

$$= \ln \frac{a}{b} \quad \text{(Ans)}$$

2. Evaluate $\lim_{x \to 0} \frac{x - \tan x}{x^3}$

Ans:

$$\lim_{x \to 0} \frac{x - \tan x}{x^3} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0} \frac{1 - \sec^2 x}{3x^2} \quad \left(\text{by } L'\text{-Hospital's Rule}\right)$$

$$= \lim_{x \to 0} - \frac{\tan^2 x}{3x^2}$$

$= -\frac{1}{3} \cdot \lim_{x \to 0} \frac{\tan^2 x}{x^2} = -\frac{1}{3} \cdot \lim_{x \to 0} \left(\frac{\tan x}{x}\right)^2$

$= -\frac{1}{3} \times \left(\lim_{x \to 0} \frac{\tan x}{x}\right)^2 = -\frac{1}{3} \times 1$

$= -\frac{1}{3}$ (Ans)

3. Evaluate $\lim_{x \to \infty} \frac{x^2+2}{3x^2+1}$

Ans: $\lim_{x \to \infty} \frac{x^2+2}{3x^2+1}$ $\left(\frac{\infty}{\infty} \text{ form}\right)$

$= \lim_{x \to \infty} \frac{2x+0}{3 \times 2x + 0}$ (by L'-Hospital's Rule)

$= \lim_{x \to \infty} \frac{2\cancel{x}}{6\cancel{x}} = \lim_{x \to \infty} \frac{1}{3}$

$= \frac{1}{3}$ (Ans)

4. Evaluate $\lim_{x \to \infty} \frac{x^n}{e^x}$ where n is +ve integer.

Ans $\lim_{x \to \infty} \frac{x^n}{e^x}$ $\left(\frac{\infty}{\infty} \text{ form}\right)$

$= \lim_{x \to \infty} \frac{n\, x^{n-1}}{e^x}$ (by L'Hospital's rule and $\frac{\infty}{\infty}$ form again)

$= \lim_{x \to \infty} \frac{n(n-1) \cdot x^{n-2}}{e^x}$ (by applying L'Hospital's rule again)

$$= \lim_{x \to \infty} \frac{n(n-1).(n-2) \cdots 1 . x^0}{e^x}$$

(by applying L' Hospital's Rule n times in total)

$$= \lim_{x \to \infty} \frac{\lfloor n}{e^x}$$

$$= 0 \quad \left(\because \lim_{x \to \infty} \frac{1}{e^x} = 0 \right)$$

~~B~~. $0 \times \infty$ form

Suppose we have to find $\lim_{x \to a} f(x). g(x)$

where $\lim_{x \to a} f(x) = 0$

$\lim_{x \to a} g(x) = \infty$

Now we can make it $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form

by expressing one of the function in

term of reciprocal.

$$\left\{ \text{i.e} \quad \lim_{x \to a} \frac{f(x)}{\frac{1}{g(x)}} \left(\frac{0}{0} \text{ form} \right) \right.$$

$$\left. \text{or} \quad \lim_{x \to a} \frac{g(x)}{\frac{1}{f(x)}} \left(\frac{\infty}{\infty} \text{ form} \right) \right\}$$

Then apply L' Hospital's Rule)

Ex :

Find $\lim_{x \to \frac{\pi}{2}^-} \sin 2x \cdot \log \cos x$

Ans :-

$$\lim_{x \to \frac{\pi}{2}^-} \sin 2x \cdot \log \cos x \quad (0 \cdot \infty \text{ form})$$

$$= \lim_{x \to \frac{\pi}{2}^-} \frac{\log \cos x}{\operatorname{cosec} 2x} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \lim_{x \to \frac{\pi}{2}^-} \frac{\frac{1}{\cos x} \cdot (-\sin x)}{-2 \operatorname{cosec} 2x \cdot \cot 2x} \quad \left(\text{by L'Hospital's rule}\right)$$

$$= \lim_{x \to \frac{\pi}{2}^-} \frac{\frac{\sin x}{\cos x}}{2 \times \frac{1}{\sin 2x} \cdot \frac{\cos 2x}{\sin 2x}}$$

$$= \lim_{x \to \frac{\pi}{2}^-} \frac{\sin^2 2x \times \sin x}{2 \cos 2x \cdot \cos x}$$

$$= \lim_{x \to \frac{\pi}{2}^-} \frac{\sin 2x \cdot \cancel{2} \sin x \cdot \cancel{\cos x} \cdot \sin x}{\cancel{2} \cancel{\cos x} \cdot \cos 2x}$$

$$= \lim_{x \to \frac{\pi}{2}^-} \frac{\sin 2x}{\cos 2x} \cdot \sin^2 x$$

$$= \lim_{x \to \frac{\pi}{2}^-} \tan 2x \cdot \sin^2 x$$

$$= 0 \times 1$$

$$= 0$$

Evaluate

$$\lim_{x \to 0+} x \cdot e^{\frac{1}{x}}$$

Ans: $\lim_{x \to 0+} x \cdot e^{\frac{1}{x}}$ ($0 \cdot \infty$ form)

$$= \lim_{x \to 0+} \frac{e^{\frac{1}{x}}}{\frac{1}{x}} \quad \left(\frac{\infty}{\infty} \text{ form} \right)$$

$$= \lim_{x \to 0+} \frac{e^{\frac{1}{x}} \cdot \cancel{\left(\frac{-1}{x^2}\right)}}{\cancel{\left(\frac{-1}{x^2}\right)}} \quad \text{(By L'Hospital's rule)}$$

$$= \lim_{x \to 0+} e^{\frac{1}{x}}$$

$$= \infty$$

~~$\infty - \infty$ form~~

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10. $\lim_{x \to \infty} \frac{x^3 - 3x + 1}{2x^3 - 7x^2 + 5}$ $\left(\frac{\infty}{\infty} \text{ form} \right)$

$$= \lim_{x \to \infty} \frac{3x^2 - 3}{6x^2 - 14x} \quad \text{(By L-Hospital rule)}$$

$$= \lim_{x \to \infty} \frac{3 \times 2x}{12x - 14}$$

$$= \lim_{x \to \infty} \frac{6}{12}$$

$$= \frac{1}{2}$$

$\infty - \infty$ form

22.02.2K

Suppose we have to find

$$\lim_{x \to a} \{f(x) - g(x)\}$$ where

$$\lim_{x \to a} f(x) = \infty$$

and $$\lim_{x \to a} g(x) = \infty$$

Now $$\lim_{x \to a} (f(x) - g(x))$$

$$= \lim_{x \to a} \left\{ \frac{1}{\frac{1}{f(x)}} - \frac{1}{\frac{1}{g(x)}} \right\}$$

$$= \lim_{x \to a} \left\{ \frac{1}{\phi(x)} - \frac{1}{\psi(x)} \right\}$$ where $\phi(x) = \frac{1}{f(x)}$

$$= \lim_{x \to a} \left\{ \frac{\psi(x) - \phi(x)}{\phi(x) \cdot \psi(x)} \right\}$$ $\psi(x) = \frac{1}{g(x)}$

which is $\frac{0}{0}$ form. Hence in this form we express each function in ~~ratiproe~~ reciprocal form and then taking L.C.M and simplifying it can be reduce to $\left(\frac{0}{0}\right)$ form and L' Hospital's rule can be applied.

Ex - Evaluate $\lim_{x\to 0}\left(\frac{1}{x} - \cot x\right)$

Ans $\lim_{x\to 0}\left(\frac{1}{x} - \cot x\right)$

$= \lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{\tan x}\right)$

$= \lim_{x\to 0}\left(\frac{\tan x - x}{x \tan x}\right) \quad \left(\frac{0}{0} \text{ form}\right)$

$= \lim_{x\to 0}\left\{\frac{\sec^2 x - 1}{x.\sec^2 x + \tan x}\right\} \quad \left(\text{by L' Hospital's rule}\right)$

$= \lim_{x\to 0}\left(\frac{\tan^2 x}{x\sec^2 x + \tan x}\right)$

$= \lim_{x\to 0}\left(\frac{\frac{\sin^2 x}{\cos^2 x}}{\frac{x.\cancel{8}}{\cos^2 x} + \frac{\sin x}{\cos x}}\right) = \lim_{x\to 0}\left(\frac{\frac{\sin^2 x}{\cancel{\cos^2 x}}}{\frac{x + \sin x.\cos x}{\cancel{\cos^2 x}}}\right)$

$= \lim_{x\to 0}\left(\frac{\sin x}{\frac{x}{\sin x} + \cos x}\right) = \frac{0}{1+1} = \frac{0}{2} = 0$ (Ans)

Note :

1. Some times $(\infty - \infty)$ form can be reduce to $0 \times \infty$ form.

Ex Find $\lim_{x \to \infty} \left(x - x^2 \log\left(1+\frac{1}{x}\right) \right)$

Ans: $\lim_{x \to \infty} \left(x - x^2 \log\left(1+\frac{1}{x}\right) \right)$

$= \lim_{x \to \infty} \left(x - \left(x \log\left(1+\frac{1}{x}\right)^x \right) \right)$ $(\infty - \infty$ form$)$

$= \lim_{x \to \infty} x \left(1 - \log\left(1+\frac{1}{x}\right)^x \right)$

$= \lim_{x \to \infty} \frac{\left(1 - \log\left(1+\frac{1}{x}\right)^x \right)}{\frac{1}{x}}$ $\left(\frac{0}{0}\text{ form}\right)$

$$= \lim_{x\to\infty} \frac{0 - \left\{ 1\cdot \log\left(1+\frac{1}{x}\right) + x\cdot \frac{1}{1+\frac{1}{x}}\left(\frac{-1}{x^2}\right) \right\}}{-\frac{1}{x^2}}$$ (by L'Hospital's rule)

$$= \lim_{x\to\infty} \frac{\log\left(1+\frac{1}{x}\right) - \frac{1}{x}\times\frac{1}{1+\frac{1}{x}}}{\frac{1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{\log\left(1+\frac{1}{x}\right) - \frac{1}{\cancel{x}}\times\frac{\cancel{x}}{(1+x)}}{\frac{1}{x^2}}$$ $\left(\frac{0}{0} \text{ form}\right)$

$$= \lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x}}\cdot\left(\frac{-1}{x^2}\right) + \left(\frac{1}{1+x}\right)^2}{\frac{-2}{x^3}}$$

$$= \lim_{x\to\infty} \frac{\frac{x}{(1+x)}\times\frac{-1}{x\,x^2} + \frac{1}{(1+x)^2}}{\frac{-2}{x^3}}$$

$$= \lim_{x\to\infty} \frac{-(1+x)+x}{x(1+x)^2}\times\frac{x^3}{-2}$$

$$= \lim_{x\to\infty} \frac{\cancel{-}1 - \cancel{x} + \cancel{x}\times x^2}{2(1+x)^2}$$ $\left(\frac{\infty}{\infty} \text{ form}\right)$

$$\lim_{x\to\infty} \frac{2x}{\cancel{2}.\,2(1+x)} \quad (\text{by L' Hospital's rule})$$

$$= \lim_{x\to\infty} \frac{x}{2(1+x)} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \lim_{x\to\infty} \frac{1}{2} \quad (\text{by L' Hospital's rule})$$

$$= \frac{1}{2}$$

— o —

0^0, 1^∞, ∞^0 forms or power form

When these above forms occure in the limit, we take the function as y and take logarithim and then it will be reduce to previous forms. Then we find the limit. Let the limit be 'l' then the actual limit is e^l

Ex:- Find $\lim_{x \to 0} (\sin x)^{\sin x}$

Ans: $\lim_{x \to 0} (\sin x)^{\sin x}$ (0^0 form)

Let $y = (\sin x)^{\sin x}$

$\Rightarrow \ln y = \sin x \ln \sin x.$

$\Rightarrow \lim_{x \to 0} \ln y = \lim_{x \to 0} \sin x \cdot \ln \sin x$ $(0 \cdot \infty \text{ form})$

$$= \lim_{x \to 0} \frac{\ln \sin x}{\operatorname{cosec} x} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \lim_{x \to 0} \frac{\frac{1}{\sin x} \cdot \cos x}{-\operatorname{cosec} x \cdot \cot x} \quad \left(\text{by L'Hospital's rule}\right)$$

$$= \lim_{x \to 0} \frac{\cancel{\cot x}\ 1}{-\operatorname{cosec} x \cdot \cancel{\cot x}}$$

$$= \lim_{x \to 0} -\sin x$$

$$= 0$$

$\therefore \lim_{x \to 0} y = e^0 = 1$

$\Rightarrow \lim_{x \to 0} (\sin x)^{\sin x} = 1$ (Ans)

Find $\lim_{x \to 0} (x)^x$

~~Let y~~

Ans

$$\lim_{x \to 0} (x)^x \quad (0^0 \text{ form})$$

$\Rightarrow$ Let $y = x^x$

$\Rightarrow \ln y = x \ln x$

$$\Rightarrow \lim_{x \to 0} \ln y = \lim_{x \to 0} x \ln x \quad (0 \times \infty \text{ form})$$

$$= \lim_{x \to 0} \frac{\ln x}{\frac{1}{x}} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \lim_{x \to 0} \frac{\frac{1}{x}}{\frac{-1}{x^2}} \quad (\text{by L'Hospital rule})$$

$$= \lim_{x \to 0} \frac{1}{x} \times \frac{x^2}{-1}$$

$$= \lim_{x \to 0} -x$$

$$= 0$$

$\boxed{\text{Here } \ln y = 0 \text{ so } y = e^0 = 1}$

$$\Rightarrow \lim_{x \to 0} y = e^0 = 1$$

$\lim_{x \to 0} x^x = 1$ (Ans)

Imp. Q. Find the value of 'a' such that $\lim_{x \to 0} \frac{a \sin 2x - \sin 3x}{x^3}$ is finite and hence find the limit.

Ans: $\lim_{x \to 0} \frac{a \sin 2x - \sin 3x}{x^3}$ $\left(\frac{0}{0} \text{ form}\right)$

$= \lim_{x \to 0} \frac{2a \cos 2x - 3\cos 3x}{3x^2}$ (by L' Hospital rule)

~~$= \lim_{x \to 0}$~~ Now the denominator is zero for $x = 0$, the fraction will tend to a finite limit iff numerator is also zero when $x = 0$. For this

$2a - 3 = 0. \Rightarrow a = \frac{3}{2}$

Or $a = \frac{3}{2}$, then

$\Rightarrow \lim_{x \to 0} \frac{\cancel{3} \cos 2x - \cancel{3} \cos 3x}{\cancel{3} x^2}$ $\left(\frac{0}{0} \text{ form}\right)$

$= \lim_{x \to 0} \frac{\cancel{3} \cdot (-2 \sin 2x) + 3 \sin 3x}{2x}$ (by L'hospital rule)

$=$

$$= \lim_{x \to 0} \frac{-4\cos 2x + 9\cos 3x}{2}$$

$$= \frac{-4+9}{2}$$

$$= \frac{5}{2} \quad \text{(Ans)}$$

2. Q. Ans: Find the value of a and b such that $\lim_{x \to 0} \frac{x(2+a\cos x) - b\sin x}{x^3} = 1$ is finite and ~~[illegible]~~.

