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4/7/48

Calculus

Calculus is a latin word.

Its meaning is pebbles (small stones)

Calculus helps to measure speed in any time.

Newton and Leibniez are the inventors of calculus.

Newton (Britain, 17th Century)
Leibniez (German)

Calculus (History)

The latin word calculus means ~~small~~ pebbles. means small stones. Small stones are used as aids of calculation. Hence Calculus is a topic where we will calculate something. Calculus was discovered by two mathematician Newton and Leibnitz - During 17th century separately. British mathematician Newton was treating of motion of objects in physics. To calculate the speed of a particle at some instant of time, ~~he~~ he discovered calculus.

German mathematician Leibnitz was treating of direction of an object along a circular path in geometry. To find the direction of a moving particle along a circular path he discovered Calculus at the same time.

Constant

A Constant is a quantity which retains (keeps) the same value during any set of mathematical operations. Constants are denoted by first hand letters. (a, b, c, d - - - - - -)
(α, β, γ - - - - - - -)

Variable :- A variable is a quantity which does not retain (keep) the same value during any set of mathematical operation but is capable of assuming different values. variables are denoted by last hand letters. (such as :- r, s, t - - - - - x, y, z)

ξ, η, ζ . . . etc (Greek letters)
(exi) (eta) (zeta)

Ex :- (a) Area of a circle is a variable

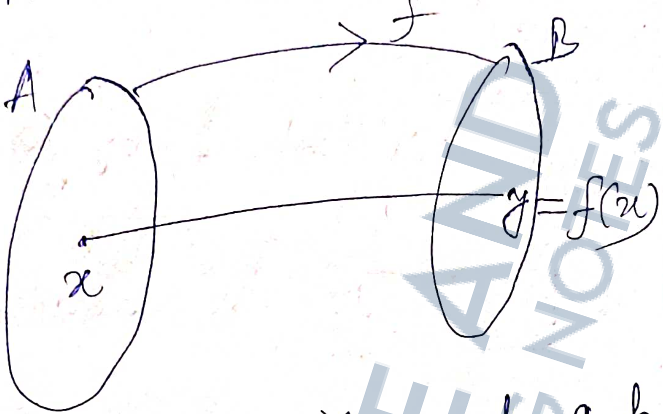
(b) ~~Some~~ Sum of angles of triangle is constant.

Function :- If x and y be two variable so related that corresponding to every value of x within a defined domain we get a unique value of y then y is called a function of x defined in its domain. we write $y = f(x)$

Here x is independent variable and y is dependent variable.

The set of values of x is domain.

And the set of values of y is Range.



Ex: we write f, g, h, ϕ, ψ (Sai) for functions.

Ex: Area of a circle is a function of its radius, i.e. that is $A = f(r) = \pi r^2$

Ex: Velocity is the function of time. $V = f(t)$

Intervals: (a) Closed intervals:—

A closed interval is denoted by $[a, b]$ & is defined as

$$[a,b] = \{x \in \mathbb{R} \mid a \le x \le b\}$$

Here a & b are included.

Open interval:-

A open interval is denoted by (a, b)

or]a, b[and is defined by

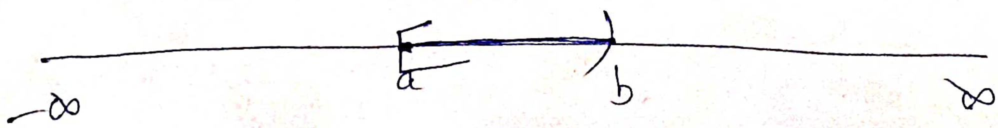
$$(a,b)=\{x\in\mathbb{R} \mid \cancel{a \le x}\ a<x<b\}$$

Here a and b are excluded.

Semi - closed ~~and~~ or semi-open intervals

A semi-closed or semi-open interval may be denoted by [a, b) or (a, b] and ~~as~~ are defined by

$$[a,b)=\{x\in\mathbb{R} \mid a\le x<b\}$$

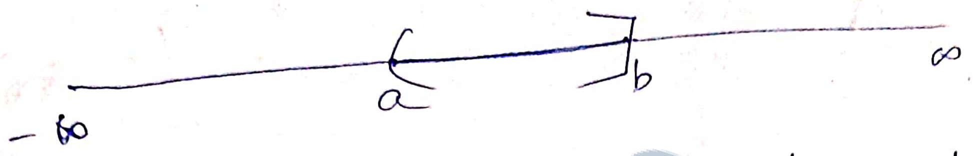


Here a is included and b is excluded

$$[a, b) = \{x \in \mathbb{R} \mid a \le x < b\}$$

Here a is included and b is excluded

$$(a, b] = \{x \in \mathbb{R} \mid a < x \le b\}$$



Here a is excluded and b is included

Various functions :-

(1) Polynomial function.

A function which is expressed in terms of a polynomial is called a polynomial function.

Ex :- $y = f(x) = a_0 + a_1x + a_2x^2 - - - - - - - a_nx^n$ is a polynomial function of degree n.

(2) Rational function :-

A function which is expressed as ratio of two polynomials is called rational function provided the denominator is not '0' $(\neq 0)$

Ex :- $y = f(x) = \dfrac{x^2+x+1}{x-3}$ where $x \neq 3$

(3) Algebraic function :-

A function which can be generated by x by a finite number of algebraic operations (addition, subtraction, division, multiplication, root (any root), square etc} is called algebraic function

Ex : All polynomial and Rational functions are algebraic $\rightarrow$ (square, Multiplication, 2 additions)

Ex : $x^2+5x+3 \rightarrow$ 4 algebraic function

Ex $\sqrt{x+1}$ is a algebraic function.
(2 function)

~~Transder~~

Transcendental function :

A function which is not algebric is called transcendental.

Ex : $y = f(x) = \sin x$

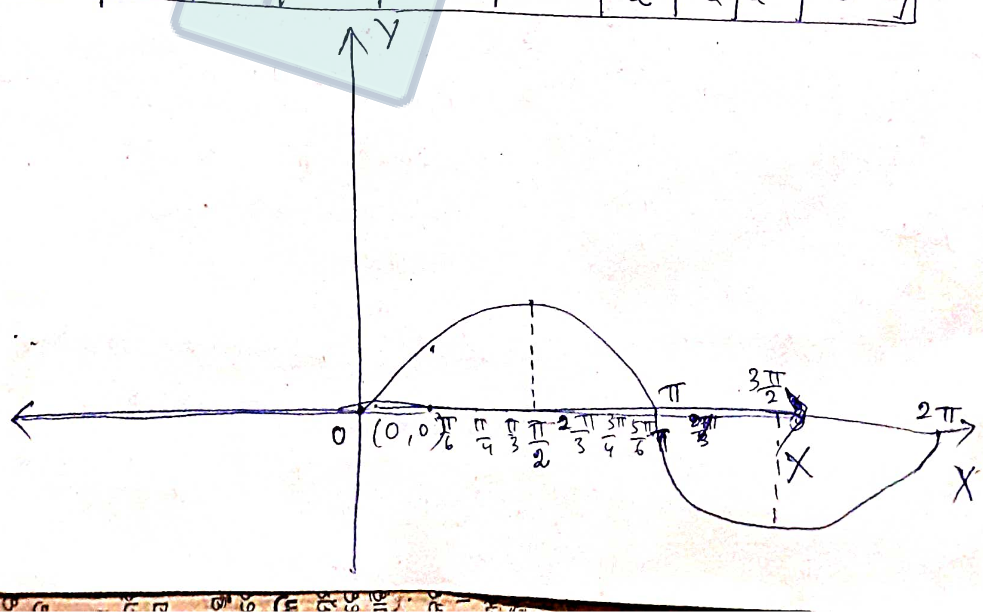
All the trigonometric function.