3. Q. Ans: Find the value of a, b, c such that $\lim_{x \to 0} \frac{ae^x - b\cos x + ce^{-x}}{x\sin x} = 2$ finite.

--- 11-(f)

2 Ans: lim[illegible]

$$\lim_{x \to 0} \frac{x(2+a\cos x) - b\sin x}{x^3} \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x \to 0} \frac{2x + a\,x\cos x - b\sin x}{x^3}$$

$$= \lim_{x \to 0} \frac{2 + a\{x(-\sin x) + \cos x\} - b\cos x}{3x^2} \quad \left(\text{by L'Hospital rule}\right)$$

$$= \lim_{x \to 0} \cancel{2 + \{a\cos x - ax\sin x\} - b\cos x}$$

Now denominator is zero for $x=0$, the fraction will tend to finite limit iff numerator is also zero when $x=0$, for this $2+a-b=0$

$$\Rightarrow a-b=-2 \quad -(i)$$

$$\Rightarrow b = a+2 \quad -(ii)$$

$$\therefore \lim_{x \to 0} \frac{2 + a\cos x - ax\sin x - (a+2)\cos x}{3x^2}$$

$$= \lim_{x \to 0} \frac{2 + a\cancel{\cos x} - ax\sin x - a\cancel{\cos x} - 2\cos x}{3x^2}$$

For the limit is finit numerator will zero when $x=0$

$$\therefore \quad 2-a=0$$

$$\Rightarrow a=2$$

Putting this value in eqn (i), we get

$$b = a+2 = 2+2 = 4$$

$\therefore$ $a=2$, $b=4$ then

$$= \lim_{x \to 0} \frac{0 - a\cos x - \{a(\cos x - x\sin x)\} + (a+2)\cos x}{6x} \quad \text{(by L'Hospital rule)}$$

$$= \lim_{x \to 0} \frac{-a\cancel{\cos x} - a\cos x + ax\sin x + a\cancel{\cos x} + 2\cos x}{6x}$$

$$\lim_{x \to 0} \frac{a\{(\sin x \cos x)\} - 2\sin x}{6} \quad \left(\begin{array}{c}\text{by} \\ \text{L'Hospital's} \\ \text{rule}\end{array}\right)$$

$$= \frac{a}{6}$$

$$= \frac{2}{6}$$

$$= \frac{1}{3}$$ (Ans)

3. $\lim_{x \to 0} \frac{ae^x - b\cos x + ce^{-x}}{x\sin x}$

Now denominator is zero for $x=0$

the fraction will tend to finite limit if

the numerator is also zero when $x=0$,

for this $a - b + c = 0$ ——(i)

$$= \lim_{x \to 0} \frac{ae^x + b\sin x - ce^{-x}}{x\cos x + \sin x} \quad \left(\begin{array}{c}\text{by L'Hospital's} \\ \text{rule}\end{array}\right)$$

when numerator is zero, then $a - c = 0$

$\Rightarrow a = c$ ——(ii)

$a - b + c = 0$

$\Rightarrow a - b + a = 0$

$\Rightarrow b = 2a$ ——(iii)

$$= \lim_{x \to 0} \frac{ae^x + 2a\sin x - ae^{-x}}{x\cos x + \sin x}$$

$$= \lim_{x \to 0} \frac{a e^{x} + 2a \cos x + a e^{-x}}{-x \sin x + \cos x + \cos x} \quad \left(\text{by L'Hospital rule} \right)$$

$\frac{4a}{2}$ (Ans)

→ For numerator is zero, ~~then~~ when $x \to 0$

then $a + 2a + a = 0$

$\Rightarrow 4a = 0$

$\Rightarrow a = 0$

$$= \cancel{\lim_{x \to 0}} = \frac{0}{2}$$

$= 0$ (Ans)

$a = 0$, $c = 0$, $b = 2a = 0$

12, 03, 2x

Derivatives of higher orders

(Successive differentiations)

Let $y = f(x)$ be the function

$$y_1 = \frac{dy}{dx} = f'(x) = Dy = f^{(1)}(x)$$

is called derivative of f or first order. Again taking derivative we have

$$y_2 = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = f''(x)$$

$$= f^{(2)}(x) = D^2y$$

It is called derivative 2nd order.

$$y_3 = \frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = f'''(x) = f^{(3)}(x) = D^3y$$

is called derivative of 3rd order.

$$y_4 = \frac{d^4y}{dx^4} = \frac{d}{dx}\left(\frac{d^3y}{dx^3}\right) = f^{iv}(x) = f^{(4)}(x) = D^4y$$

is called derivative of 4th order.

$$y_5 = \frac{d^5 y}{dx^5} = \frac{d}{dx}\left(\frac{d^4 y}{dx^4}\right) = f^{v}(x) = f^{(5)}(x) = D^5 y$$

is called derivative of fifth order.

$$y_n = \frac{d^n y}{dx^n} = \frac{d}{dx}\left(\frac{d^{n-1} y}{dx^{n-1}}\right) = f^{(n)}(x) = D^n y$$

is called derivative of nth order.

<u>Question :</u>

If $y = e^{m \sin^{-1} x}$ then prove that

$$(1-x^2) \cdot \frac{d^2 y}{dx^2} - x \frac{dy}{dx} = m^2 y$$

<u>Proof :</u>

$$y = e^{m \sin^{-1} x}$$

$$\Rightarrow \frac{dy}{dx} = e^{m \sin^{-1} x} \cdot \frac{d}{dx} m \sin^{-1} x$$

$$= e^{m \sin^{-1} x} \cdot (m) \cdot \frac{d}{dx} \sin^{-1} x$$

$$= e^{m \sin^{-1} x} \cdot (m) \cdot \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{e^{m \sin^{-1} x} \cdot m}{\sqrt{1-x^2}}\right)$$

$$= m \cdot \frac{d}{dx}\left(\frac{e^{m \sin^{-1} x}}{\sqrt{1-x^2}}\right)$$

$$= m \cdot \left(\frac{\sqrt{1-x^2} \cdot \frac{d}{dx} e^{m\sin^{-1}x} - e^{m\sin^{-1}x} \cdot \frac{d}{dx}\sqrt{1-x^2}}{1-x^2} \right)$$

$$\cancel{= m \cdot \left(\sqrt{1-x^2} \cdot \left(e^{m} \frac{1}{\sqrt{1-x^2}} \right. \right.}$$

$$= m \left(\frac{\cancel{\sqrt{1-x^2}} \left(\frac{e^{m\sin^{-1}x}\, m}{\cancel{\sqrt{1-x^2}}} \right) + \frac{e^{m\sin^{-1}x} \cdot \cancel{2}x}{\cancel{2}\sqrt{1-x^2}}}{1-x^2} \right)$$

$$= \frac{m \cdot e^{m\sin^{-1}x}}{1-x^2} \cdot \left(\frac{x}{\sqrt{1-x^2}} + m \right)$$

L.H.S

$$(1-x^2) \cdot \frac{d^2y}{dx^2} - x \cdot \frac{dy}{dx}$$

$$= \cancel{(1-x^2)} \left(\frac{m \cdot e^{m\sin^{-1}x}}{\cancel{(1-x^2)}} \left(\frac{x}{\sqrt{1-x^2}} + m \right) \right) - x \cdot \frac{e^{m\sin^{-1}x}\, m}{\sqrt{1-x^2}}$$

$$= m\, e^{m\sin^{-1}x} \left(\cancel{\frac{x}{\sqrt{1-x^2}}} + m - \cancel{\frac{x}{\sqrt{1-x^2}}} \right)$$

$= m^2 \cdot e^{m \sin^{-1} x}$

$= m^2 \cdot y$ (R.H.S) (Proved)

Exercise - 11 (c)

1. (ii) $y = \cos 2x$

$y_1 = \frac{d}{dx} \cos 2x = -2 \sin 2x$

$y_2 = \frac{d}{dx}(y_1) = \frac{d}{dx}(-2 \sin 2x) = -2 \cdot (\cos 2x) \cdot 2$

$= -4 \cos 2x$

$y_3 = \frac{d}{dx}(-4 \cdot \cos 2x) = -4 \cdot (-\sin 2x) \cdot 2$

$= 8 \sin 2x$

$y_4 = \frac{d}{dx} 8 \sin 2x = 8 \cdot \cos 2x \cdot (2)$

$= 16 \cos 2x$

Derivative Or nth order

Derivative or nth order

~~Proof~~ 1. Let $y = (ax+b)^m$

where ~~a and bx~~ 'a' and 'b' are constants

m = +ve integer.

then $\cancel{\dfrac{d^n y}{dx^n}} =$

$$\frac{d^n y}{dx^n} = \begin{cases} m(m-1)(m-2)\cdots(m-n+1)\cdot a^n (ax+b)^{m-n} & \text{when } n \le m \\ 0 & \text{when } n > m \end{cases}$$

Or $\rightarrow \dfrac{d^n y}{dx^n} = \dfrac{m!\, a^n (ax+b)^{m-n}}{(m-n)!}$

$$= \frac{(m-n)!\,\{m-(n-1)\}\{m-(n-2)\}\cdots\{m-(n-(n-1))\}\{m-(n-n)\}\, a^n (ax+b)^{m-n}}{(m-n)!}$$

$$= (m-n+1)(m-n+2)\cdots(m-1)m\, a^n (ax+b)^{m-n}$$

$$= m(m-1)(m-2)\cdots(m-n+1)\, a^n (ax+b)^{m-n}$$

Proof:

Given that a and b are constants and m is +ve integer and $n \le m$

$$y = (ax+b)^m$$

$$\frac{dy}{dx} = m\cdot(ax+b)^{m-1}\cdot\frac{d}{dx}(ax+b)$$

$$= m(ax+b)^{m-1}\cdot(a)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx} m. (ax+b)^{m-1} . a$$

$$= m(m-1)(ax+b)^{m-2} . a . a$$

$$= m(m-1) . (ax+b)^{m-2} . a^2 .$$

$$\frac{d^3y}{dx^3} = m(m-1)(m-2) . (ax+b)^{m-3} . a^3$$

Proceeding like wise

$$\frac{d^n y}{dx^n} = m(m-1)(m-2) \ldots \underset{\downarrow}{\overset{m-(n-1)}{}} (m-n+1) . (ax+b)^{m-n} . a^n$$

For $n = m$

$$\frac{d^m y}{dx^m} = m(m-1)(m-2) \ldots (1) . (ax+b)^0 . a^m$$

$$= m(m-1)(m-2) \ldots (1) a^m$$

$$= \underline{|m} . a^m$$

$$= \text{Constant}$$

$$\therefore \frac{d^{m+1} y}{dx^{m+1}} = \frac{d}{dx}\left(\frac{d^m y}{dx^m}\right) = 0$$

$$\frac{d^{m+2} y}{dx^{m+2}} = 0 , \ldots \ldots \text{ So on}$$

$$\therefore \frac{d^n y}{dx^n} = 0 \quad \text{for} \quad n > m$$ (Proved)

(2.) If m is -ve integer or rational number (fraction) and $y = (ax+b)^m$

then

$$\frac{d^n y}{dx^n} = m(m-1)(m-2) \text{ ---- } (m-n+1) \cdot a^n \cdot (ax+b)^{m-n}$$

<u>Proof</u> :-

Let $y = (ax+b)^m$ where m is -ve integer or fraction.

$$\frac{dy}{dx} = m \cdot (ax+b)^{m-1} \cdot \frac{d}{dx}(ax+b)$$

$$= m \cdot (ax+b)^{m-1} \cdot a$$

$$\frac{d^2 y}{dx^2} = \frac{d}{dx} m \cdot (ax+b)^{m-1} \cdot a$$

$$= m \cdot (m-1)(ax+b)^{m-2} \cdot a^2$$

$$\frac{d^3y}{dx^3} = m(m-1)(m-2)\,(ax+b)^{m-3}\cdot a^3$$

Proceeding like wise, we get

$$\frac{d^ny}{dx^n} = m(m-1)(m-2)\text{------}(m-n+1)\,(ax+b)^{m-n}\cdot a^n$$

Here since m is $-ve$ integer or rational number (fraction), $m-n$ can not be Zero because n is $+ve$ integer.

$\therefore$ for no $+ve$ integer n, $\frac{d^ny}{dx^n}$ is constant.

$\therefore$ for no $+ve$ integer n, $\frac{d^ny}{dx^n}$ is Zero.

$$\therefore \frac{d^ny}{dx^n} = m(m-1)(m-2)\text{---}(m-n+1)\,(ax+b)^{m-n}\cdot a^n$$

(Proved)

3. Let $y = \log(ax+b)$

then $$\frac{d^ny}{dx^n} = \frac{(-1)^{n-1}\cdot a^n\cdot \underline{|n-1}}{(ax+b)^n}$$

Proof

Let

$$y = \log(ax+b)$$

To prove that $\dfrac{d^n y}{dx^n} = \dfrac{(-1)^{n-1} \cdot a^n \cdot \lfloor n-1}{(ax+b)^n}$

We will prove it by method of induction.

Now $y = \log(ax+b)$

$$\frac{dy}{dx} = \frac{a}{ax+b} = \frac{(-1)^{1-1} \cdot a^1 \cdot \lfloor 1-1}{(ax+b)^1}$$

$\therefore$ It is true for $n = 1$

Let it be true for $n = K$

i.e $\dfrac{d^K y}{dx^K} = \dfrac{(-1)^{K-1} \cdot a^K \cdot \lfloor K-1}{(ax+b)^K}$

is true.

Now $\dfrac{d^{K+1} y}{dx^{K+1}} = \dfrac{d}{dx}\left(\dfrac{d^K y}{dx^K}\right)$

$$= \frac{d}{dx}\left(\frac{(-1)^{k-1} \cdot a^k \cdot \underline{|k-1}}{(ax+b)^k}\right)$$

by induction hypothesis.

$$= (-1)^{k-1} \cdot a^k \, \underline{|k-1} \cdot \frac{d}{dx}\left(ax+b\right)^{-k}$$

$$= (-1)^{k-1} \cdot a^k \, \underline{|k-1} \cdot (-k) \cdot (ax+b)^{-k-1} \cdot a$$

$$= (-1)^{k-1} \cdot a^k \cdot \underline{|k-1} \cdot k \cdot (-1)^1 \cdot (ax+b)^{-(k+1)} \cdot a$$

$$= (-1)^k \, a^{k+1} \, \underline{|k} \, (ax+b)^{-(k+1)}$$

$$= \frac{(-1)^{k+1-1} \cdot a^{k+1} \, \underline{|k+1-1}}{(ax+b)^{k+1}}$$

$\therefore$ It is true for $n = k+1$.

$\therefore$ It is true for $\forall n \in N$

by method of induction. (proved)

(4). Let $y = e^{(ax+b)}$

To prove $\frac{d^n y}{dx^n} = a^n \cdot e^{ax+b}$

Let $y = e^{ax+b}$

To prove that $\frac{d^n y}{dx^n} = a^n \cdot e^{(ax+b)}$

We will prove it by method of induction.

Now $y = e^{ax+b}$

$\frac{dy}{dx} = a \cdot e^{ax+b} = a^1 \cdot e^{(ax+b)}$

$\therefore$ It is true for $n = 1$

Let it be true for $n = K$

i.e $\frac{d^K y}{dx^K} = a^K \cdot e^{(ax+b)}$ is true

Now $\frac{d^{K+1} y}{dx^{K+1}} = \frac{d}{dx}\left(\frac{d^K y}{dx^K}\right)$

$= \frac{d}{dx}\left(a^K \cdot e^{(ax+b)}\right)$

$= a^K \cdot e^{(ax+b)} \cdot a$

$= a^{k+1} \cdot e^{ax+b}$

$\therefore$ It is true for $n = k+1$

$\therefore$ It is true for $\forall n \in N$ by

method of induction. (proved)

(5). Let $y = a^x$

then $\dfrac{d^n y}{dx^n} = (\log a)^n \cdot a^x$

Proof : Let $y = a^x$.

To prove that $\dfrac{d^n y}{dx^n} = (\log a)^n \cdot a^x$

we will prove it by method of induction

Now $y = a^x$

$\dfrac{dy}{dx} = a^x \cdot \log a = (\log a)^1 \cdot a^x$

$\therefore$ It is true for $n = 1$

Let it be true for $n = k$

i.e $\dfrac{d^k y}{dx^k} = a^x \cdot (\log a)^k$ is true

Now $\dfrac{d^{K+1}y}{dx^{K+1}} = \dfrac{d}{dx}\left(\dfrac{d^K y}{dx^K}\right)$

$= \dfrac{d}{dx}\left(a^x . (\log a)^K\right)$

$= (\log a)^K . \dfrac{d}{dx} a^x$

$= (\log a)^K . (a^x . \log a)$

$= a^x . (\log a)^{K+1}$

$\therefore$ It is true for $n = K+1$

$\therefore$ It is true for $\forall n \in N$

by method of Induction.

(Proved)

(6) Let $y = \sin(ax+b)$

then $$\frac{d^n y}{dx^n} = a^n . \sin\left(ax+b+\frac{n\pi}{2}\right)$$

Proof : Let $y = \sin(ax+b)$

To prove that $$\frac{d^n y}{dx^n} = a^n . \sin\left(ax+b+\frac{n\pi}{2}\right)$$

we will prove it by method of induction

$$y = \sin(ax+b)$$

$$\frac{dy}{dx} = a.\cos(ax+b)$$

$$= a^1 . \sin\left(ax+b+1.\frac{\pi}{2}\right)$$

$\therefore$ It is true for $n = 1$

Let it be true for $n = K$

i.e $$\frac{d^K y}{dx^K} = a^K . \sin\left(ax+b+\frac{K\pi}{2}\right)$$
is true

Now $$\frac{d^{K+1} y}{dx^{K+1}} = \frac{d}{dx}\left(\frac{d^K y}{dx^K}\right)$$ P.T.O

$$= \frac{d}{dx}\left(a^K . \sin\left(ax+b+\frac{K\pi}{2}\right)\right)$$

$$= a^K . \cos\left(ax+b+\frac{K\pi}{2}\right) . a$$

$$= a^{K+1} . \sin\left(ax+b+\frac{K\pi}{2}+\frac{\pi}{2}\right)$$

$$= a^{K+1} \sin\left(ax+b+(K+1)\frac{\pi}{2}\right)$$

$\therefore$ It is true for $n = K+1$

$\therefore$ It is true for $\forall n \in N$

by method of induction

Upadhyay

7) Let $y = \cos(ax+b)$

then $\frac{d^n y}{dx^n} = a^n . \cos(ax+b+\frac{n\pi}{2})$

Proof Let $y = \cos(ax+b)$

To prove that $\frac{d^n y}{dx^n} = a^n . \cos(ax+b+\frac{n\pi}{2})$

we will prove it by method of induction

Now $y = \cos(ax+b)$

$\frac{dy}{dx} = -a\sin(ax+b)$

$= a \cos(ax+b+\frac{\pi}{2})$

$= a^1 \cos(ax+b+1 . \frac{\pi}{2})$

$\therefore$ It is true for $n=1$

Let it be true for $n=k$

i.e $\frac{d^k y}{dx^n} = a^k . \cos(ax+b+\frac{k\pi}{2})$ is

true

Now

$$\frac{d^{K+1}y}{dx^{K+1}} = \frac{d}{dx}\left(\frac{d^K y}{dx^K}\right)$$

$$= \frac{d}{dx}\left(a^K . \cos\left(ax+b+\frac{K\pi}{2}\right)\right)$$

$$= a^K . \left(-\sin\left(ax+b+\frac{K\pi}{2}\right). a\right)$$

$$= -a^{K+1} . \sin\left(ax+b+\frac{K\pi}{2}\right)$$

$$= a^{K+1} . \cos\left(ax+b+\frac{K\pi}{2}+\frac{\pi}{2}\right)$$

$$= a^{K+1} \cos\left(ax+b+(K+1)\frac{\pi}{2}\right)$$

$\therefore$ It is true for $n = K+1$

$\therefore$ It is true for $\forall n \in N$

by method of induction

(8). Leibnitz theorem

$$\frac{d^n}{dx^n}(uv) = u_n . v + n\, u_{n-1} . v_1 + \frac{n(n-1)}{\lfloor 2}\, . u_{n-2} . v_2 + \frac{n(n-1)(n-2)}{\lfloor 3} u_{n-3} . v_3 + \dots$$

$+ \quad u . v_n$

Problems

1. Find the nth derivative of

$\cos^2 x$

Ans :

$$\frac{d^n}{dx^n}(\cos^2 x)$$

$$= \frac{d^n}{dx^n}\left(\frac{1+\cos 2x}{2}\right)$$

$$= \frac{1}{2} \cdot \frac{d^n}{dx^n}(1+\cos 2x)$$

$$= \frac{1}{2} \left\{ \frac{d^n}{dx^n} 1 + \frac{d^n}{dx^n} \cos 2x \right\}$$

$$= \frac{1}{2} \left\{ 0 + 2^n . \cos\left(2x + \frac{n\pi}{2}\right)\right\}$$

$$= \underline{2^{n-1} \cos\left(2x+\frac{n\pi}{2}\right)}$$

Comp

2. Find the n^{th} derivative of

$x^2 \sin x$

Ans: $\frac{d^n}{dx^n}(x^2 \cdot \sin x)$

$$= \frac{d^n}{dx^n}(\sin x \cdot x^2)$$

$$= \left(\frac{d^n}{dx^n}\sin x\right) \cdot x^2 + n \cdot \left(\frac{d^{n-1}}{dx^{n-1}}\sin x\right) \cdot \frac{d}{dx}x^2$$

$$+ \frac{n(n-1)}{\lfloor 2} \cdot \left(\frac{d^{n-2}}{dx^{n-2}}\sin x\right) \cdot \frac{d^2}{dx^2}x^2 + 0$$

by Leibnitz theorem

$$= \sin\left(x + \frac{n\pi}{2}\right) \cdot x^2 + n \cdot \left(\sin\left(x + (n-1)\frac{\pi}{2}\right)\right) \cdot 2x$$

$$+ \frac{n(n-1)}{\cancel{\lfloor 2}} \cdot \sin\left(x + (n-2)\frac{\pi}{2}\right) \cdot \cancel{2}$$

$$= x^2 \sin\left(x + \frac{n\pi}{2}\right) + 2nx \sin\left(x + (n-1)\frac{\pi}{2}\right)$$

$$+ n(n-1)\sin\left(x + (n-2)\frac{\pi}{2}\right)$$

(Ans)

11- (c) — (n)

Exercise - 11 (c)

2.

$$\frac{d^n}{dx^n}(\cos x \cdot \sin x)$$

$$= \frac{d^n}{dx^n}\left(\frac{\sin 2x}{2}\right) = \frac{1}{2}\frac{d^n}{dx^n}\sin 2x$$

$$= \frac{1}{2} 2^n \sin\left(2x + \frac{n\pi}{2}\right)$$

$$= 2^{n-1} \cdot \sin\left(2x + \frac{n\pi}{2}\right)$$

4.

$$y = \tan^{-1} x$$

$$y_1 = \frac{1}{1+x^2}$$

$$\Rightarrow (1+x^2) y_1 = 1$$

Differenciating both the sides w.r.t x again

$\Rightarrow (1+x^2) \cdot y_2 + x \cdot y_1 = 0$

Again differenciating $(n-1)$ times, we

get by Leibtnz theorem

$\Rightarrow y_{n+1}(1+x^2) + (n-1) \cdot y_n \, 2x$

$+ \frac{(n-1)(n-2)}{\lfloor 2} \cdot y_{n-1} \cdot 2 + 0 + 0 + y_n \cdot 2x$

$+ \cancel{\frac{(n-1)(n-2)}{\lfloor 2}} + (n-1) \cdot y_{n-1} \cdot 2 + 0 + 0 \ldots$

$= 0$

$\Rightarrow (1+x^2) y_{n+1} + y_n \cdot x (n-1+1) = 0$

$+ (n-1) y_{n-1} \left(\frac{n-\cancel{2}}{} + \cancel{2} \right)$

$\Rightarrow (1+x^2) y_{n+1} + 2nx \, y_n + n(n-1) \, y_{n-1} = 0$

$\cos x \cdot \sqrt{1-\sin^2 x}$

(iii) $y = e^{m\cos^{-1}x}$

$$y_1 = e^{m\cos^{-1}x} \cdot \frac{-m}{\sqrt{1-x^2}}$$

$$\Rightarrow y_1\sqrt{1-x^2} = -m \cdot e^{m\cos^{-1}x} = -my$$

$$\Rightarrow y_1^2 (1-x^2) = m^2 y^2$$

Differenciating both sides w.r.t x, we get

$$\Rightarrow (1-x^2) \cdot \cancel{2}y_1 \cdot y_2 + y_1^2 (-\cancel{2}x) = m^2 \cdot \cancel{2}yy_1$$

$$\Rightarrow (1-x^2)\, \cancel{y_1} \cdot y_2 - x y_1^{\cancel{2}} = m^2 y \cancel{y_1}$$

$$\Rightarrow (1-x^2)\, y_2 - xy_1 = m^2 y$$

Differenciating n times, we get

$$y_{n+2} \cdot (1-x^2) + n \cdot y_{n+1} \cdot (-2x)$$
$$+ \frac{n(n-1)}{2} \cdot y_n (-2) - \{ y_{n+1} x + n y_n \cdot 1 + 0 \}$$

$$= m^2 y_n$$

$$\Rightarrow (1-x^2) \cdot y_{n+2} - x y_{n+1} (2n+1)$$
$$- y_n (n(n-1) + n + m^2) = 0$$

$\Rightarrow (1-x^2)\, y_{n+2} - x \cdot y_{n+1} \cdot (2n+1)$

$- y_n\,(n^2+m^2) = 0$ Conne...

---

4. (ii) $2y = x\left(1+\frac{dy}{dx}\right)$

$\Rightarrow 2y = x + x\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{2y-x}{x}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{2y-x}{x}\right)$

$= \frac{x \cdot \frac{d}{dx}(2y-x) - (2y-x)\cdot 1}{x^2}$

$= \frac{x \cdot \left(2 \cdot \frac{dy}{dx} - 1\right) - (2y-x)}{x^2}$

$= \frac{x\left(2 \cdot \frac{2y-x}{x} - 1\right) - 2y + x}{x^2}$

$$= \frac{4y - 2x - \cancel{x} - 2y + \cancel{x}}{x^2}$$

$$= \frac{2y - 2x}{x^2}$$

$$= \frac{2(y-x)}{x^2}$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{2\cdot(y-x)}{x^2}\right)$$

$$= 2\cdot\left(\frac{x^2\cdot\frac{d}{dx}(y-x) - (y-x)\cdot 2x}{x^4}\right)$$

$$= 2\left(\frac{x^2\left(\frac{dy}{dx} - 1\right) - 2xy + 2x^2}{x^4}\right)$$

$$= 2\left(\frac{\cancel{x^2}\,x\left(\frac{2y-x}{\cancel{x}} - \cancel{1}\right) - 2xy + 2x^2}{x^4}\right)$$

$$= 2\left(\frac{2xy \cancel{+ x^2} - x^2 - \cancel{2xy} + \cancel{2x^2}}{x^4}\right)$$

$$= 0$$

$\therefore \frac{d^2y}{dx^2}$ is constant (Proved)

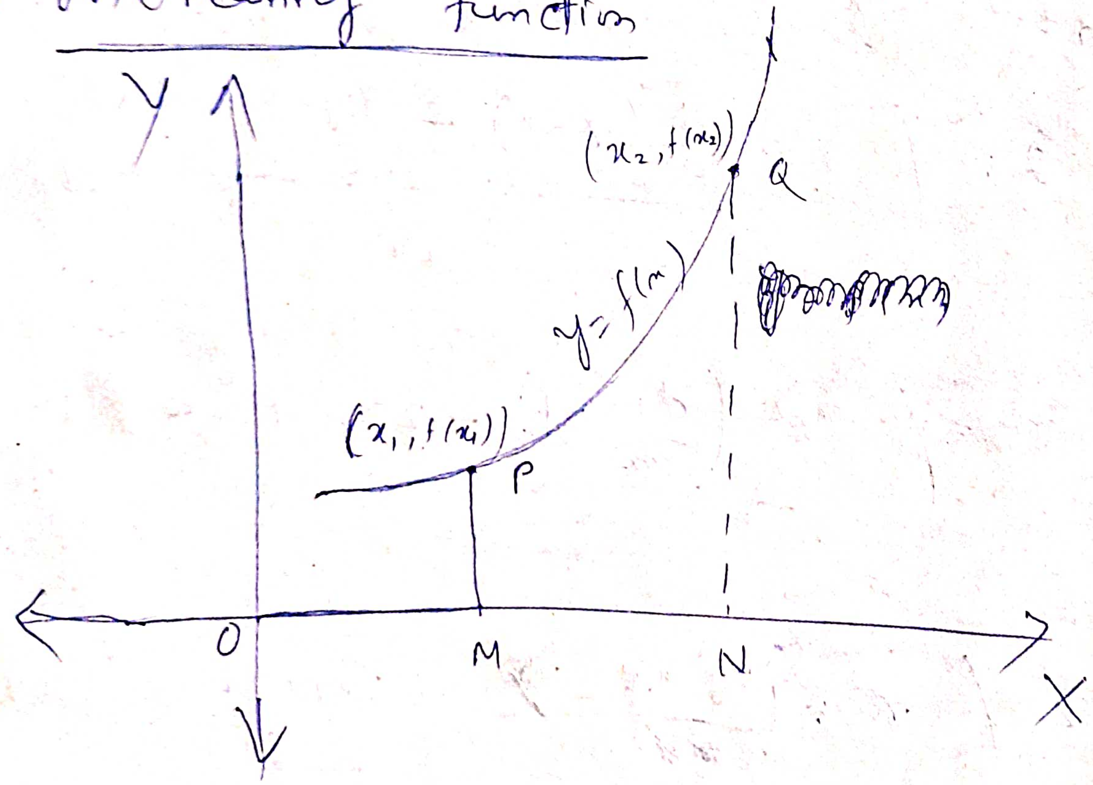
Imp

Increasing or decreasing functions.

or monotonic functions

Monotonic function means either increasing function or decreasing function

Increasing function



$OM = x_1$, $ON = x_2$

$PM = f(x_1)$, $QN = f(x_2)$

A function $f(x)$ is said to be monotonic increasing (m.i) in on interval I

(Principle of mechanics by seeing and graph)

if $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2) \ \forall \ x_1, x_2 \in I$

$\Rightarrow \cancel{f(x_1) \leq f(x_2) \ \forall \ x_1, x_2 \in I}$

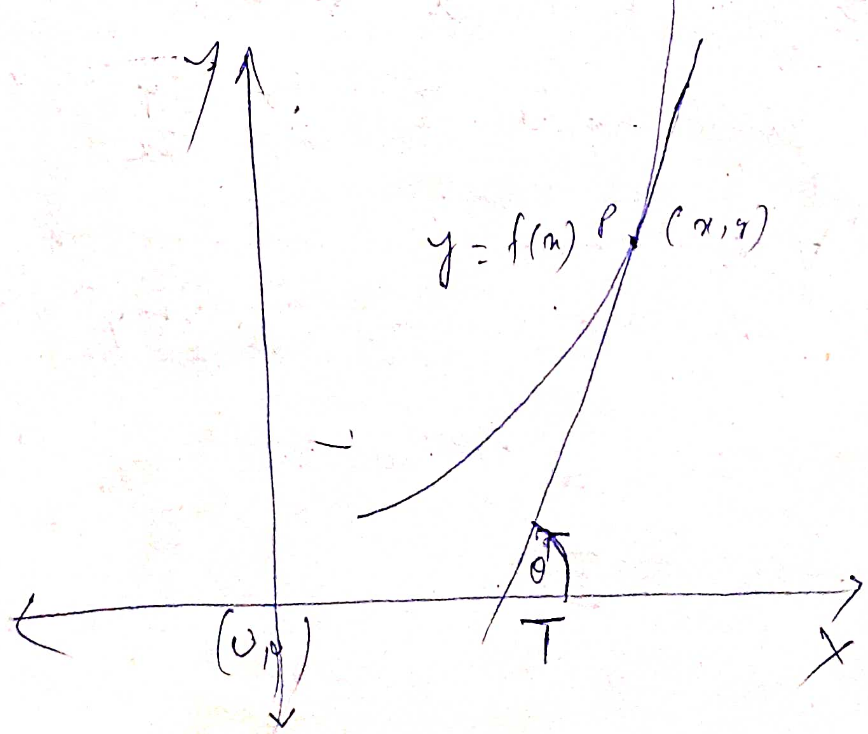
A function $f(x)$ is said to be strictly increasing in an interval I

if $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ for $\forall x_1, x_2 \in I$

$\Rightarrow \cancel{f(x_1) < f(x_2)}$

<u>Note</u> :

All strictly increasing functions are monotonic increasing but converse is not true



Let $y = f(x)$ be increasing function

Let θ be inclination of the tangent

at $P(x, y)$.

$\therefore$ Slope of the tangent at $P = \tan\theta$

but slope of the tangent at $P = \frac{dy}{dx}$ or $f'(x)$

$$\therefore \quad \tan\theta = \frac{dy}{dx} \text{ or } f'(x).$$

Now θ is acute

$$\therefore \quad \tan\theta \geq 0$$

i.e $\frac{dy}{dx} \geq 0$

Hence $f(x)$ is said to be m.i.

in an interval I. if $\frac{dy}{dx}$ or $f'(x) \geq 0$

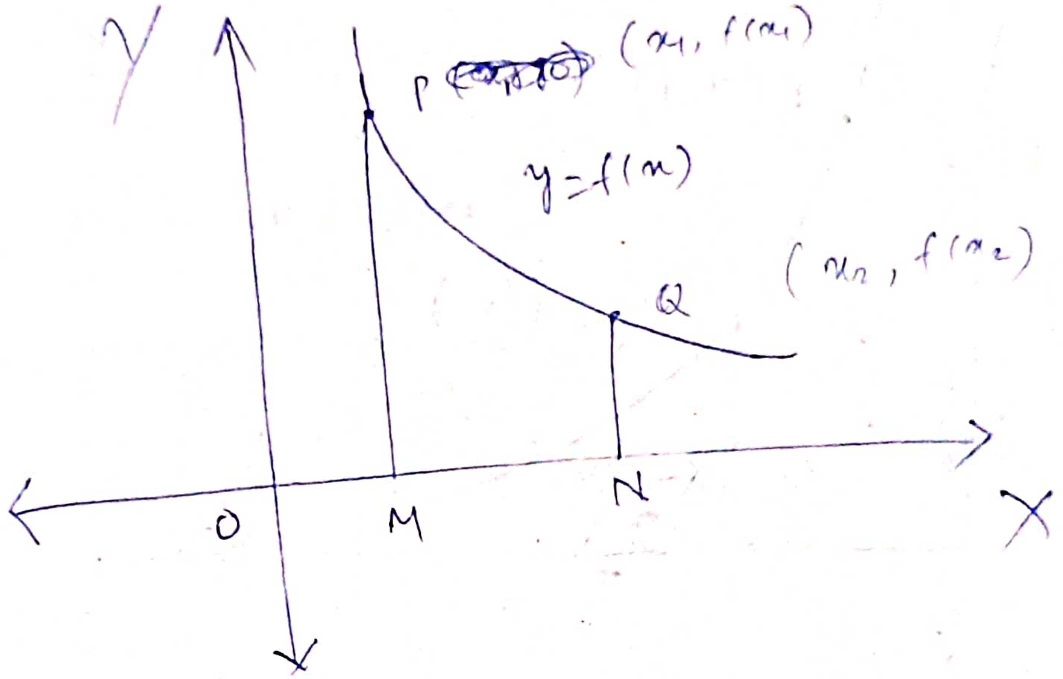
$\forall x \in I$.

$f(x)$ is said to be strictly increasing

in an interval I if $\frac{dy}{dx}$ or $f'(x) > 0$

$\forall x \in I$.