Trigonometric function :

$y = f(x) = \sin x, \cos x, \tan x$ - - - - - -

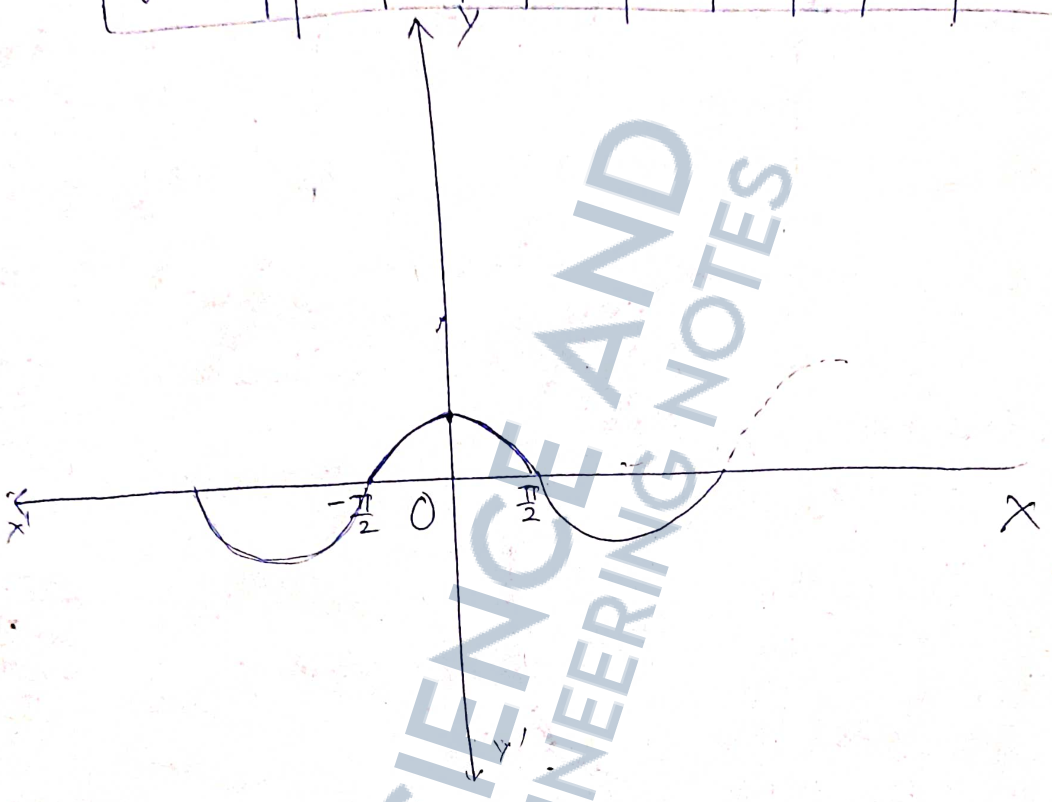
Graph of sinx

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$y = \sin x$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0



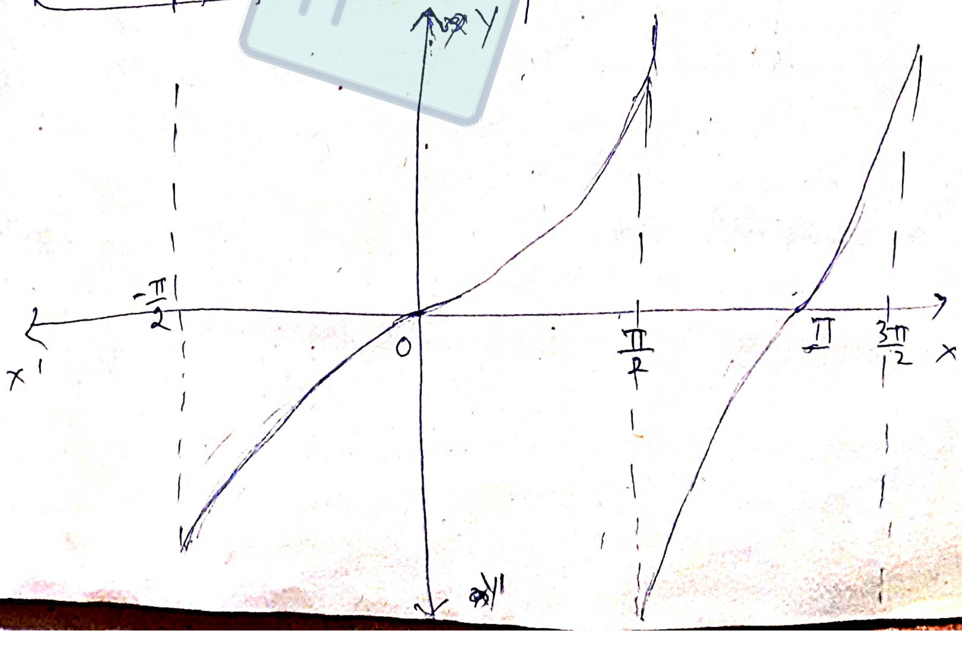
graph of cos x

x	$-\frac{\pi}{2}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$y = \cos x$	0	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1

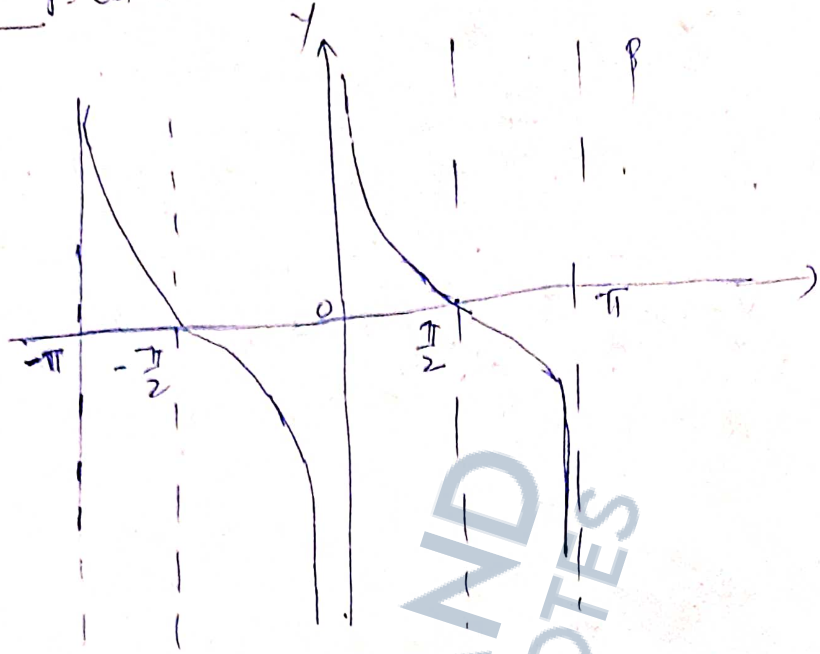


graph of tan x

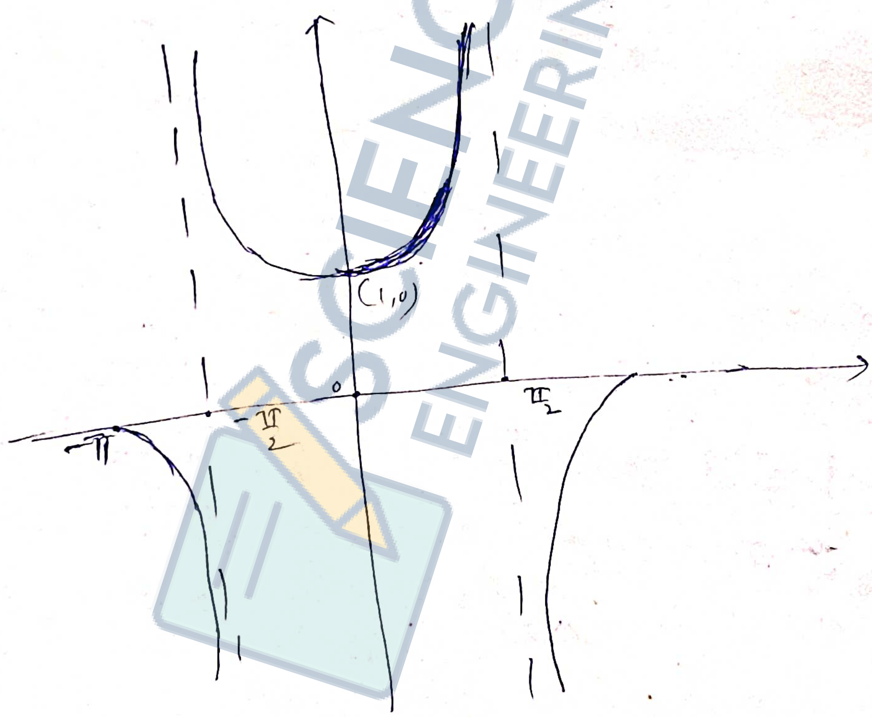
x	$-\frac{\pi}{2}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$
$y = \tan x$	$-\infty$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	$-\sqrt{3}$	$-\frac{1}{\sqrt{3}}$	0	



Graph of $y = \cot x$

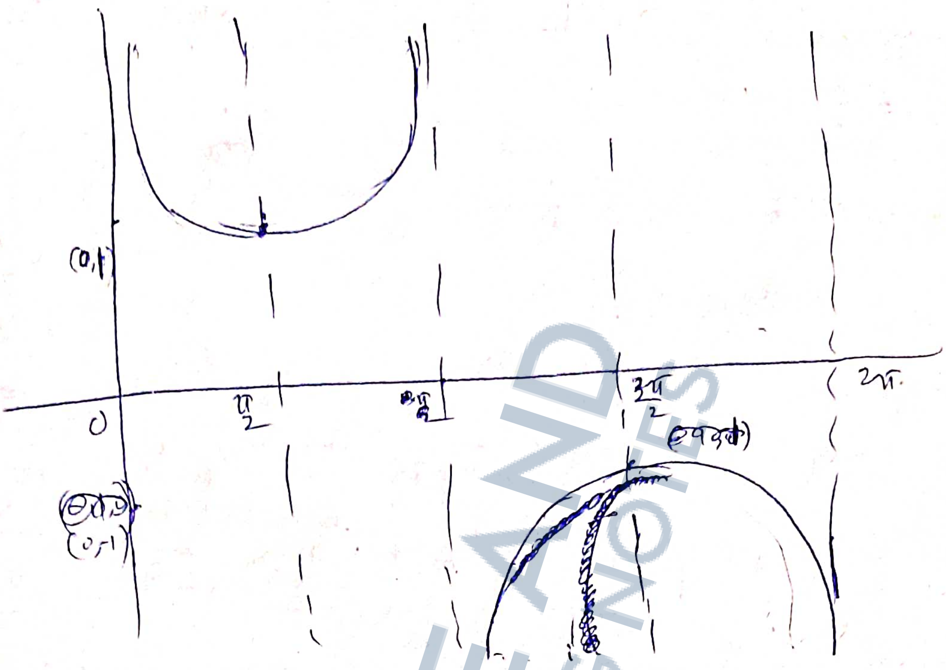


Graph of $y = \sec x$



Graph of Cosec x

Graph of $y = \operatorname{cosec} x$



Inverse trigonometric function :-

$y = f(x) = \sin^{-1}x, \cos^{-1}x$ - - - etc

Exponential function

$y = f(x) = a^x$ where $a > 0$ | $\boxed{0^x \neq 0}$

~~y = f(x)~~ π is defined as the ratio of circumference (পরিধি) and (ব্যাস) diametre. of a circle.

Mecury → [illegible]

π = It is greek alhabate.