Decreasing function



$OM = x_1$, $PM = f(x_1)$

$ON = x_2$, $QN = f(x_2)$

~~Now~~

A function $f(x)$ is said to be monotonic decreasing (m.d) in an interval I

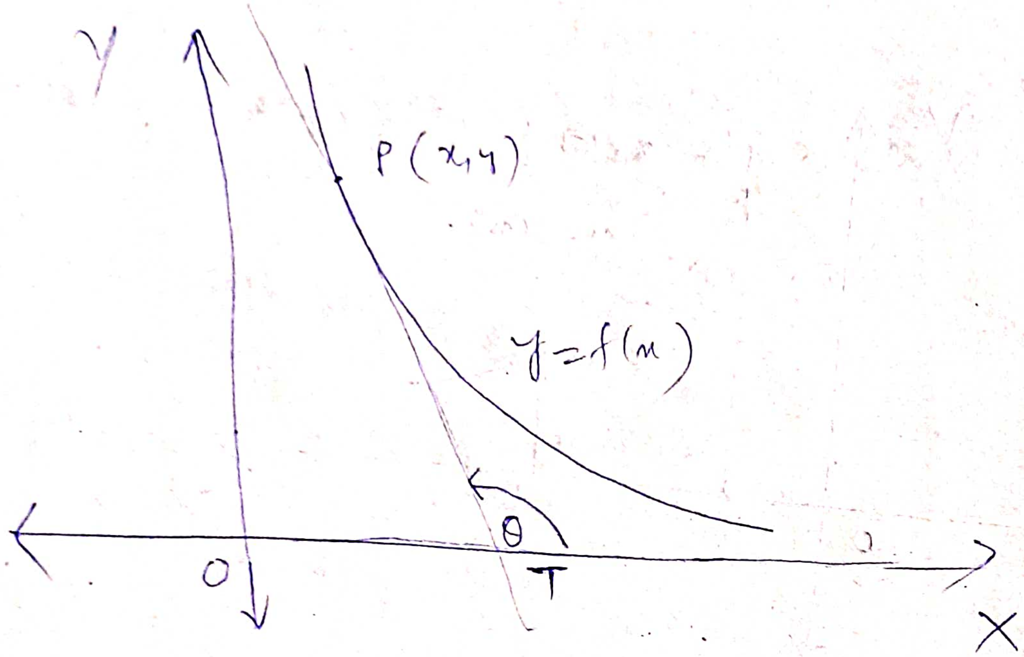
if $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2) \;\forall x_1, x_2 \in I$

A function $f(x)$ is said to be strictly decreasing in an interval I

if $x_1 < x_2 \Rightarrow f(x_1) > f(x_2) \;\forall x_1, x_2 \in I$

Note

All strictly decreasing functions are monotonic decreasing but not conversely.



Let $y = f(x)$ be decreasing function

Let θ be inclination of the tangent at $P(x, y)$

$\therefore$ Slope of the tangent at $P = \tan\theta$.

But slope of the " " $P = \frac{dy}{dx}$ or $f'(x)$

$\therefore \tan\theta = \frac{dy}{dx}$ or $f'(x)$

Now θ is obtuse

$\therefore \tan\theta \leqslant 0$

i.e $\frac{dy}{dx} \leq 0$

Hence $f(x)$ is said to be m.d in an interval I if

$\frac{dy}{dx}$ or $f'(x) \leq 0 \ \forall x \in I$

$f(x)$ is said to be strictly decreasing in an interval I if $\frac{dy}{dx}$ or $f'(x) < 0 \ \forall x \in I$.

Critical points or stationary points

The values of x for which $f'(x)$ equal to zero or $f(x)$ is not differentiable are called critical points of the function $f(x)$. At these points the function is increasing and decreasing both.

$x_1 > x_2 \Rightarrow f(x_1) > f(x_2)$ increasing.

$x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$ decreasing

Problems

1. Find the intervals where the function $f(x) = 2x^3 - 9x^2 + 12x$ is monotonic increasing or m.d.

Ans :

$$f(x) = 2x^3 - 9x^2 + 12x$$

$$f'(x) \; \cancel{(2x^3 - 9x^2 + 12x)}$$

$$= 6x^2 - 18x + 12$$

$f'(x) = 0$ for critical point.

$$\Rightarrow 6x^2 - 18x + 12 = 0$$

$$\Rightarrow x^2 - 3x + 2 = 0$$

$$\Rightarrow x^2 - 2x - x + 2 = 0$$

$$\Rightarrow x(x-2) - 1(x-2) = 0$$

$$\Rightarrow (x-2)(x-1) = 0$$

$$x = 1 \quad \text{or} \quad x = 2$$

$x = 1, 2$ are the critical points

Consider $x \in (-\infty, 1]$ ie

$x \leq 1$

Here $f'(x) = 6x^2 - 18x + 12$

$= 6(x^2 - 3x + 2)$

$= 6(x-1)(x-2) \geq 0$

$\Big(\because x \leq 1 \Rightarrow x - 1 \leq 0$ and $x \leq 1$

$\Rightarrow x < 2$

$\Rightarrow x - 2 < 0$

$\therefore (x-1)(x-2) \geq 0 \Big)$

$\therefore f$ is m.i in $(-\infty, 1]$

Consider $x \in [1, 2]$

i.e $1 \leq x \leq 2$

Here $f'(x) = 6(x-1)(x-2) \leq 0$

$\Big(\because x \geq 1 \Rightarrow x - 1 \geq 0$ and

$x \leq 2 \Rightarrow x - 2 \leq 0$

$\therefore (x-1)(x-2) \leq 0 \Big)$

$\therefore f$ is m.d in $[1, 2]$

Consider $x \in [2, \infty)$

i.e $x \geq 2$

Here $f'(x) = 6(x-1)(x-2) \geq 0$

$\because x \geq 2$, $x - 2 \geq 0$

$x \geq 2$, $x > 1$, $x - 1 > 0$

$\therefore (x-1)(x-2) \geq 0$

$\therefore f$ is m.i in $[2, \infty)$

$\therefore f$ is m.i in $(-\infty, 1]$

and $[2, \infty)$ and m.d in $[1, 2]$

$\therefore f$ is m.i in $(-\infty, 1] \cup [2, \infty)$

and m.d in $[1, 2]$

Q. Find the intervals where the

function $f(x) = x^3 - 6x^2 + 15x + 10$

is m.i or m.d

Ans: I $\quad f(x) = x^3 - 6x^2 + 15x + 10$

$f'(x) = 3x^2 - 12x + 15$

$f'(x) = 0$ for critical points

$\Rightarrow 3x^2 - 12x + 15 = 0$

$\Rightarrow x^2 - 4x + 5 = 0$

$x = \dfrac{-(-4) \pm \sqrt{16 - 4\cdot(1)\cdot(5)}}{2\cdot(1)}$

$= 4 \pm \sqrt{}$

$f'(x) = 3(x^2 - 4x + 5)$

$= 3(x^2 - 2\cdot 2\cdot x + 4 + 5 - 4)$

$= 3((x-2)^2 + 1) > 0$

$\forall x \in \mathbb{R}$

$(\because (x-2)^2 \geq 0)$

$\therefore f$ is m.i in $\mathbb{R}$

3. Find the intervals where

$f(x) = -x^3 + 3x^2 - 6x + 4$ is

m. I or m.d

Ans:

$f(x) = -x^3 + 3x^2 - 6x + 4$

$f'(x) = -3x^2 + 6x - 6$

$f'(x) = -3(x^2 - 2x + 2)$

$= -3\{(x-1)^2 + 1) < 0$

$\forall x \in R$

$(\because (x-1)^2 \geq 0)$

$\therefore f$ is m. d in R

4. Find the interval $f(x)$

$= e^x + e^{-x}$ is m. I or

m. d

Ans: $f(x) = e^{x} + e^{-x}$

$f'(x) = e^{x} - e^{-x}$

$f'(x) = 0$ for critical points.

$\Rightarrow e^{x} - e^{-x} = 0$

$\cancel{\Rightarrow e^{x} = e^{-x}}$

$\Rightarrow e^{x} - \dfrac{1}{e^{x}} = 0$

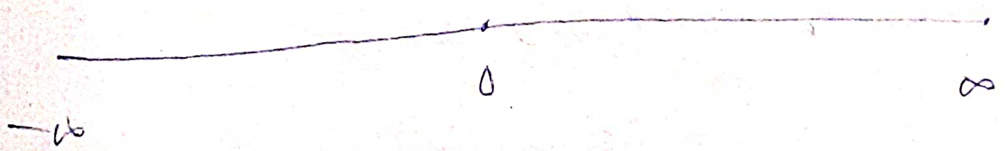
$\Rightarrow \dfrac{e^{2x} - 1}{e^{x}} = 0$

$\Rightarrow e^{2x} - 1 = 0$

$\Rightarrow e^{2x} = 1$

$\Rightarrow e^{2x} = e^{0}$

$\Rightarrow x = 0$ is the critical point



Consider $x \in [-\infty, 0]$

i.e. $x \leq 0$

$\therefore \quad f'(x) = \cancel{e^{x} - e^{-x}}$

$= e^{x} - \frac{1}{e^{x}}$

Since $\quad x \leq 0$

$-x \geq 0$

$\therefore \quad e^{-x} \geq e^{x}$

$\Rightarrow \quad e^{x} - e^{-x} \leq 0$

$\Rightarrow \quad f'(x) \leq 0$

$\therefore \quad f$ is m.d in $(-\infty, 0]$

Consider $\quad x \in [0, \infty]$

i.e $\quad x \geq 0$

$f'(x) = e^{x} - e^{-x}$

Since $\quad x \geq 0$

$-x \leq 0$

$e^{x} \geq e^{-x}$

$\Rightarrow e^{x} - e^{-x} \geq 0$

$\Rightarrow f'(x) \geq 0$

$\therefore$ f is m.i in $[0, \infty)$

Elements $\rightarrow$ 11(γ)

Exercise- 11 (γ)

Ans:

x.

$$y = \frac{\ln x}{x}, \quad x > 0$$

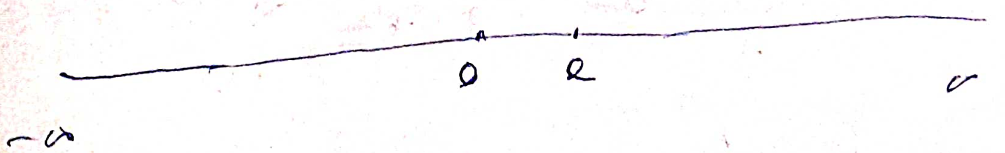
$$\frac{dy}{dx} = \frac{x \cdot \frac{1}{x} - \ln x}{x^2} = \frac{1 - \ln x}{x^2} = 0$$

for critical points.

$\Rightarrow \ln x = 1 = \ln e$

$\Rightarrow x = e$ is the critical point

But given that $x > 0$,



Consider $x \in (0, e]$

i.e. $0 < x \leq e$

Since $x \leq e$

$\therefore \ln x \leq \ln e$

i.e $\ln x \leq 1$

i.e $1 - \ln x \geq 0$

$$\therefore \frac{dy}{dx} = \frac{1-\ln x}{x^2} \geq 0$$

i.e $y = \frac{\ln x}{x}$ is m.i in $(0, e]$.

Consider $x \in [e, \infty)$

i.e $x \geq e$

$\Rightarrow \ln x \geq \ln e$

i.e $\ln x \geq 1$

$\Rightarrow 1 - \ln x \leq 0$

$$\therefore \frac{dy}{dx} = \frac{1-\ln x}{x^2} \leq 0$$

i.e $y = \frac{\ln x}{x}$ is m.d in $[e, \infty)$

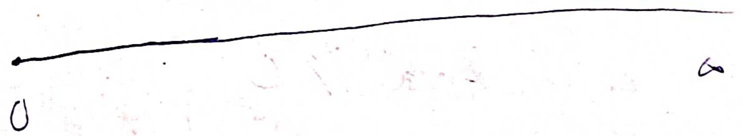
1.(i) Ans $y = \ln x, \quad x \in R_+$

$$\frac{dy}{dx} = \frac{1}{x}$$

At $x = 0$, $\frac{dy}{dx}$ does not exist.

$\therefore$ $x = 0$ is a critical point.

Given that $x \in R_+$ i.e $x > 0$ i.e $x \in (0, \infty)$



$\therefore$ $\frac{dy}{dx} = \frac{1}{x} > 0$

i.e $y = \ln x$ is m.i or strictly increasing in R_+ and decreasing nowhere in R_+

ii) ~~$y = a^x$~~, ~~$y = a^x$~~

Given that $a > 0$, $x \in R$

~~dy~~ Now $y = a^x$

$$\Rightarrow \frac{dy}{dx} = a^x . \ln a$$

Consider $0 < a \leq 1$

$\therefore \ln a \leq \ln 1$

i.e. $\ln a \leq 0$

$\therefore \frac{dy}{dx} = a^x . \ln a \leq 0 \ \forall x \in R$

$\therefore$ y is m.d in R where

$0 < a \leq 1$

Consider $a \geq 1$

$\therefore \ln a \geq 0$

$\therefore \frac{dy}{dx} = a^x . \ln a \geq 0 \ \forall x \in R$

$\therefore$ y is m.i in R where

$a \geq 1$

(vi) $y = \frac{1}{x-1}$, $x \neq 1$

$\Rightarrow \frac{dy}{dx} = \frac{(x-1).(0) - (1).(-1)}{(x-1)^2}$

$= \frac{-1}{(x-1)^2} < 0$

$(\because (x-1)^2 > 0)$

$\therefore$ y is strictly decreasing in

~~...~~

$R - \{1\}$ or $\mathbb{R}$

and increasing no where.

(vii) $$y = \begin{cases} x^2+1, & x \le -3 \\ x^3-8x+13, & x > -3 \end{cases}$$

$$f'(-3) = \lim_{x \to -3} \frac{f(x) - f(-3)}{x+3}$$

~~Consider~~ $$= \lim_{x \to -3} \frac{f(x) - 10}{x+3}$$

~~Consider~~

$\left(\because \text{ at } x = -3, \; f(x) = x^2+1 \; \cancel{9+1} \; \therefore f(-3) = 10 \right)$

R.H.D

$$= \lim_{x \to -3^+} \frac{x^3 - 8x + 13 - 10}{x+3}$$

$$= \lim_{x \to -3^+} \frac{x^3 - 8x + 3}{x+3}$$

$$= \lim_{x \to -3^+} \frac{x^3 + 3x^2 - 3x^2 - 9x + x + 3}{x+3}$$

$$= \lim_{x \to 3^+} \frac{x^2(x+3x) - 3x(x+3) + 1(x+3)}{(x+3)}$$

$$= \lim_{x \to -3^+} \frac{\cancel{(x+3)}\{(x^2-3x)+1\}}{\cancel{(x+3)}}$$

$$= \lim_{x \to -3^+} x^2 - 3x + 1$$

$$= 9 + 9 + 1$$

$$= 19$$

L.H.D

$$\lim_{x \to -3^-} \frac{x^2 + 1 - 10}{x+3}$$

$$= \lim_{x \to -3^-} \frac{x^2 - 9}{x+3}$$

$$= \lim_{x \to -3^-} \frac{\cancel{(x+3)}(x-3)}{\cancel{(x+3)}}$$

$$= -3 - 3$$

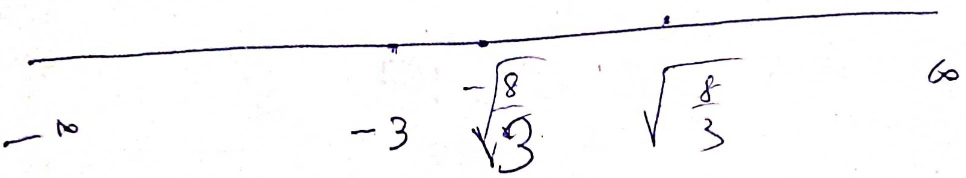
$$= -6$$

$$L.H.D \neq R.H.D$$

$\therefore$ $y = f(x)$ is not differentiable at

$x = -3$

$\therefore$ $x = -3$ is the critical point.