May I have a ~~[illegible]~~ large container of coffee.
3 . 1 4 1 5 9 2 6

3 . 1415926

π is irretional. Its approximate value is 3.141

$\boxed{\pi < \frac{22}{7}}$

'e' is called exponential number

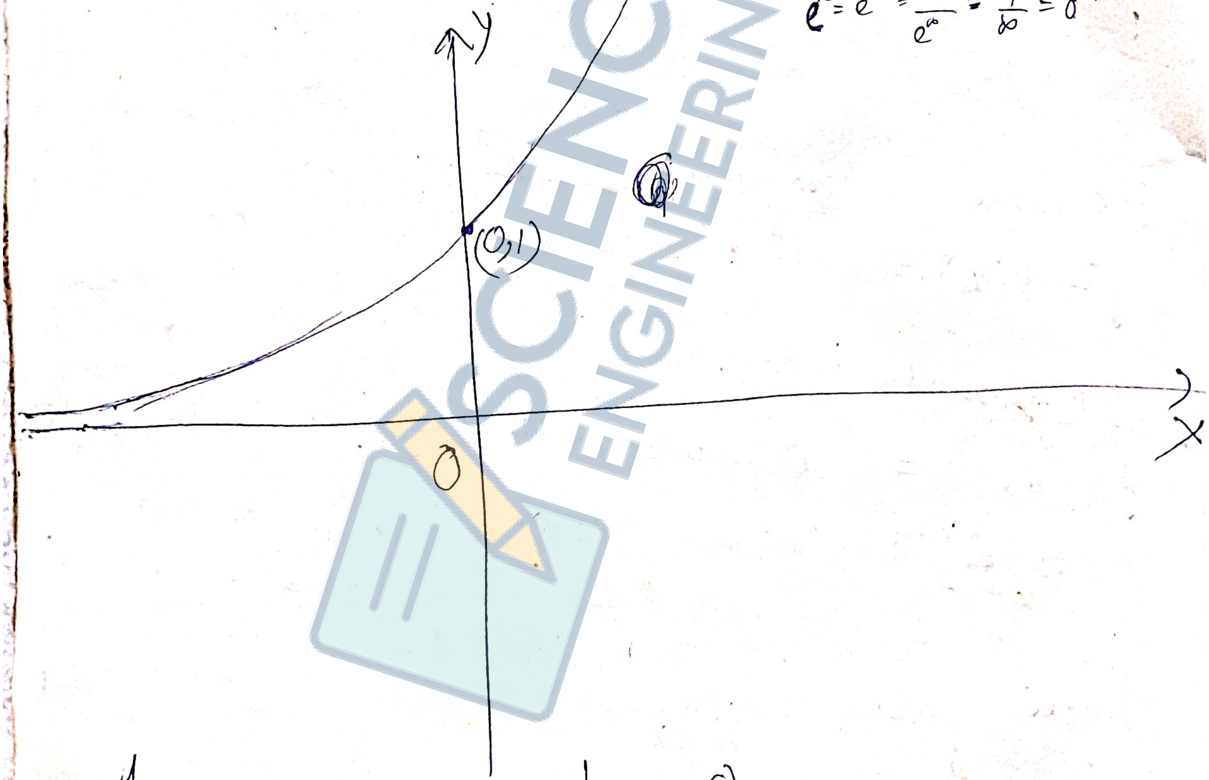
e is irrational. Its approxmate value is 2.718. It lies in between 2 and 3

$y = f(x) = e^x$ is an exponetial function.

graph of e^x

x	0	1	2	∞	-1	$-2 \ldots$	$-\infty$
$y=e^x$	1	2.718	$(2.718)^2$	∞	$\frac{1}{2.718}$	$\frac{1}{(2.718)^2}$	0

$e^x = e^{-\infty} = \frac{1}{e^\infty} = \frac{1}{\infty} = 0$



$\frac{1}{0} = \infty$, $\frac{1}{\infty} = 0$

$2+3=5$ function?

Logarithmic function :

$y = f(x) = \log_a x$

$\Rightarrow \quad x = a^y$

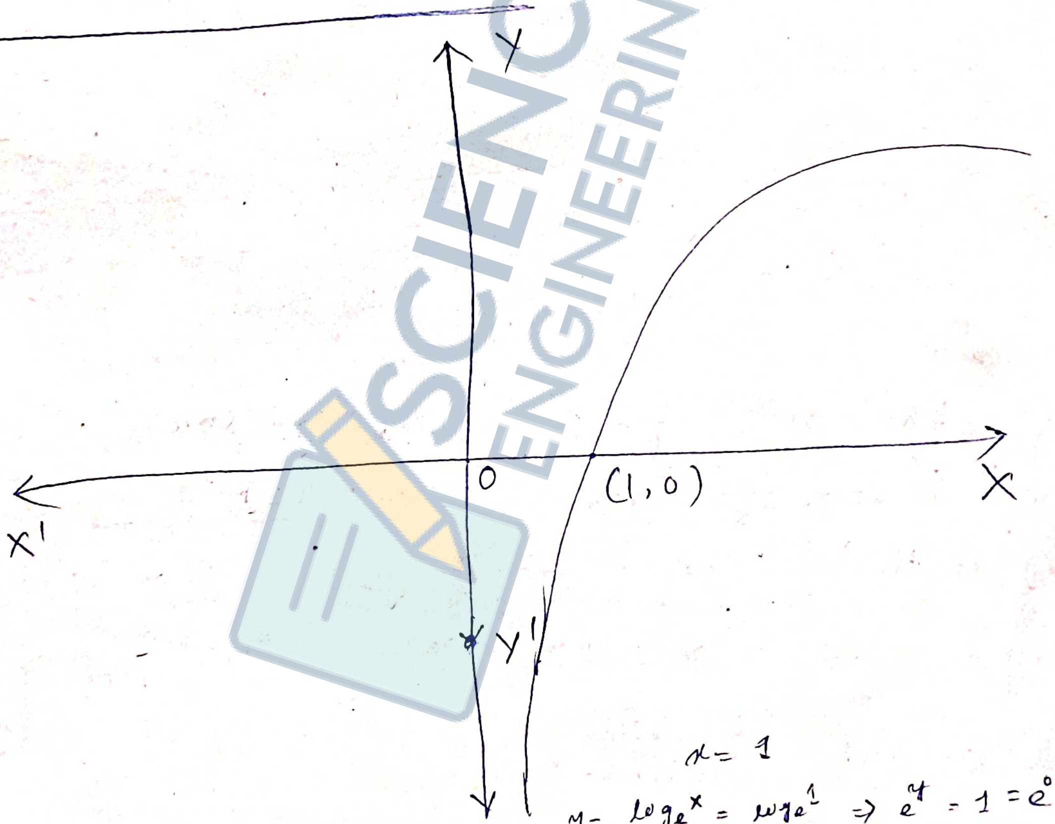
$\log_{10} x$ is called common logarithm.