Consider $x \in (-\infty, -3)$

i.e $x < -3$

$\therefore$ $y = f(x) = x^2 + 1$

$$\frac{dy}{dx} = 2x < 0$$

$$(\because x < -3)$$

$\therefore$ y is m.d in $x \in (-\infty, -3)$

Consider $x \in (-3, \infty)$

i.e $x > -3$

$\therefore$ $y = f(x) = x^3 - 8x + 13$

$$\frac{dy}{dx} = 3x^2 - 8 = 0 \text{ for critical points}$$

$$\Rightarrow 3x^2 = 8$$
$$\Rightarrow x^2 = \frac{8}{3}$$
$$\Rightarrow x = \pm\sqrt{\frac{8}{3}}$$

Consider $x \in \left(-3, -\sqrt{\frac{8}{\sqrt{3}}}\right]$

i.e $-3 < x \leq -\sqrt{\frac{8}{\sqrt{3}}}$

$$\frac{dy}{dx} = 3x^2 - 8 \geq 0$$

$\Bigg(\because -3 < x \leq \sqrt{\frac{8}{3}}$

$\therefore x$ is $-ve$

and $x \leq -\sqrt{\frac{8}{3}}$

$\Rightarrow x^2 \geq \frac{8}{3}$

$\Rightarrow 3x^2 \geq 8$

$\Rightarrow 3x^2 - 8 \geq 0 \Bigg)$

$\therefore$ y is m.i in $\left(-3, -\sqrt{\frac{8}{3}}\right]$

Consider $x \in \left[-\sqrt{\frac{8}{3}}, \sqrt{\frac{8}{3}}\right]$

i.e $-\sqrt{\frac{8}{3}} \leq x \leq \sqrt{\frac{8}{3}}$

$\Rightarrow |x| \leq \sqrt{\frac{8}{3}}$

$\Rightarrow |x|^2 \leq \frac{8}{3}$

$\Rightarrow x^2 \leq \frac{8}{3}$

$\Rightarrow 3x^2 \leq 8$

$\Rightarrow 3x^2 - 8 \leq 0$

$\therefore \frac{dy}{dx}$ is $3x^2 - 8 \leq 0$

$\therefore y$ m.d in $\left[-\sqrt{\frac{8}{3}}, \sqrt{\frac{8}{3}}\right]$

Consider $x \in \left[\sqrt{\frac{8}{3}}, \infty\right)$

i.e $x \geq \sqrt{\frac{8}{3}}$

$\Rightarrow x^2 \geq \frac{8}{3}$

$\Rightarrow 3x^2 \geq 8$

$\Rightarrow 3x^2 - 8 \geq 0$

$\therefore \frac{dy}{dx} = 3x^2 - 8 \geq 0$

$\therefore y = $ m.i in $\left[\sqrt{\frac{8}{3}}, \infty\right)$

$\therefore$

y is m.i in $\left(-3, -\sqrt{\frac{8}{3}}\right] \cup \left[\sqrt{\frac{8}{3}}, \infty\right)$

and m.d in $(-\infty, -3) \cup \left[-\sqrt{\frac{8}{3}}, \sqrt{\frac{8}{3}}\right]$

3. $f(x) = \frac{e^x}{x^p}$

$$f'(x) = \frac{x^p . e^x - e^x . p . x^{p-1}}{x^{2p}}$$

$$= \frac{x^{p-1} e^x (x - p)}{x^{2p}}$$

$$= \frac{\cancel{x^p} . e^x (x - p)}{x . \cancel{x^p} . x^p}$$

$$= \frac{e^x (x - p)}{x^{p+1}}$$

Or $x > p > 0$ then

$f'(x) > 0$

ie f is strictly increasing.

for $p = 0$

$f(x) = e^x$ which is strictly increasing in R

$\cancel{\frac{d}{dx} e^x}$

because $f'(x) = e^x > 0$.

$\therefore$ $f(x)$ is strictly increasing for $x > p \geq 0$

4. Let $f(x) = 2\sin x + \tan x - 3x$

$f'(x) = 2\cos x + \sec^2 x - 3 = g(x)$ (say)

$\therefore g'(x) = -2\sin x + 2\sec^2 x \cdot \tan x$.

$= -2\sin x + 2\sec^2 x \cdot \frac{\sin x}{\cos x}$

$= 2\sin x(\sec^3 x - 1) > 0$

$\left\{ \text{For } 0 < x < \frac{\pi}{2} \quad \sec^3 x > 1 \text{ and } \sin x > 0 \right)$

$\therefore$ g is m.i in $(0, \frac{\pi}{2})$

$\therefore$ $g(x) > g(0)$ $(\because x > 0)$

$\Rightarrow g(x) > 0$

$(\because g(0) = 0)$

$\Rightarrow f'(x) > 0$ for $x \in (0, \frac{\pi}{2})$

$\Rightarrow$ f is ~~[illegible]~~ strictly increasing in $(0, \frac{\pi}{2})$.

$\therefore f(x) > f(0) \quad (\because x > 0)$

$\Rightarrow f(x) > 0 \quad (\because f(0) = 0)$

$\Rightarrow 2\sin x + \tan x - 3x > 0$

$\Rightarrow 2\sin x + \tan x > 3x$ (Proved)

Note that in $0 < x < 1$

$\ln x$ value is negative

Ex: $\ln \frac{1}{2} = \ln 1 - \ln 2 = 0 - .3010 = -0.\text{ve}$

~~$\ln \frac{2}{3} = \ln \left(\frac{1}{1-\frac{1}{3}}\right)$~~

$\ln\left(\frac{2}{3}\right) = \ln\left(\frac{1}{\frac{3}{2}}\right) = \ln\left(\frac{1}{1 \ldots}\right.$

1) Give an example of a function which is discontinuous at two points.

Ans) $f(x) = \begin{cases} 2x+1, & \text{if } x \leq 0 \\ 2x-1 & \text{if } 0 < x < 1 \\ 3x+5 & \text{if } x \geq 1 \end{cases}$

Here $f(x)$ is discontinuous at

$x = 0$, and $x = 1$

2. $f(x) = [x]$, S C $(-1, 1)$

$f'(x) = 0$

3. Q. $c > \sqrt{a^2+b^2}$, then is the eqn

$a\cos\theta + b\sin\theta = c$ consistent?

$\cos c$

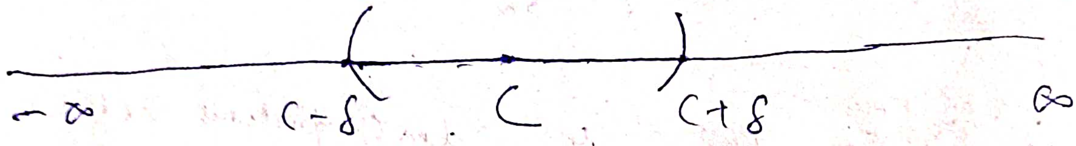
$\sqrt{a^2+b^2}$

Imp

Maxima and Minima

Neighbourhood of a Point

Let $c \in \mathbb{R}$ then $(c-\delta, c+\delta)$ is called a neighbourhood of c (nbd, nhd)



Absolute max^m and Absolute min^m

A function f is said to have absolute max^m or global max^m at c if $f(c) \geq f(x)$ $\forall x \in \text{dom} f$

A function f is said to have absolute min^m or global min^m at c if

$$f(c) \leq f(x) \quad \forall x \in \text{dom} f$$

Local max^m and local min^m

A function f is said to have

local max^m or relative max^m or max^m at c if $\exists$ a neighbourhood $(c-\delta, c+\delta)$ such that $f(c) \geq f(x)$ $\forall x \in (c-\delta, c+\delta)$

A function f is said to have local min^m or relative min^m or min^m at c if $\exists$ a ~~nob~~ nbd $((c-\delta, c+\delta)$ such that $f(c) \leq f(x)$ $\forall x \in (c-\delta, c+\delta)$

Notes

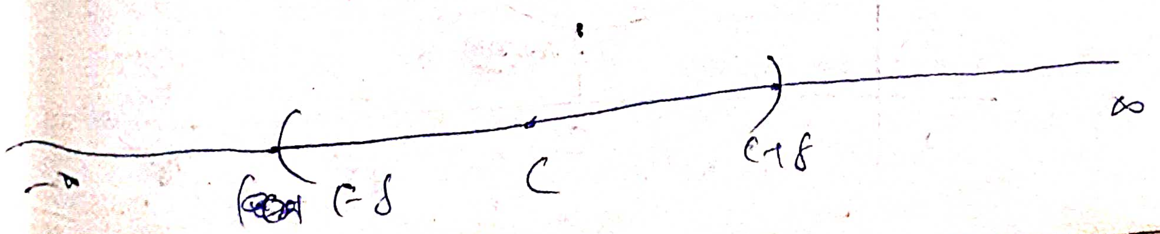
1. Local max^m $\leq$ Absolute max^m.

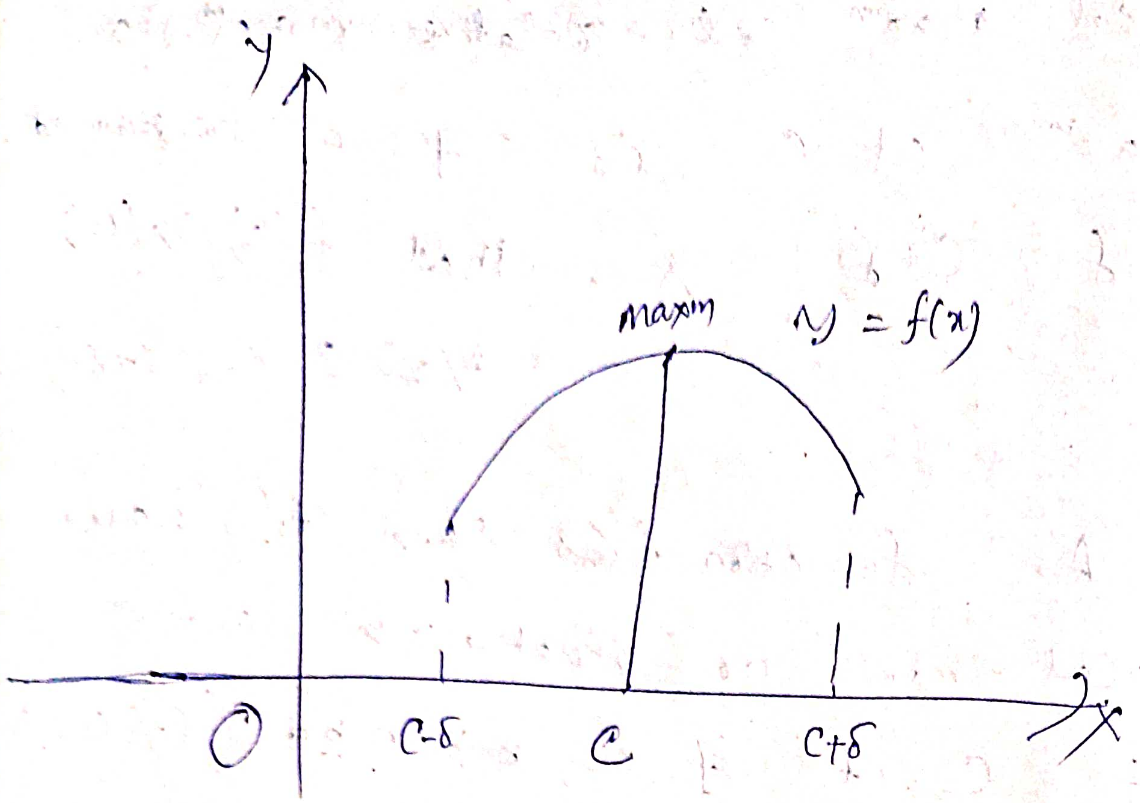
~~2.~~ Local min^m $\cancel{\geq}$ Absolute min^m.

2. Extrema (Calculus) or Optima (Other application) means either maxima or minima.

Process of finding local extrema

1st derivative test



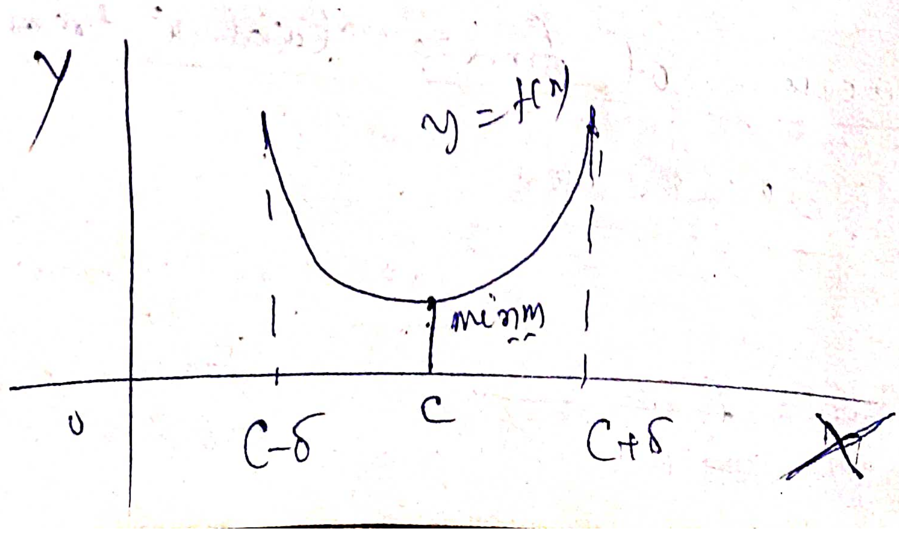


Suppose a function f is increasing in $(c-\delta, c]$ and it decrease in $[c, c+\delta)$ then f has local $\max^m$ value at $x=c$

i.e if $f'(x) \geq 0 \ \forall x \in (c-\delta, c]$

and $f'(x) \leq 0 \ \forall x \in [c, c+\delta)$

then f has local $\max^m$ value at $x=c$.



Suppose a function f is decreasing

$(c-\delta, c]$ and it is increasing

in $[c, c+\delta)$

then f has local min^m value at $x=c$

i.e if $f'(x) \leq 0 \ \forall x \in (c-\delta, c]$

and $f'(x) \geq 0 \ \forall x \in [c, c+\delta)$

then has f has local

min^m value at $x=c$

2nd derivative test

Suppose we have to find local

extrema of a function $f(x)$

we proceed as per the following steps

(i) Find $f'(x)$

(ii) Put $f'(x)=0$ and solve it

The values of x obtained are

called critical points or stationary

points.

iii) Find $f''(x)$

Suppose $x = c$ is a Critical point

If $f''(c) > 0$ then f is local min^m at $x = c$. and $f(c)$ is the local min^m value.

If $f''(c) < 0$ then f is local max^m at $x = c$, and $f(c)$ is the local max^m value.

If $f''(c) = 0$, and $f'''(c) \neq 0$ then f is neither local max^m nor local min^m at $x = c$.

Here $x = c$ is called 'point of ~~inflection~~' (inflexion) or 'saddle point'

If $f''(c) = 0$ and $f'''(c) = 0$ then find $f^{IV}(c)$. If $f^{IV}(c) < 0$ then f is local max^m at $x = c$ and if $f^{IV}(c) > 0$ then f is local min^m at $x = c$ and so on.

To find Point of inflexion if ~~ff~~ to find f''' is difficult

Problem (I) Find the points where $f''(x) = 0$ or $f''(x)$ does not exist.

(II) Test wheather $f''(x)$ changes sign on two sides of these point.

1. Find the extrema of the function $f(x) = 2x^3 - 3x^2 - 12x + 6$

Ans:
~~$f'(x) = 2$~~

$f(x) = 2x^3 - 3x^2 - 12x + 6$

$f'(x) = 6x^2 - 6x - 12 = 0$

for critical points.

$x^2 - x - 2 = 0$

$\Rightarrow (x-2)(x+1) = 0$

$\Rightarrow x = 2, -1$ are the critical points.

$f''(x) = 12x - 6$

$f''(2) = 18 > 0$

$\therefore f(x)$ local minm at $x = 2$.

Local minm value $= f(2)$

$2 \cdot 8 - 3 \cdot 4 - 2 \cdot 12 + 6 = -14$

$f''(-1) = -18 < 0$

$\therefore f$ is local max^m at $x = -1$

$\therefore$ Local max^m value is $f(-1)$

$= 2.(-1)^3 - 3(-1)^2 - 12(-1) + 6$

$= -2 - 3 + 12 + 6$

$= 13$ (Ans)

$\therefore$ Local max^m value is equal 13 which occures at $x=-1$

Local min^m value is equal $= -17$ which occures at $x=2$

Find the absolute max^m and absolute min^m of $f(x) = x^2 - 6x + 2$ in $[0, 4]$

Ans :

$$f(x) = x^2 - 6x + 2$$

$$f'(x) = 2x - 6 = 0$$

for critical points

$\Rightarrow x = 3$ is the critical point, and $3 \in [0, 4]$

$$f''(x) = 2$$

$\therefore f''(3) = 2 > 0$

$\therefore f$ is local min^m at $x = 3$

Local $\min^m_n$ value $= f(3) =$

$$3^2 - 6.(3) + 2$$

$$= -7$$

Also $f(0) = 2$

$f(4) = 16 - 24 + 2 = -6$

Here absolute $\max^m$ value $= 2$ which occurs at $x = 0$

Absolute $\min^m$ value $= -7$ which occures at $x = 3$.

3. Find the extrema of the function $f(x) = 4x^3 - 18x^2 + 27x - 7$

Ans: $f(x) = 4x^3 - 18x^2 + 27x - 7$

$f'(x) = 12x^2 - 36x + 27 = 0$ for critical points

$\Rightarrow 3(4x^2 - 12x + 9) = 0$

$\Rightarrow 3(2x - 3)^2 = 0$

$\Rightarrow 2x = 3$

$\Rightarrow x = \frac{3}{2}$ is the critical point

$f''(x) = 24x - 36 = 12(2x-3)$

$f''\left(\frac{3}{2}\right) = 24.\frac{3}{2} - 36 = 0$

$f'''(x) = 24$

$f'''\left(\frac{3}{2}\right) = 24 \neq 0$

At $x = \frac{3}{2}$, f ~~been~~ is neither max^m nor min^m.

i.e $x = \frac{3}{2}$ is the point of inflection.