$\log_e x$ is called natural logarithm or it is also Napierian logarithm. and denoted by $\ln x$ or $\log x$

$y = f(x) = \log x$ or $\ln x$ or $\log_e x$

$\Rightarrow \quad x = e^y$

Graph of $y = \ln x$



$x = 1$

$y = \log_e x = \log_e 1 \Rightarrow e^y = 1 = e^0$

$\therefore y = 0$

$y = \log e^0 \Rightarrow e^y = 0 = e^{-\infty} \Rightarrow y = -\infty$

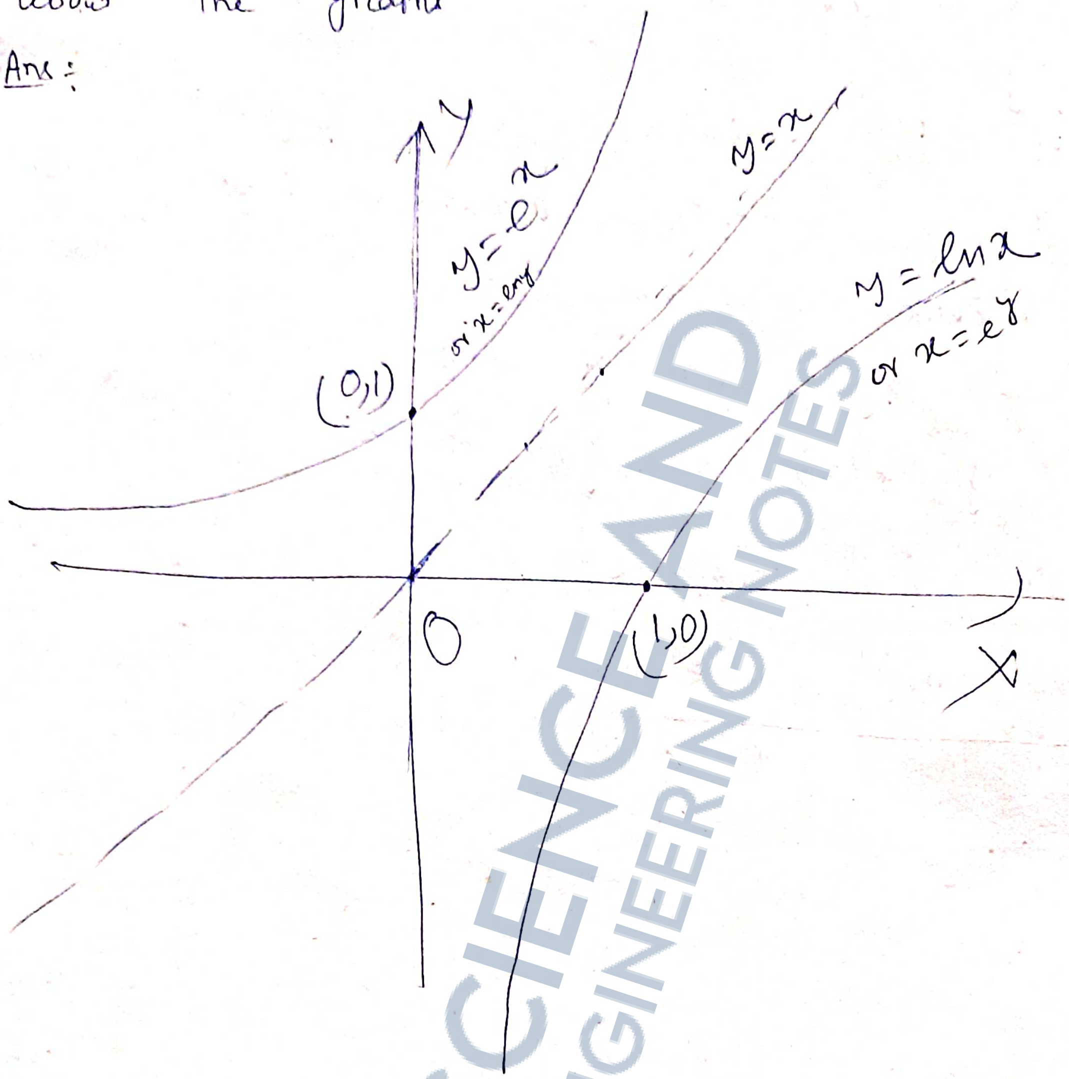
$y = \log e^\infty \Rightarrow e^y = \infty = e^\infty \Rightarrow y = \infty$

x	1	0	∞		
$y = \ln x$	0	$-\infty$	∞		

$e^{-\infty} = 0 \Rightarrow \log 0 = -\infty$, $\Rightarrow e^\infty = \infty \Rightarrow \log e^\infty = \infty$

Q/ ~~Q.~~ Draw the graphs of $y = e^x$ and $y = \ln x$ and write your conclusion about the graphs

Ans:

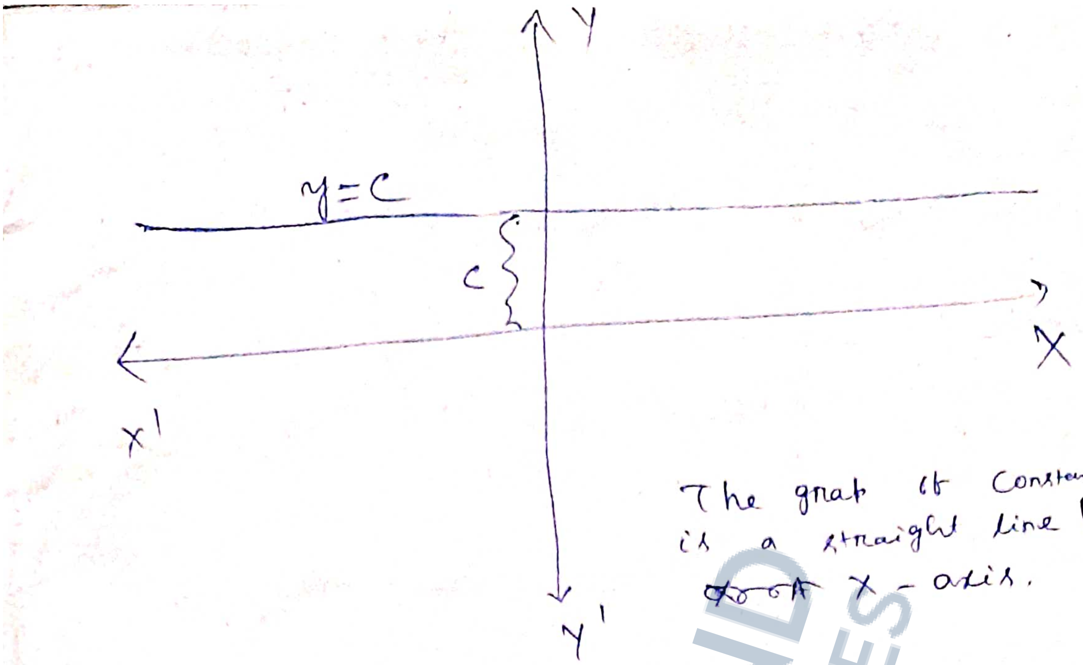


Graphs of $y = e^x$ and $y = \ln x$ are ~~same~~ symmetric with respect to (w.r.to) the line $y = x$. ~~Then~~ i.e they are mirror images of each other.

Constant function:

$y = f(x) = c$

Graph of $y = c$



The graph of constant function is a straight line || to ~~x-of~~ x - axis.

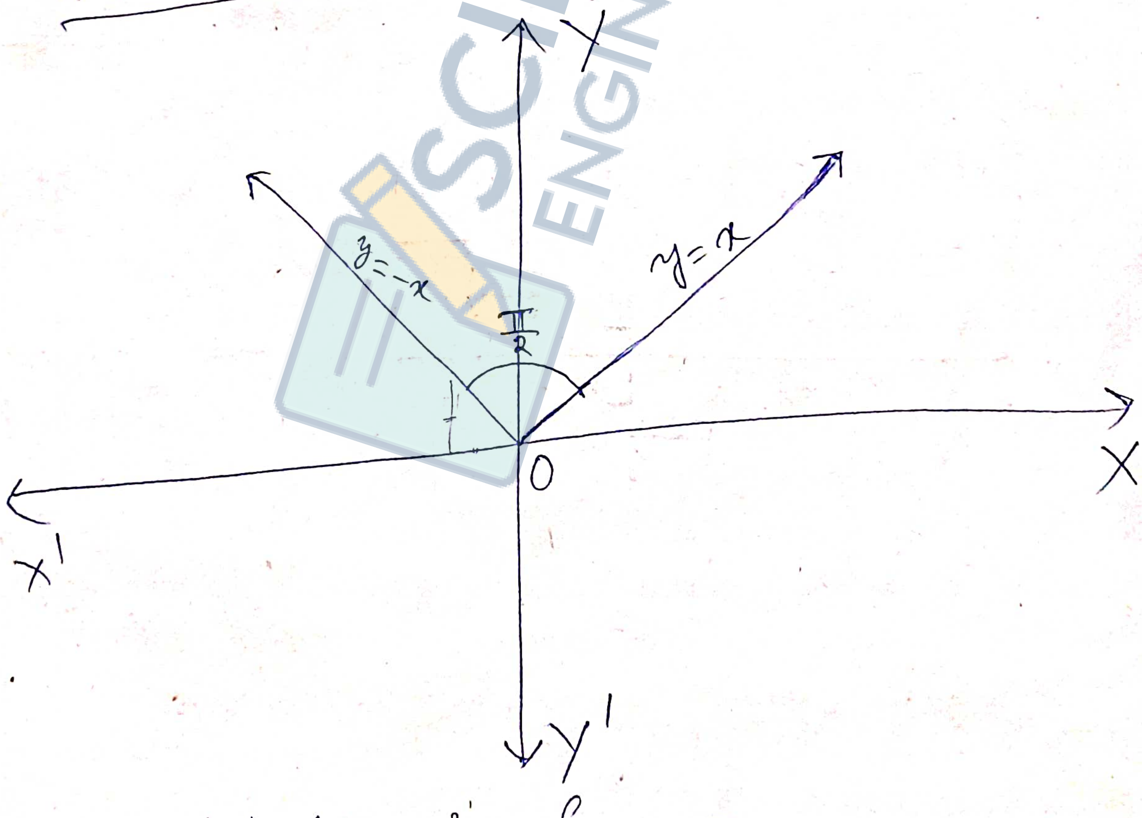
Absolute Value function or ~~Rd~~ Modulus function:

$$y = f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Ex: $|5| = 5$, $|-5| = -(-5) = 5$

$|0| = 0$

Graph of absolute value function or $y = |x|$



$$y = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Greatest integer function or step function or bracket function or postage function (a step is here)

$y = f(x) = [x]$ = Greatest integer $\leq x$

ex: $[5.2] = 5$

$[5.9]$ = greatest integer ≤ 5.9
$= 5$

$[5] = 5$

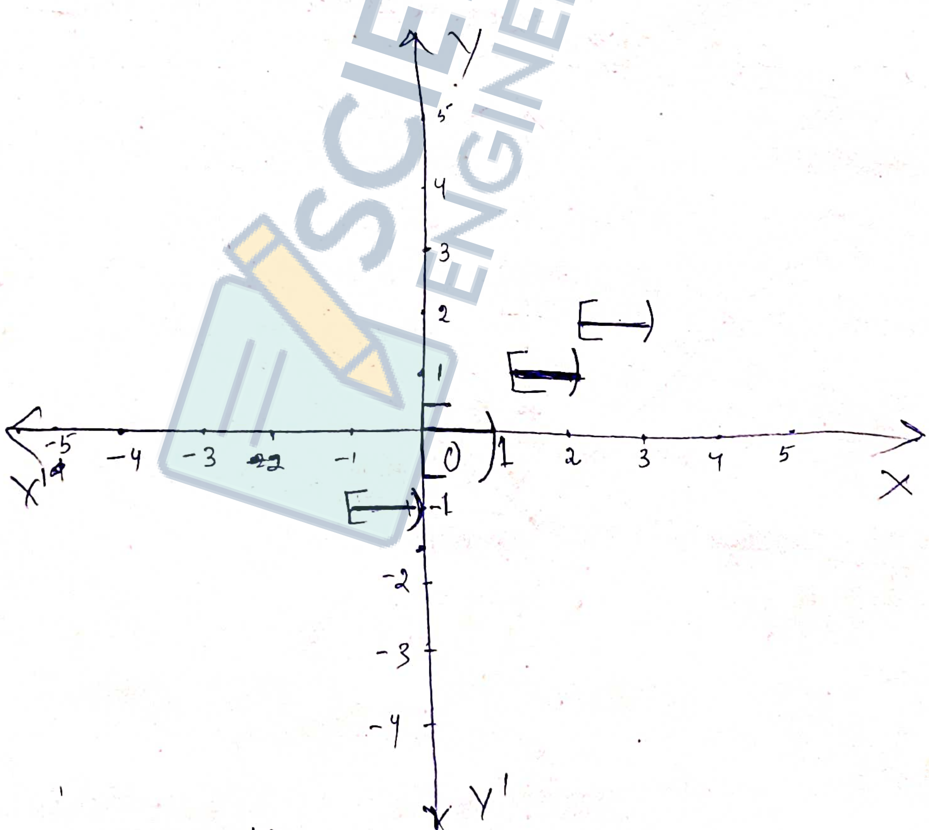
$[-5.9] = -6$

$[-5.2] = -6$

$[0.3] = 0$

$[-0.3] = -1$

Graph of $y = [x]$



$0 \leq x \leq 1$ then $y = [x] = 0$

If $x \in [0, 1)$ then $y = [x] = 0$

If $1 \leq x < 2$ i.e $x \in [1,2)$ then

$y = [x] = 1$

If $2 \leq x < 3$ i.e $x \in [2,3)$ then $y = [x] = 2$

If $-1 \leq x < 0$ i.e $x \in [-1,0)$ then $y = [x] = -1$

Doubt : $x = \infty$, $y = \infty$ $\}$ e^x, $\ln x$ graph

Implicit function and explicit function

If the dependent variable (y) and independent variable (x) are written separately then the function is explicit function.

If x and y are written together then the function is implicit function.

Ex : $y = \sin x + e^x$ (explicit function)

$x^3 y + xy^2 + y^3 + 5 = 0$ (implicit function)

Even function and odd function :

A function $f(x)$ is called even if $f(-x) = f(x)$.