Q.8 $\boxed{9, 18 \quad 2010, 2014 \text{ question}}$

10. Let the two +ve numbers be

x and y

Given that $x + y$ = constant $= K$ (say)

$\Rightarrow y = K - x$

Product = $xy = x.(K-x) = Kx - x^2$

$= f(x)$ (say)

Now $f(x) = Kx - x^2$

$f'(x) = K - 2x = 0$ for critical points

~~Critical~~ $\Rightarrow x = \frac{K}{2}$ is the critical point.

$f''(x) = -2$

$f''\left(\frac{K}{2}\right) = -2 < 0$

$\therefore$ $f(x)$ is maximum at $x = \frac{K}{2}$

i.e. product is max^m when $x = \frac{K}{2}$

and $y = K - x$

$= K - \frac{K}{2}$

$= \frac{K}{2}$

$\therefore$ Product is max^m when $x = y$ (proved)

9. Given that $x + y = 15$

$\Rightarrow y = 15 - x$

$xy^2 = x \cdot (15-x)^2 = x(225 + x^2 - 30x)$

$= 225x + x^3 - 30x^2$

$= f(x)$ (say)

$f(x) = 225x + x^3 - 30x^2$

$f'(x) = 225 + 3x^2 - 60x = 0$

for critical points

$\Rightarrow 3x^2 - 60x + 225 = 0$

$\Rightarrow x^2 - 20x + 75 = 0$

$\Rightarrow x^2 - 15x - 5x - 75 = 0$

$\Rightarrow x(x-15) - 5(x-15) = 0$

$\Rightarrow (x-15)(x-5) = 0$

$\therefore x = 15, 5$ ~~for critical points~~ are critical points

$f''(x) = 6x - 60$

$f''(15) = 30 > 0$

$\therefore f$ is min^m at $x = 15$

$f''(x) = 6x - 60$

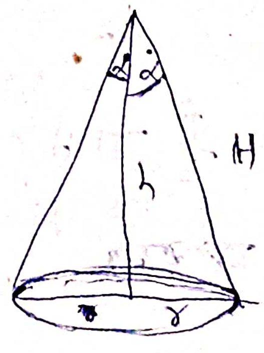
$f''(5) = -30 < 0$

$\therefore f$ is max^m at $x = 5$.

$\therefore f(x)$ is max^m at $x = 5$ and $y = 15 - 5 = 10$

$\sin = \frac{r}{H}$

14.



Let α be semivertical angle of cone and H be its slant height and h be its height.

$\therefore \frac{h}{H} = \cos\alpha$

$\Rightarrow h = H\cos\alpha$

Let r be it's radius

$$\frac{r}{H} = \sin\alpha$$

$$\Rightarrow r = H\sin\alpha$$

Given that H is Constant

$$\text{Volume} = \frac{1}{3}\cdot\pi r^2 h$$

$$= \frac{1}{3}\cdot\pi H^2\sin^2\alpha \cdot H\cos\alpha$$

$$= \frac{1}{3}\pi H^3 \sin^2\alpha\cdot\cos\alpha$$

$$= \frac{1}{3}\pi H^3 (1-\cos^2\alpha)\cos\alpha$$

$$= \frac{1}{3}\pi H^3 \{\cos\alpha - \cos^3\alpha\}$$

$$= V \text{ (Say)}$$

$$\frac{dv}{d\alpha} = \frac{d\left(\frac{1}{3}\pi H^3(\cos\alpha - \cos^3\alpha)\right)}{d\alpha}$$

$$\Rightarrow \frac{1}{3}\pi H^3\{-\sin\alpha + 3\cos^2\alpha\sin\alpha\} = 0$$

for critical points

$$\Rightarrow 3\cos^2\alpha\sin\alpha - \sin\alpha = 0$$

$$\Rightarrow \sin\alpha(3\cos^2\alpha - 1) = 0$$

$3\cos^2\alpha - 1 = 0 \quad (\because \sin\alpha \neq 0)$

$\Rightarrow \cos^2\alpha = \frac{1}{3}$

Rerun

$\cancel{3}\left(\frac{-\cancel{\cos}\,1^{3}}{3\sqrt{3}}\right)$

$\Rightarrow \sec^2\alpha = 3$

$\Rightarrow \tan^2\alpha = \sec^2\alpha - 1 = 3 - 1 = 2$

$\Rightarrow \tan\alpha = \pm\sqrt{2}$

$\Rightarrow$ But α is acute

$\therefore \tan\alpha$ is +ve

$\frac{-\cancel{3}\,1}{3\sqrt{3}}$

$\therefore \tan\alpha = \sqrt{2}$

$\Rightarrow \alpha = \tan^{-1}\sqrt{2}$

$$\frac{d^2V}{d\alpha^2} = \frac{1}{3}\pi H^3\left(-\cos\alpha + 3\left(-2\cos\alpha\sin^2\alpha + \cos^3\alpha\right)\right)$$

$$\left.\frac{d^2V}{d\alpha^2}\right]_{\alpha=\tan^{-1}\sqrt{2}} = \frac{1}{3}\pi H^3\left(-\frac{1}{\sqrt{3}} + 3\left(-2\cdot\frac{1}{\sqrt{3}}\cdot\frac{2}{3} + \frac{1}{3\sqrt{3}}\right)\right)$$

$\left(\because \sin^2\alpha = 1 - \cos^2\alpha = 1 - \frac{1}{3} = \frac{2}{3}\right)$

$$= \frac{1}{3}\pi H^3\left(-\cancel{\frac{1}{\sqrt{3}}} - \frac{4}{\sqrt{3}} + \cancel{\frac{1}{\sqrt{3}}}\right)$$

$$= -\frac{4}{3\sqrt{3}}\pi H^3 < 0$$

$\therefore$ V is max^m when $\alpha = \tan^{-1}\sqrt{2}$

7. $f(x) = \begin{cases} (x+1)^2 , & x \leq 0 \\ (x-1)^2 , & x > 0 \end{cases}$

on $[-1, 1]$

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$$

At $x = 0$
$f(x) = (x+1)^2$
$f(0) = 1$.
$\therefore f(0) = 1$

$$= \lim_{x \to 0} \frac{f(x) - 1}{x}$$

R.H.D $\lim_{x \to 0+} \frac{f(x) - 1}{x}$

$$= \lim_{x \to 0+} \frac{(x-1)^2 - 1}{x}$$

$$= \lim_{x \to 0+} \frac{x^2 - 2x}{x}$$

$$= \lim_{x \to 0+} \frac{\cancel{x}(x - 2)}{\cancel{x}}$$

$$= -2$$

L.H.D $\lim_{x \to 0-} \frac{f(x) - 1}{x}$

$$= \lim_{x \to 0^-} \frac{x^2 \cancel{-1} + 2x \cancel{-1}}{x}$$

$$= \lim_{x \to 0^-} x + 2$$

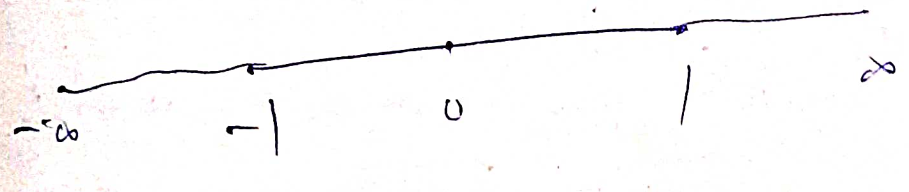
$$= 2$$

R.H.D $\neq$ L.H.D

$\therefore f'(0)$ does not exist.

$\therefore x = 0$ is a critical point.

and $0 \in [-1, 1]$.



Consider $\cancel{x \in [-1, 0]}$

$\therefore x \in (-1, 0)$ i.e $x < 0$

$\therefore f(x) = (x+1)^2$

$f'(x) = 2(x+1) > 0$ $\left(\because x > -1 \atop \Rightarrow x+1 > 0\right)$

$\therefore f$ is m.i in $(-1, 0)$

Consider $x \in (0, 1)$ i.e $0 < x < 1$

$$f(x) = (x-1)^2$$

$$f'(x) = 2(x-1) < 0$$

$\left(\because x < 1 \atop x-1 < 0\right)$

$\therefore f$ is m.d in $(0, 1)$

$\therefore f$ has local max^m at $x = 0$

and local max^m value of

$$f(0) = 1.$$

$f(1) = 0$

$f(-1) = 0$

$\therefore$ Absolute max^m value $= 1$ which occurs at $x = 0$

and absolute min^m value $= 0$ which occurs at $x = \pm 1$

11-(s)

$$\text{How} = m(m-1)(m-2) \text{------} (m-n+1)$$

$$= \frac{m!}{(m-n)!}$$

9.04.2K

Derivative as rate measure

Let $y = f(x)$ be the function. Let ~~δx,~~ $\delta x, \delta y$ be small changes in x and y respectively.

$\frac{\delta y}{\delta x}$ represents the average rate of change of y with respect to x in $(x, x+\delta x)$. $\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{dy}{dx}$ is called instantaneous rate of change of y with respect x. Thus the rate of change of y with respect to x represents the derivative.

Velocity and acc^n

Let a particle moves a distance S in time 't' such that $S = f(t)$ then

Speed or magnitude of velocity

$$= \frac{ds}{dt} = v$$

Magnitude of accn $= a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$ at

any instant t.

11-(P)

1\.

(iv) $S = t^3 - 6t^2 + 15t + 12$

$$v = \frac{ds}{dt} = \frac{d}{dt}(t^3 - 6t^2 + 15t + 12)$$

$$= 3t^2 - 12t + 15$$

At the end of two seconds

$$V = \left.\frac{ds}{dt}\right]_{t=2} = 12 - 24 + 15$$

$$= 3 \text{ unit/sec}$$

$$a = \frac{dv}{dt} = 6t - 12$$

At the end of two sec

$$a = \left.\frac{dv}{dt}\right]_{t=2} = 0$$

2\. Let each side of equilateral triangle

be x

Given that $x = 4$ cm

The rate of increase of side

$= \frac{dx}{dt} = \sqrt{3}$ cm. (Given)

Area of the triangle $= \frac{\sqrt{3}}{4} x^2$

Rate of increase of area

$= \frac{d}{dt}\left(\frac{\sqrt{3}}{4} x^2\right)$

$= \frac{\sqrt{3}}{4} \times 2x \times \frac{dx}{dt}$

$= \frac{\sqrt{3}}{\cancel{4}} \times 2 \times \cancel{4} \times \sqrt{3}$

$= 6$ cm^2/sec.

4. Let the side of cube be x

Surface area $= 6x^2 = S$

$\frac{dS}{dt} = 12x \cdot \frac{dx}{dt}$

but given that $\frac{dS}{dt} = 15$ and $x = 5$

$\therefore$ Rate at which its edge is decreasing $= \frac{dx}{dt} = \frac{1}{12x} \times \frac{dS}{dt}$

$= \frac{1}{60} \times 15 = \frac{1}{4}$ cm/sec

Increments and differentials

Let $y = f(x)$ be a function, Here x and y are independent and dependent variables respectively. The increment in x is denoted by δx or Δx or h.

and the increment in y is denoted by

δy or Δy or K

$$\therefore \quad y + \delta y = f(x + \delta x)$$

$$\Rightarrow \quad \delta y = f(x+\delta x) - y$$

$$\Rightarrow \quad \boxed{\delta y = f(x+\delta x) - f(x)} \quad \text{——— (i)}$$

The differential in y is denoted by dy. The differential in x is denoted by dx. The differential in y is the product of its derivative and increment in x

$$\text{i.e} \quad \boxed{dy = f'(x) \times \delta x} \quad \text{——— (ii)}$$

$$\text{Also} \quad \boxed{dy \simeq \delta y} \quad \text{——— (iii)}$$

Note ÷

(i) We know $\boxed{dy = f'(x) \cdot dx}$ for any function $f(x)$

Taking $y = f(x) = x$

$\therefore f'(x) = 1$

$\therefore$ We get $dx = 1 \cdot \delta x$

$$\boxed{\therefore dx = \delta x}$$

Hence $dy = f'(x) \cdot dx$

i.e $\boxed{dy = \frac{dy}{dx} \cdot dx}$ —— (IV)

(ii) Relative error in y

$= \frac{\delta y}{y}$

% error $= \frac{\delta y}{y} \times 100$

Problem

1. Find δy if $y = x^2 - 3x + 2$,
$x = 3$
$\delta x = \cancel{0.02}\ .02$

Ans: $y = x^2 - 3x + 2$, $x = 3$, $\delta x = .02$

Let $y = f(x) = x^2 - 3x + 2$
$f(x+\delta x) = (x+\delta x)^2 - 3(x+\delta x) + 2$

$\delta y = f(x+\delta x) - f(x)$

$$\delta y = (x+\delta x)^2 - 3(x+\delta x) + \cancel{2} - x^2 + 3x - \cancel{2}$$

$$= \cancel{x^2} + (\delta x)^2 + 2x.\delta x - \cancel{3x} - 3\delta x - \cancel{x^2} + \cancel{3x}$$

$$= (\delta x)^2 + \cancel{\delta x}\ 2.x\,\delta x - 3\,\delta x$$

$$= (.02)^2 + 2.(3).(.02) - 3.(.02)$$

$$= .0004 + .12 - .06$$

$$= \cancel{.0016 - .06}$$

$$= .1204 - .06$$

$$= .0604$$

$$\begin{array}{r} .0004 \\ 12 \\ \hline .0016 \end{array}$$

$$\begin{array}{r} .0004 \\ .1200 \\ \hline .1204 \\ .0600 \\ \hline .0604 \end{array}$$

2. Find <u>differential dy</u>

if $y = x^2 - 3x + 2$

$x = 3$

$\delta x = .02$

<u>Ans</u>

Let $y = f(x) = x^2 - 3x + 2$, when

$x = 3$, $\delta x = .02$

$f'(x) = 2x - 3$

$$dy = \cancel{dy}\ f'(x).\,\delta x$$

$$= (2x-3) \times .02$$

$$= 3 \times .02$$

$$= .06$$

~~3. Find the approximate value of~~

3. Find the differential dy

ii) $y = \dfrac{x^2}{1+x^2}$

Ans: Let $y = f(x) = \dfrac{x^2}{1+x^2}$

$$f'(x) = \frac{(1+x^2)\,2x - (x^2)(2x)}{(1+x^2)^2}$$

$y = \sqrt{x}$

$x = 4,\ \delta x = 5$

$$= \frac{2x(\cancel{1+x^2} - x^2)}{(1+x^2)^2}$$

$$= \frac{2x}{(1+x^2)^2}$$

$\therefore\ dy = f'(x)\cdot dx$

$$= \frac{2x}{(1+x^2)^2}\cdot dx$$

4. Find the approximate value of $\sqrt{4.5}$ using differential

Imp

Ans: Let $y = f(x) = \sqrt{x}$,

$x = 4,\ \delta x = 0.5$

$$f'(x) = \frac{1}{2\sqrt{x}}$$

We know $\delta y \approx dy$

$$\Rightarrow f(x+\delta x) - f(x) \approx f'(x) \cdot \delta x$$

$$\Rightarrow f(x+\delta x) \simeq f(x) + f'(x) \cdot \delta x$$

$$\Rightarrow \sqrt{x+\delta x} \simeq \sqrt{x} + \frac{1}{2\sqrt{x}} \cdot \delta x$$

$$\Rightarrow \sqrt{4.5} \simeq \sqrt{4} + \frac{1}{2\sqrt{4}} \cdot (0.5)$$

$$\Rightarrow \sqrt{4.5} \approx 2 + \frac{1}{4} \cdot 0.5$$

$$\approx 2 + .125$$

$$\approx 2.125$$

4. Here diameter $= y = 14$.

$\delta y = .02$.

$\therefore$ radius $= r = 7$.

$\delta r = .01$

Surface area $= S = 4\pi r^2 = 4\pi \cdot 7^2$

$= \cancel{4\pi \times 7 \times 7}$

$$\delta s \approx ds = 8\pi r \cdot \delta r$$

$= 8\pi \cdot 7 \cdot (\cdot 01) =$

~~$5 \cdot 7\pi$~~

% error $\frac{\delta s}{s} \times 100$

$= \frac{8\pi \cdot \cancel{7}^{2} \times \cdot 01}{\cancel{4}\cancel{\pi}\cancel{7}^{2} \cdot 7} \times 100$

$= \frac{2}{7}$ % (Ans)

5. Let side of the cube be x

Let V be its volume

$= x^3$

$\therefore f(x) = x^3$

$f'(x) = 3x^2$

Here $x = 3$

$\delta x = 0.04$

$\delta V \approx dV = f'(x) \cdot \delta x$

$= 3x^2 \cdot \delta x$

$= 3 \cdot 3^2 \times (\cdot 04)$

$= 1.08 \text{ cm}^3$

Function ~~and~~ of single variable

If in a function there is one independent variable and one dependent variable then the function is called function of single variable

e.g $y = f(x)$

Here x is independent variable and y is dependent variable.

Function of several variables

If in a function there are more than

one independent variable then it is

called function of several variables

Here there is ~~dos~~ one dependent variable.

e.g. $z = f(x_1, x_2, ------ x_n)$ is a

function of several variables.

Here $x_1, x_2, ---$ x_n are independent

variables and z is dependent variable.

Here we say that it is

a function of n - variables.