A function is call odd if $f(-x) = -f(x)$

Ex : $y = f(x) = \sin x$

$f(-x) = \sin(-x) = -\sin x = -f(x)$

$\therefore f(x) = \sin x$ is odd function.

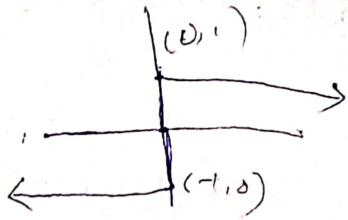
$y = f(x) = \cos x$

$\Rightarrow f(-x) = \cos(-x) = \cos x = f(x)$

$\therefore f(x) = \cos x$ is even function.

Signum function

$$y = f(x) = \begin{cases} \frac{|x|}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

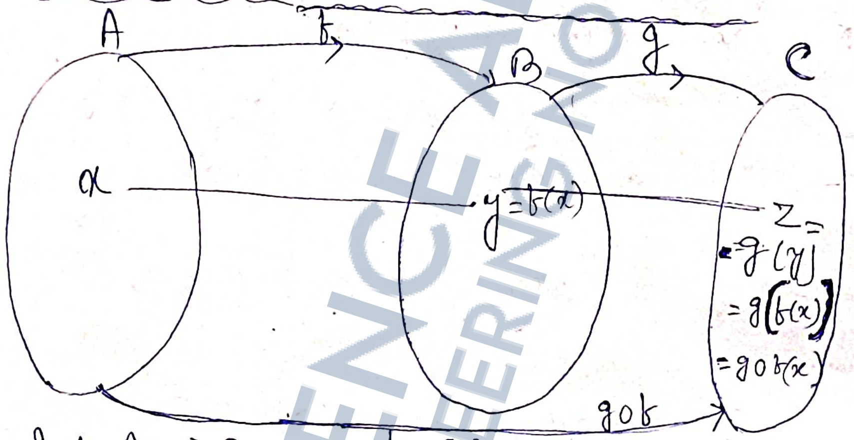


or $f(x) = 1, x>0$, $f(x) = -1, x<0$, $f(x) = 0, x=0$

Characteristic function or dirichlet's function

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational.} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Composite function : or product function



f is a function from A to B

Let $f : A \to B$ and $g : B \to C$

$\therefore$ $gof : A \to C$

gof is called Composite function.

$$gof(x) = g(f(x)) \quad \forall x \in A$$

Ex : $f(x) = x^2$, $g(x) = \sin x$

$$gof(x) = g(f(x)) = g(x^2) = \sin x^2$$

$$fog(x) = f(g(x)) = f(\sin x) = (\sin x)^2 = \sin^2 x$$

$$\therefore \quad gof(x) \neq fog(x)$$

Commutative law is not satisfied.

Algebra of function:

$\log_e x \} \log_e x$

$(f+g)(x) = f(x) + g(x)$

$(f-g)(x) = f(x) - g(x)$

$(fg)(x) = f(x) \cdot g(x)$

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ provided $g(x) \neq 0$

$(cf)x = c\,f(x)$ where c is constant

Question

1) Find $g \circ f$ and $f \circ g$ for each function of the following pair of function

(i) $f(x) = |x|$, $g(x) = \sin x$

(ii) $f(x) = x^2$, $g(x) = [x]$

(iii) $f(x) = \log x$, $g(x) = \frac{|x|}{x}$

(iv) $f(x) = e^x$, $g(x) = \log_e x$

(v) $f(x) = x^2+3x+5$, $g(x) = \tan x$

2) Find $(f+g)$, $(f \cdot g)$, $(\alpha f + \beta g)$, for each of the following pair of functions along with their domains.

(i) $f(x) = \sin x$, $g(x) = e^x$.

(ii) $f(x) = \sqrt{x^2-2}$, $g(x) = \log(1+x)$

(iii) $f(x) = \tan x$, $g(x) = e^{\frac{1}{x^2}}$

(iv) $f(x) = x^3+x^2+x+1$, $g(x) = \frac{1}{x^2-5x+6}$

(v) $f(x) = \cos x$, $g(x) = \sqrt{\left(\frac{\pi}{2}\right)^2 - x^2}$

1.(i) $f(x) = |x|$

$g(x) = \sin x$

$f \circ g(x) = f(g(x)) = f(\sin x) = |\sin x|$

$g \circ f(x) = g(f(x)) = g(|x|) = \sin |x|$

$\therefore$ ~~fog~~ $f \circ g \neq g \circ f$

2.(B)(i) $f(x) = \sin x$, $g(x) = e^x$

$(f+g)(x) = f(x) + g(x)$

$= \sin x + e^x$

which is defined $\forall x \in \mathbb{R}$

$\therefore$ domain of

$(f+g) = R = (-\infty, \infty)$

$(fg)x = f(x) \cdot g(x)$

$= \sin x \, e^x$

which is defined $\forall x \in R$

$\therefore$ domain of $(fg) = R = (-\infty, \infty)$

$(\alpha f + \beta g)x = \alpha f(x) + \beta g(x)$

$= \alpha \sin x + \beta e^x$

which is defined $\forall x \in R$

$\therefore$ domain of $\alpha f + \beta g = R$

(iv) $f(x) = x^3 + x^2 + x + 1$, $g(x) = \dfrac{1}{x^2 - 5x + 6}$

$(f+g)(x) = f(x) + g(x) = (x^3 + x^2 + x + 1) + \dfrac{1}{x^2 - 5x + 6}$

Here $x^2 - 5x + 6 \neq 0 \Rightarrow (x-2)(x-3) \neq 0$

$\Rightarrow x \neq 2$ and $x \neq 3$

$\therefore$ Domain $= \mathbb{R} - \{2, 3\}$

Limit of a function:

~~$\log e^{-\infty} = ?$~~ Note:

$$e^{\log x} = x$$

Limit of a function:

$x \to a$ (x tends to a or x approaches to a)

$x \to a$ means x is very closed to a.

But $x \neq a$

$x \to 2$ means x is very closed to 2

but $x \neq 2$

$\therefore x = 1.99, 1.98, 1.97$ ....etc
$2.01, 2.001, 2.02$ ---- etc

The limit of $f(x)$ is l as x tends to a is denoted by the symbol ~~limit f(x)~~

$$\lim_{x \to a} f(x) = l$$ limit of $f(x)$ is l as $x \to a$

or $$\lim_{x \to a} f(x) = l$$

or $$\underset{x \to a}{\mathrm{Lt}}\, f(x) = l$$

or $$\underset{x \to a}{\Lambda}\, f(x) = l$$

It means that $f(x)$ is close to l whenever x is close to a but $x \neq a$.