$z = f(x, y)$ is called function

of '2' variables.

$w = f(x, y, z)$ is called function

of 3 variables

Here w is dependendent variable

and x, y, z are independent variable.

In case of function of

single variable $y = f(x)$ i.e

we differentiate y w.r.t x then

the derivatives $\frac{dy}{dx}, \frac{d^2y}{dx^2}$ etc are

called ordinary derivatives.

In case of function of several variables $Z = f(x_1, x_2 \cdots, x_n)$

If we differentiate Z with respect to x_1 when all other variables are treated as constant, the derivative is called partial derivative of Z with respect to x_1 and we denote it as ~~(diva)~~ ∂ (diva)

$$\frac{\partial z}{\partial x_1}$$

or $\frac{\partial f}{\partial x_1}$ or f_{x_1}

Suppose $Z = f(x, y)$ be a function of two variables.

$$\frac{\partial Z}{\partial x} \text{ or } \frac{\partial f}{\partial x} \text{ or } f_x,$$

$\frac{\partial Z}{\partial y}$ or $\frac{\partial f}{\partial y}$ or f_y are called partial derivatives of first order

In $\frac{\partial z}{\partial x}$ we take y as Constant

In $\frac{\partial z}{\partial y}$ we take x as Constant.

Partial derivatives of 2nd order

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) \text{ or } f_{xx} \text{ or } f_{x^2}$$

$$\frac{\partial^2 z}{\partial x . \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) \text{ or } f_{xy}$$

$$\frac{\partial^2 z}{\partial y . \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) \text{ or } f_{yx}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) \text{ or } f_{y.y} \text{ or } f_{y^2}$$

For $u = f(x, y, z)$

~~There~~ There are 3 partial derivatives of first order

i.e $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ and 9 second order

or $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$ partial derivatives.

Problem : $f(x,y) = ax^2 + 2hxy + by^2$.

Find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial x \partial y}$,

$\frac{\partial^2 f}{\partial y . \partial x}$, $\frac{\partial^2 f}{\partial y^2}$

Ans : $f(x,y) = ax^2 + 2hxy + by^2$.

$$\frac{\partial f}{\partial x} = a.2x + 2hy + 0$$

$$= 2ax + 2hy$$

$$\frac{\partial f}{\partial y} = 2hx + b2y.$$

$$= 2hx + 2by$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}\left(2ax + 2hy\right)$$

$$= 2a$$

$$\frac{\partial^2 f}{\partial x . \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$\frac{\partial}{\partial x}\left(2hx + 2by\right)$$

$$= 2h \cdot 1 + 0$$

$$= 2h$$

$$\frac{\partial^2 f}{\partial y \cdot \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$$= \frac{\partial}{\partial y}\left(2ax + 2hy\right)$$

$$= 0 + 2h \cdot 1$$

$$= 2h$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial y}\left(2hx + 2by\right)$$

$$= 0 + 2b \cdot 1$$

$$= 2b$$

$$\cancel{\frac{-2(x-y)}{-\frac{2}{x}(x-y)}}$$

Homogeneous functions

If in a function $Z = f(x,y)$ the sum of the powers of x and y in each term in $f(x,y)$ remains same. Then the function is called homogeneous function. If the sum of the powers of x and y in each term is n then it is called homogeneous function of degree n.

Ex: $f(x,y) = x^2 + xy + y^2$ is a homogeneous function of 2nd degree

Notes:

. A function f is said to be homogeneous in x and y of degree n if ~~$f(tx, ty)$~~ $f(tx, ty) = t^n f(x,y)$ where t is +ve.

Ex :- Prove that $f(x,y) = x^2+xy+y^2$ is a homogeneous function of 2nd degree.

Proof :-

$$f(x,y) = x^2+xy+y^2$$

$$f(tx,ty) = t^2x^2 + t^2x y + t^2y^2$$

$$= t^2(x^2+xy+y^2)$$

$$= t^2 f(x,y)$$

$\therefore$ f is a homogeneous function of 2nd degree.

2. A function $f(x,y)$ is said to be homogeneous in x and y of degree n if $f(x,y) = x^n.\phi\left(\frac{y}{x}\right)$

or if $f(x,y) = y^n.\psi\left(\frac{x}{y}\right)$

Ex : Prove that $f(x,y) = x^2+xy+y^2$ is a homogeneous function of 2nd degree.

Proof :

$$f(x,y) = x^2+xy+y^2$$

$$= x^2\left\{1+\frac{y}{x}+\left(\frac{y}{x}\right)^2\right\}$$

$= x^2 \, \phi\left(\frac{y}{x}\right)$

$\Rightarrow f(x,y)$ is homogeneous function of 2nd degree.

$$f(x,y) = x^2 + xy + y^2$$

$$= y^2\left(\left(\frac{x}{y}\right)^2 + \frac{x}{y} + 1\right)$$

$$= y^2 \cdot \psi\left(\frac{x}{y}\right)$$

$\therefore f(x,y)$ is homogeneous function of 2nd degree

Euler's Theorem

Suppose $f(x,y)$ is a homogenous function of degree 'n' then

$$x \cdot \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = n f(x,y)$$

Ex: If $f(x,y,z) = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$

Then proove that $x \frac{\partial f}{\partial x} + y \cdot \frac{\partial f}{\partial y} + z \cdot \frac{\partial f}{\partial z}$

$= 0$

Proof :-

$$f(x,y,z) = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$$

$$\frac{\partial f}{\partial x} = \frac{1}{y} + 0 + z\left(\frac{-1}{x^2}\right)$$

$$= \frac{1}{y} - \frac{z}{x^2}$$

$$\frac{\partial f}{\partial y} = x\left(\frac{-1}{y^2}\right) + \frac{1}{z} + 0$$

$$= \frac{1}{z} - \frac{x}{y^2}$$

$$\frac{\partial f}{\partial z} = 0 + y\left(\frac{-1}{z^2}\right) + \frac{1}{x}$$

$$= \frac{1}{x} - \frac{y}{z^2}$$

L.H.S

$$= x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z}$$

$$= x\left(\frac{1}{y} - \frac{z}{x^2}\right) + y\left(\frac{1}{z} - \frac{x}{y^2}\right) + z\left(\frac{1}{x} - \frac{y}{z^2}\right)$$

$$= \frac{x}{y} - \frac{xz}{x^2} + \frac{y}{z} - \frac{yx}{y^2} + \frac{z}{x} - \frac{yz}{z^2}$$

$$= \cancel{\frac{x}{y}} - \cancel{\frac{z}{x}} + \cancel{\frac{y}{z}} - \cancel{\frac{x}{y}} + \cancel{\frac{z}{x}} - \cancel{\frac{y}{z}}$$

$= 0$ (R.H.S)

Q8

$$f(x,y,z) = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$$

~~Which is a homogeneous function in x, y, and z~~

$$f(tx,ty,tz) = \frac{tx}{ty} + \frac{ty}{tz} + \frac{tz}{tx}$$

$$= t^0\left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x}\right)$$

$$= t^0 f(x,y,z)$$

$\therefore$ $f(x,y,z)$ is a homogeneous function of degree 0.

By Euler's theorem

if $f(x,y,z)$ is a homogeneous function of degree n then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = nf(x,y,z)$$

i.e. $$x \cdot \frac{\partial f}{\partial x} + y \cdot \frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = 0 \cdot f(x,y,z)$$

$$= 0$$

(Proved)

Exercise - II(V)

4. $Z = f\left(\frac{y}{x}\right)$

$$\frac{\partial z}{\partial x} = f'\left(\frac{y}{x}\right) \cdot \left(\frac{-y}{x^2}\right)$$

$$\therefore x \cdot \frac{\partial z}{\partial x} = -f'\left(\frac{y}{x}\right) \cdot \left(\frac{y}{x}\right)$$

$$\frac{\partial z}{\partial y} = f'\left(\frac{y}{x}\right) \cdot \left(\frac{1}{x}\right)$$

$$\therefore y \cdot \frac{\partial z}{\partial y} = f'\left(\frac{y}{x}\right) \cdot \frac{y}{x}$$

L.H.S

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y}$$

$$= -f'\left(\frac{y}{x}\right) \frac{y}{x} + f'\left(\frac{y}{x}\right) \cdot \left(\frac{y}{x}\right)$$

$$= 0 \quad (R.H.S).$$

OR

$$Z = f\left(\frac{y}{x}\right)$$

$$= x^0 \cdot f\left(\frac{y}{x}\right)$$

$\therefore$ Z is a homogenous function of degree 0

By Euler's Theorem of f

f(x,y) is a homogenous function

of degree n then

$$x \cdot \frac{\partial f}{\partial x} + y \cdot \frac{\partial f}{\partial y} = n f(x,y)$$

i.e $x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = 0 \cdot z$

$= 0$

i.e $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 0$

11-W

(i) $z = x^2 y^2 + 4xy^3 - 3x^3 y = f(x,y)$

$$f(tx, ty) = t^2x^2 \cdot t^2y^2 + 4\, t x \cdot t^3y^3 - 3\, t^3x^3 \cdot ty$$

$$= t^4 (x^2y^2 + 4xy^3 - 3x^3y)$$

$$= t^4 \cdot f(x,y)$$

$\therefore z = f(x,y)$ is a homogenous function of degree 4.

By Euler's theorem

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 4 \cdot z$$

(iii) $\dfrac{\partial^2 f}{\partial x\, \partial z} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial z}\right)$ *

(iv) $\dfrac{\partial^2 f}{\partial y\, \partial z} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial z}\right)$

We will verify it.

Now $z = x^2y^2 + 4xy^3 - 3x^3y$

$$\frac{\partial z}{\partial x} = y^2 2x + 4y^3 - 3y\, 3x^2$$

$$= 2xy^2 + 4y^3 - 9x^2y$$

$$\frac{\partial z}{\partial y} = x^2 . 2y + 4x . 3y^2 - 3x^3$$

$$= 2x^2y + 12xy^2 - 3x^3$$

L.H.S $= x . \dfrac{\partial z}{\partial x} + y . \dfrac{\partial z}{\partial y}$

$= x . (2xy^2 + 4y^3 - 9x^2y) + y(2x^2y + 12xy^2 - 3x^3)$

$= 2x^2y^2 + 4xy^3 - 9x^3y + 2x^2y^2 + 12xy^3 - 3x^3y$

$= 4x^2y^2 + 16xy^3 - 12x^3y$

$= 4(x^2y^2 + 4xy^3 - 3x^3y)$

$= 4 . z$ (R.H.S) $\therefore$ Euler's theorem is verified

Jay Jagannath

1. Evaluate

$$\lim_{x\to\infty}\left(\frac{1^{\frac{1}{x}}+2^{\frac{1}{x}}+3^{\frac{1}{x}}+\cdots\cdots\rightarrow n^{\frac{1}{x}}}{n}\right)^{nx}$$

Ans :

$$\lim_{x\to\infty}\left(\frac{1^{\frac{1}{x}}+2^{\frac{1}{x}}+3^{\frac{1}{x}}\cdots\cdots+n^{\frac{1}{x}}}{n}\right)^{nx}$$

Put $\frac{1}{x}=y$.

$x\to\infty$

$\Rightarrow y\to 0$

$$=\lim_{y\to 0}\left(\frac{1^y+2^y+3^y\cdots\cdots+n^y}{n}\right)^{\frac{n}{y}}$$

$$=\lim_{y\to 0}\left(\frac{1^y+2^y+3^y\cdots\cdots n^y+n-n}{n}\right)^{\frac{n}{y}}$$

$$=\lim_{y\to 0}\left(1+\frac{1^y+2^y+3^y+\cdots\cdots+n^y-n}{n}\right)^{\frac{n}{y}}$$

$$=\lim_{y\to 0}\left(1+\frac{(1^y-1)+(2^y-1)+(3^y-1)\cdots\cdots+(n^y-1)}{n}\right)^{\frac{n}{y}}$$

$$=\lim_{y\to 0}\left(1+\frac{(1^y-1)+(2^y-1)+(3^y-1)\cdots\cdots+(n^y-1)}{n}\right)^{\frac{n\cdot\{(1^y-1)+(2^y-1)+(n^y-1)\}}{(1^y-1+2^y-1)\cdots+n^y-1)\cdot y}}$$

$$=\lim_{z\to 0}(1+z)^{\frac{1}{z}}\cdot\left(\frac{(1^y-1)+(2^y-1)+\cdots(n^y-1)}{y}\right)$$

Put $\frac{(1^y-1)+(2^y-1)+(n^y-1)}{n}=z$

$y\to 0\Rightarrow z\to 0$

$$=\left\{\lim_{z\to 0}(1+z)^{\frac{1}{z}}\right\}^{\lim_{y\to 0}\left(\frac{1^y-1}{y}+\frac{2^y-1}{y}+\frac{3^y-1}{y}+\cdots\frac{n^y-1}{y}\right)}$$

$$= e^{\ln 1+\ln 2+\ln 3+\ln 4\cdots\cdots\ln n}$$

(Ans)

2. To prove $dy = f'(x).\delta x$

Proof :

Let a function f be differentiable on an interval (a,b).

Let $y = f(x)$

$\Rightarrow y + \delta y = f(x+\delta x)$

$\Rightarrow \delta y = f(x+\delta x) - y = f(x+\delta x) - f(x)$

From defn of derivative

$$\frac{dy}{dx} = f'(x) = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} \quad \text{—— (i)}$$

where δx and δy are small increments in x and y.

From (i), we can write

$$\lim_{\delta x \to 0} \left[\frac{\delta y}{\delta x} - f'(x) \right] = 0$$

or $\delta y = \delta x f'(x) + \delta x . \alpha$ —— (ii)

(where α is a function of δx such that $\delta x \to 0$ and $f'(x)$ is independent of δx)

The first term on the r.h.s of eqn (2)

(δx) is the principal part of δy. It is denoted by 'dy' or df and is called the differential of y relative to increment δx.

Thus $dy = \delta x \cdot f'(x)$

3. Prove $d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}$, provided $v \neq 0$

Proof: Let $y = \frac{u}{v}$ where u and v are functions of x, then

$$dy = \left[\frac{d}{dx}\left(\frac{u}{v}\right)\right] dx$$

$$= \left[\frac{v\frac{du}{dx} - u\cdot\frac{dv}{dx}}{v^2}\right] dx$$

$$= \left[\frac{v\cdot\frac{du}{dx}\cdot dx - u\cdot\frac{dv}{dx}\cdot dx}{v^2}\right]$$

$$= \left[\frac{v\,du - u\,dv}{v^2}\right]$$

(Proved)

or

Let $\frac{u}{v} = z$

$$dz = \frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv$$

$$= \frac{\partial}{\partial u}\left(\frac{u}{v}\right)du + \frac{\partial\left(\frac{u}{v}\right)}{\partial v}dv$$

$$= \frac{1}{v}\cdot du + u\cdot\frac{-1}{v^2}dv$$

$$= \frac{v\,du - u\,dv}{v^2}$$

$$\therefore d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}$$

— o —

R = Real numbers

Rolle's Theorem

Let $f : [a,b] \to R$ such that

(i) f is differentiable on (a,b)

(ii) f is continuous on $[a,b]$

(iii) $f(a) = f(b)$

then $\exists$ a point $c \in (a,b)$ such that

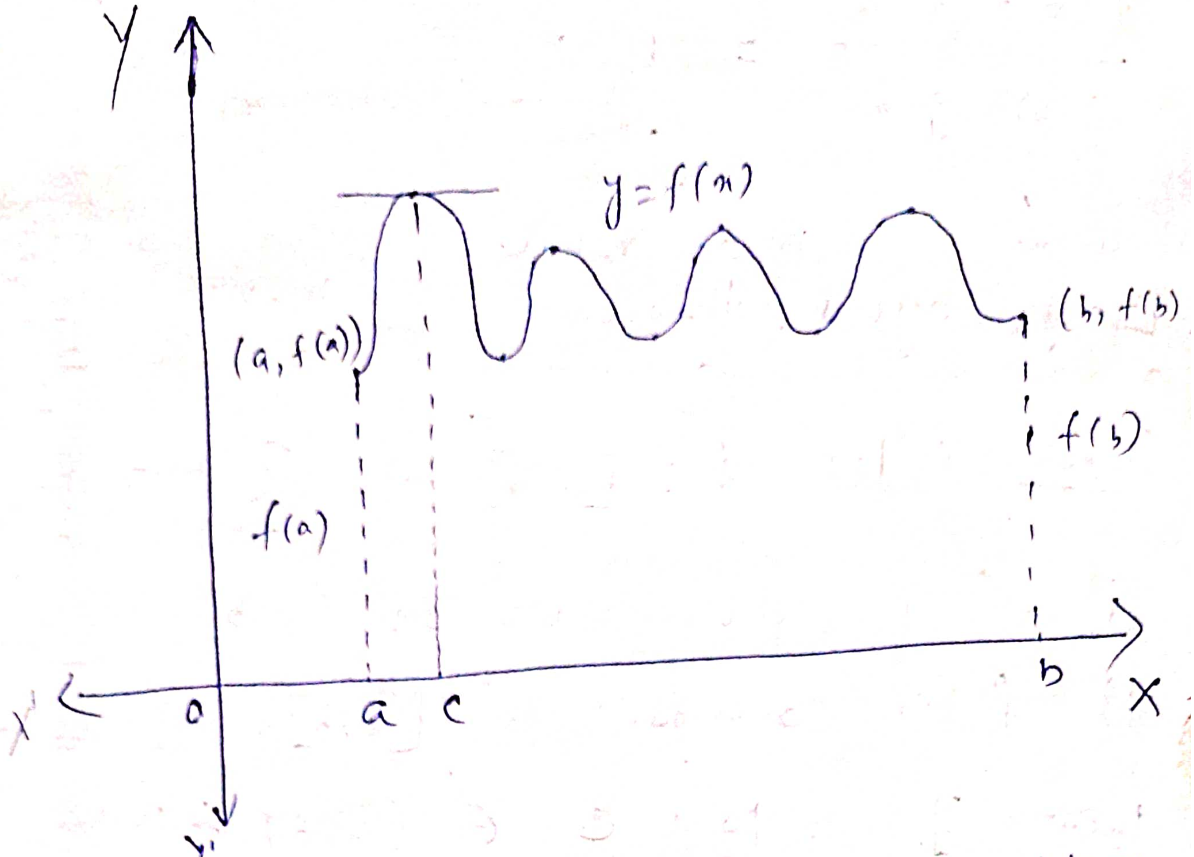
$$f'(c) = 0$$

Trick → close open – close open

$[a,b]$ – (a,b) – $[a,b]$ (a,b)

Geometrical interpretation

Between any two points $(a, f(a))$ and $(b, f(b))$ with equal ordinates on



the graph $y=f(x)$ there is at least One point $c \in (a,b)$ where the tangent to the curve is $\parallel$ to X-axis.

([illegible]) mean value
Cauchy's theorem :-

Statement :- Let $f : [a,b] \to R$ and $g : [a,b] \to R$ such that

(i) f and g are differentiable on (a,b)

(ii) f and g are continuous on $[a,b]$

(iii) $g'(x) \neq 0$ $\forall x \in (a,b)$ then

$\exists$ a point $c \in (a,b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

([illegible])

Lagrange's mean value theorem (m.v.t) OR (mean value theorem)

Statement: let $f : [a,b] \to \mathbb{R}$ such that

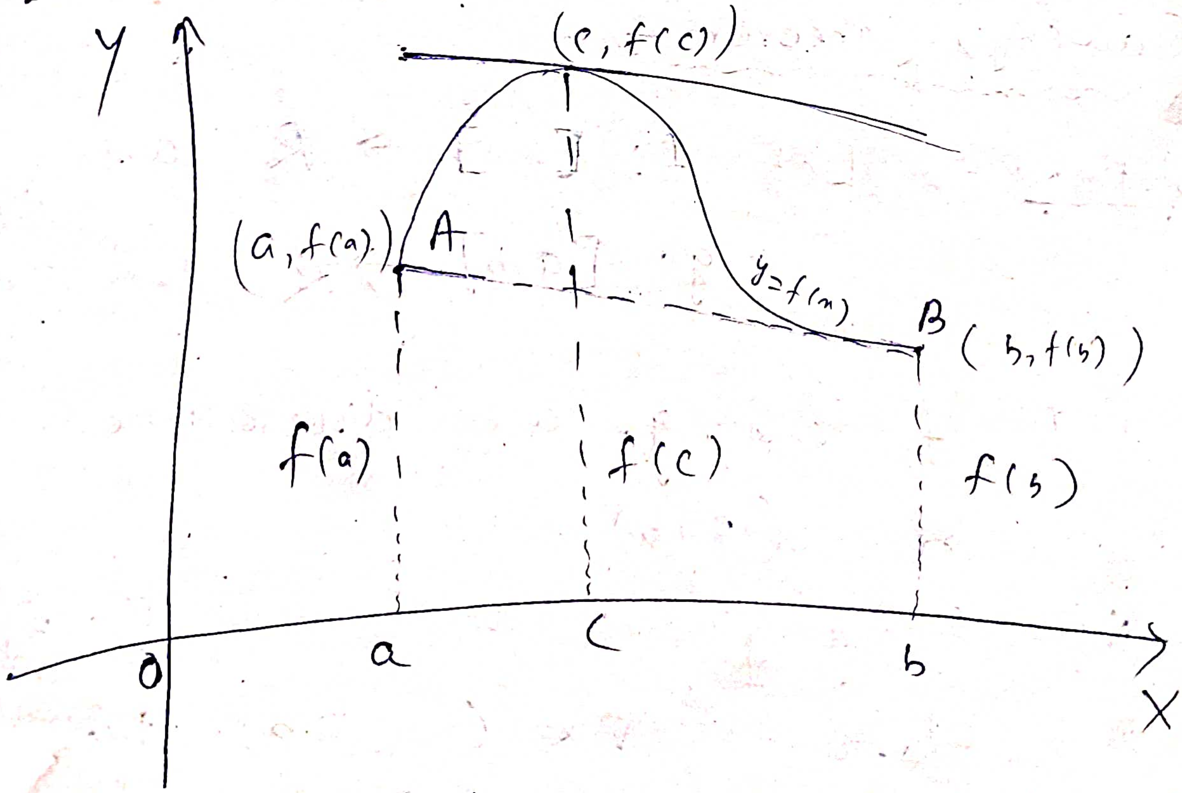
(i) f is differentiable on (a,b)

(ii) f is continuous on $[a,b]$

then $\exists$ a point $c \in (a,b)$ such that

$$\frac{f(b) - f(a)}{b-a} = f'(c)$$

Geometrical interpretation



Between any two points $(a, f(a))$

and $(b, f(b))$ on the graph $y = f(x)$

$\exists$ at least one point $c \in (a, b)$

where the tangent to the curve is $\parallel$ to

the line joining $(a, f(a))$ and $(b, f(b))$

Note :-

Lagrange's m.v.t can be obtained from

Cauchy's m.v.t by taking $g(x) = x$

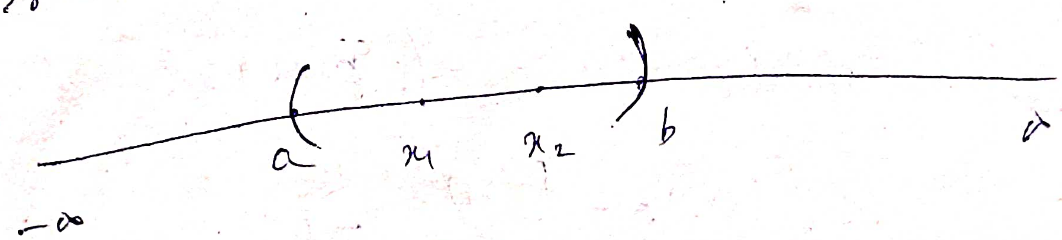
Ex : ~~66~~ 63

Prove that if ~~the~~ derivative of ~~the~~ a function is zero in an interval then the function is constant in that interval

Proof :- Let $f'(x) = 0 \quad \forall x \in (a, b)$

Let x_1, x_2 be any two number in (a, b) such that $a < x_1 < x_2 < b$

Since $f'(x) = 0 \quad \forall x \in (a, b)$.



$\therefore f'(x)$ exists in (a, b)

But $[x_1, x_2] \subset (a, b)$

$\therefore$ $f'(x)$ ~~exists~~ exists in $[x_1, x_2]$

$\Rightarrow$ f is differentiable in $[x_1, x_2]$

$\Rightarrow$ f is continuous in $[x_1, x_2]$

($\because$ Differentiability $\Rightarrow$ Continuity)

Now f is differentiable in $[x_1, x_2]$

$\Rightarrow$ f is differentiable in (x_1, x_2)

$\therefore$ Conditions of Lagranges ~~M.V.T~~ m.v.t are satisfied for f in $[x_1, x_2]$.

$\therefore$ There $\exists$ a point $c \in (x_1, x_2)$

such that $\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c)$

But $f'(c) = 0$ (given)

$\therefore$ $\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} = 0$

$\Rightarrow \quad f(x_2) - f(x_1) = 0$

$\Rightarrow \quad f(x_2) = f(x_1)$

where $x_1 \neq x_2$

$\therefore$ $f(x)$ has the same value at any 2 distinct points of the interval (a,b). Hence $f(x)$ is constant in (a,b). (proved)

Problems

1. Verify the ~~validity~~ validity of rolle's theorem and find the point $c \in (a,b)$ where $f'(c) = 0$ if Rolle's theorem holds good

(i) $f(x) = [x]$, $a = -\frac{1}{2}$, $b = \frac{3}{2}$

(ii) $f(x) = |x|$, $a = -1$, $b = 1$

(iii) $f(x) = (x-a)^m \cdot (x-b)^m$

(iv) $f(x) = 3x - x^3$, $a = 0$, $b = \sqrt{3}$

(v) $f(x) = \cos x$, $a = -\frac{\pi}{2}$, $b = \frac{\pi}{2}$

(vi) $f(x) = x^3 - 3x$, $a = -\sqrt{3}$, $b = 0$

(vii) $f(x) = x - 1$, $a = -\frac{1}{2}$, $b = \frac{1}{2}$

2. Using Lagrange's m.v.t find c in the following Cases.

(i) $f(x) = -x^2 + 3x + 2$, $a = 0$, $b = 1$

(ii) $f(x) = \ln x$, $a = 1$, $b = 2$

(iii) $f(x) = x^3 - 2x$, $a = 0$, $b = 2$

3. Verify Cauchy's m.v.t for $f(x) = \sin x$, $g(x) = \cos x$

$a = \frac{\pi}{4}$, $b = \frac{\pi}{3}$

4. f is continuous in $[a, b]$ and differentiable in (a, b) and $|f'(x)| \leq 1$ in (a, b) then prove that for any pair of points x_1, x_2 in (a, b),

$$|f(x_1) - f(x_2)| \leq |x_1 - x_2|$$

Class Ex $\sin\frac{\pi}{3} - \sin\frac{\pi}{6} \leq (\frac{\pi}{3} - \frac{\pi}{6})$

5. Using m.v.t. prove that for any pair of real numbers x, y,

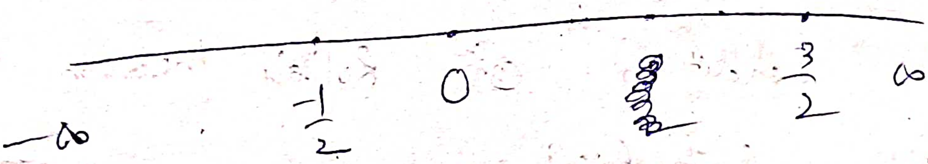
$$|\sin x - \sin y| \leq |x - y|$$

Ans

1.(i) $f(x) = [x]$

$a = -\frac{1}{2}$, $b = \frac{3}{2}$

$\therefore f : [-\frac{1}{2}, \frac{3}{2}] \to \mathbb{R}$



Here $\lim\limits_{x \to 0+} [x] = 0$

$\lim\limits_{x \to 0-} [x] = -1$

$\therefore \lim\limits_{x \to 0+} [x] \neq \lim\limits_{x \to 0-} [x]$

$\therefore \lim\limits_{x \to 0} [x]$ does not exist.

$\therefore f$ is discontinuous at $x = 0$

$\therefore f$ is not differentiable at $x = 0$

Similarly f is not differentiable at $x = 1$

But $0, 1 \in \left(-\frac{1}{2}, \frac{3}{2}\right)$

$\therefore$ ~~f is not~~

$\therefore$ Conditions of Rolle's theorem are not satisfied.

(ii) $f(x) = |x|$ which is not differentiable at $x = 0$

and $0 \in (-1, 1)$

Here $f : [-1, 1] \to \mathbb{R}$

$\therefore$ Conditions of Rolle's theorem are not satisfied.

(iii) $f(x) = (x-a)^n \cdot (x-b)^m$

Here $f : [a, b] \to \mathbb{R}$

$$f'(x) = (x-a)^n \cdot m(x-b)^{m-1} + (x-b)^m n (x-a)^{n-1}$$

$$= n(x-b)^m (x-a)^{n-1} + m(x-a)^n \cdot (x-b)^{m-1}$$

$f'(x)$ exists on (a, b)

Since $f(x)$ is a polynomial, it is continuous on $[a, b]$

Also

$f(a) = 0$.

$f(b) = 0$

$\therefore f(a) = f(b)$

$\therefore$ Conditions of Rolle's theorem are satisfied.

$\therefore$ ~~There~~ $\exists$ a point $c \in (a,b)$.

such that $f'(c) = 0$

$\Rightarrow n(c-b)^m \cdot (c-a)^{n-1} + m(c-b)^{m-1}(c-a)^n = 0$

$\Rightarrow (c-b)^{m-1} \cdot (c-a)^{n-1}\left(n(c-b) + m(c-a)\right) = 0$

But $c \in (a,b)$

$\therefore c \neq a, \quad c \neq b$

$\therefore c-a \neq 0, \quad c-b \neq 0$

$\therefore n(c-b) + m(c-a) = 0$

$\Rightarrow cn + cm = ma + nb$

$\Rightarrow c(n+m) = ma + nb$

$\Rightarrow c = \dfrac{ma+nb}{m+n}$

(v) $f(x) = \cos x$, $a = -\frac{\pi}{2}$, $b = \frac{\pi}{2}$

Here $f: [-\frac{\pi}{2}, \frac{\pi}{2}] \to R$

(i) $f'(x) = -\sin x$

$f'(x)$ exists on $\cancel{(a,b)}$ $(-\frac{\pi}{2}, \frac{\pi}{2})$

f is differentiable on $\cancel{(a,b)}$ $(-\frac{\pi}{2}, \frac{\pi}{2})$

(ii) $f(x) \overset{= \cos x}{\cancel{is}}$ Continous $[-\frac{\pi}{2}, \frac{\pi}{2}]$

Also $f(\frac{\pi}{2}) = 0$

$f(-\frac{\pi}{3}) = 0$

$f(\frac{\pi}{2}) = f(-\frac{\pi}{3})$

Conditions of Rolle's theorem are satisfied.

$\therefore \exists$ a point $c \in (-\frac{\pi}{2}, \frac{\pi}{2})$

such that $f'(c) = 0$

$\Rightarrow -\sin c = 0$

$\Rightarrow \sin c = 0$

$\therefore c = 0$ and $0 \in (-\frac{\pi}{2}, \frac{\pi}{2})$

2. (ii) $f(x) = \ln x$, $a=1$, $b=2$

Here $f : [1,2] \to R$

$f'(x) = \frac{1}{x}$ which exists in $(1,2)$

also $f(x) = \ln x$ is continuous $[1,2]$

Conditions of Lagrange's m.v.t are satisfied.

$\therefore \exists$ a point $c \in (1,2)$

such that

$$\Rightarrow \frac{f(2)-f(1)}{2-1} = f'(c)$$

$$\Rightarrow \frac{\ln 2 - \ln 1}{1} = f'(c)$$

$$\Rightarrow \ln 2 = \frac{1}{c}$$

$$\Rightarrow c = \frac{1}{\ln 2}$$

3. $f(x) = \sin x$, $g(x) = \cos x$

$a = \frac{\pi}{4}$, $b = \frac{\pi}{3}$

$\therefore f : [\frac{\pi}{4}, \frac{\pi}{3}] \to R$

and $g : [\frac{\pi}{4}, \frac{\pi}{3}] \to R$

Here f and g are continuous

on $\left[\frac{\pi}{4}, \frac{\pi}{3}\right]$

and f and g are differentiable

on $\left(\frac{\pi}{4}, \frac{\pi}{3}\right)$

$\therefore f'(x) = \cos x,\ g'(x) = -\sin x$

Also $g'(x) \neq 0$ in $\left(\frac{\pi}{4}, \frac{\pi}{3}\right)$

$\therefore$ Conditions of Cauchy's ~~mean~~ m.v.t are satisfied.

$\therefore \exists$ a point $c \in \left(\frac{\pi}{4}, \frac{\pi}{3}\right)$ such

that

$$\frac{f\left(\frac{\pi}{3}\right) - f\left(\frac{\pi}{4}\right)}{g\left(\frac{\pi}{3}\right) - g\left(\frac{\pi}{4}\right)} = \frac{f'(c)}{g'(c)}$$

$$\Rightarrow \frac{\sin\left(\frac{\pi}{3}\right) - \sin\left(\frac{\pi}{4}\right)}{\cos\left(\frac{\pi}{3}\right) - \cos\left(\frac{\pi}{4}\right)}$$

$$\Rightarrow \frac{\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}}{\frac{1}{2} - \frac{1}{\sqrt{2}}} = \frac{\cos c}{-\sin c} = -\cot c$$

$$\Rightarrow \frac{\sqrt{3}-1}{\sqrt{2}-2} = -\cot c$$

$$\Rightarrow \frac{\sqrt{3}-\sqrt{2}}{1-\sqrt{2}} = -\cot c$$

$$\Rightarrow \cot c = \frac{\sqrt{3}-\sqrt{2}}{\sqrt{2}-1}$$

$$\Rightarrow c = \cot^{-1}\left(\frac{\sqrt{3}-\sqrt{2}}{\sqrt{2}-1}\right)$$