Ex ⇒ $$\lim_{x \to 2} 2x = 4$$

$x = 1.99,\ 1.98,\ 1.97,\ 1.96$ ... etc
$2.01,\ 2.001,\ 2.002$ ... etc

$\Rightarrow 2x = 3.98,\ 3.96,\ 3.94,\ 3.92$ ... etc
$4.02,\ 4.002$ ... etc

$$\lim_{x \to 2} (2x) = 4$$

<u>Rule</u> :- Put the value of x to which it tends to then the value of limit is obtained.

<u>Algebra of limit</u>

1. $$\lim_{x \to a} \left(f(x) + g(x)\right) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

2. $\lim_{x\to a}\left(f(x)-g(x)\right) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$

3. $\lim_{x\to a}\left(f(x).g(x)\right) = \lim_{x\to a} f(x) . \lim_{x\to a} g(x)$

4. $\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ Provided $\lim_{x\to a} g(x) \neq 0$

5. $\lim_{x\to a}\left(K.f(x)\right) = K . \lim_{x\to a} f(x)$ (where K is constant)

6. $\lim_{x\to a} K = K$ (where K is constant)

Problem : 1 Evaluate $\lim_{x\to 0}(x^2+2x-1)(x+2)$

Solution :

$$\lim_{x\to 0}(x^2+2x-1)(x+2)$$

$$= \lim_{x\to 0}(x^2+2x-1) . \lim_{x\to 0}(x+2)$$

$$= \left\{\lim_{x\to 0}(x^2) + 2.\lim_{x\to 0}(x) - \lim_{x\to 0}(1)\right\}\left\{\lim_{x\to 0}(x) + \lim_{x\to 0}(2)\right\}$$

$$= \{0 + 0 - 1\}\{0 + 2\}$$

$$= (-1).(2)$$

$$= -2 \quad \text{(ans)}$$

$\frac{0}{0}$ = ~~indeterminate~~ indeterminate form

Indeterminate forms

1. $\frac{0}{0}$ 2. $\frac{\infty}{\infty}$ 3. $\infty - \infty$ 4. $0 \times \infty$

5. 0^0 6. 1^∞ 7. ∞^0

Rule :- When inderminate form occur first simplify the function by factorising, rationalising, or taking L.C.M as far as practicable and then take the limit

Ex:1

$$\lim_{x\to 2} \frac{(x^2-4)}{x-2}$$

$$= \lim_{x\to 2} \frac{(x+2)(x-2)}{(x-2)}$$

$$= \lim_{x\to 2} (x+2)$$

$$= 2+2$$

$$= 4$$

Ex-2

$$\lim_{x\to 0} \frac{2-\sqrt{4-x}}{x}$$

$$= \lim_{x\to 0} \frac{(2-\sqrt{4-x})(2+\sqrt{4-x})}{x(2+\sqrt{4-x})}$$

$$= \lim_{x\to 0} \frac{\{4-(4-x)\}}{x(2+\sqrt{4-x})}$$

$$= \lim_{x\to 0} \frac{x}{x(2+\sqrt{4-x})}$$

$$= \lim_{x\to 0} \frac{1}{2+\sqrt{4-x}}$$

$$= \frac{1}{2+\sqrt{4-0}}$$

$$= \frac{1}{2+2}$$

$$= \frac{1}{4} \text{ (ans)}$$

Ex
3. $\lim_{x \to 1} \left(\frac{1-x^5}{1-x}\right)$

$= \lim_{x \to 1} \left(\frac{x^5-1}{x-1}\right)$

$= \lim_{x \to 1} \frac{\cancel{(x-1)}(x^4+x^3+x^2+x+1)}{\cancel{(x-1)}}$

$= \lim_{x \to 1} (x^4+x^3+x^2+x+1)$

$= 1+1+1+1+1$

$= 5$

divisor	divident	quotient
	Remainder	

$$
\begin{array}{r|l}
x-1 & x^5 - 1 \quad (x^4+x^3+x^2+x+1 \\
& \underline{\cancel{x^5} - x^4} \\
& \quad x^4 - 1 \\
& \underline{\cancel{x^4} - x^3} \\
& \quad x^3 - 1 \\
& \underline{\cancel{x^3} - x^2} \\
& \quad x^2 - 1 \\
& \underline{\cancel{x^2} - x} \\
& \quad x - 1 \\
& \underline{x - 1} \\
& \quad 0
\end{array}
$$

4. $\lim_{x \to 2} \frac{\left(\frac{1}{x^2} - \frac{1}{4}\right)}{(x-2)}$

$= \lim_{x \to 2} \frac{(4-x^2)}{4x^2(x-2)}$

$= \lim_{x \to 2} \frac{-(x^2-4)}{4x^2(x-2)}$

$= -\lim_{x \to 2} \frac{(x+2)(x-2)}{4x^2(x-2)}$

$= -\lim_{x \to 2} \frac{x+2}{4x^2}$

$= -\left(\frac{4}{16}\right)$

$= -\frac{1}{4}$

Formulas

1. $$\lim_{x \to a} \frac{x^n - a^n}{x-a} = n \times a^{n-1}$$

Where n is rational

derivation : Case - I

Suppose n is positive integer.

$$\frac{x^n - a^n}{x-a}$$

$$x-a \,\big)\, x^n - a^n \,\big(\, x^{n-1} + a x^{n-2} + a^2 x^{n-3} + \cdots + a^{n-1}$$

$$\underline{\overset{(-)}{x^n} \;\; \overset{(+)}{- a x^{n-1}}}$$

$$a x^{n-1} - a^n$$

$$\underline{\overset{(-)}{a x^{n-1}} \;\; \overset{(+)}{- a^2 x^{n-2}}}$$

$$a^2 x^{n-2} - a^n$$

$$\lim_{x \to a} \frac{x^n - a^n}{x-a}$$

$$= \lim_{x \to a} \frac{\cancel{(x-a)}\left(x^{n-1} + a x^{n-2} + a^2 x^{n-3} \cdots + a^{n-1}\right)}{\cancel{(x-a)}}$$

$$= \lim_{x \to a} \left(x^{n-1} + a x^{n-2} + a^2 x^{n-3} + \cdots + a^{n-1}\right)$$

$$= a^{n-1} + a \cdot a^{n-2} + a^2 \cdot a^{n-3} + \cdots + a^{n-1}$$

$$= a^{n-1} + a^{n-1} + a^{n-1} \cdots + n \text{ times}$$

$$= n \cdot a^{n-1}$$

Case - II

Suppose $n = 0$

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = \lim_{x \to a} \frac{1 - 1}{x - a}$$

$$= \lim_{x \to a} 0$$

(because 0 is constant) $\lim K = K$

$$= 0$$

$$= 0 \cdot a^{0-1}$$

$$= n \cdot a^{n-1}$$

Case - III Suppose n is negative integer.

Let $n = -m$ where m is positive.

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = \lim_{x \to a} \frac{x^{-m} - a^{-m}}{x - a}$$

$$= \lim_{x \to a} \frac{\frac{1}{x^m} - \frac{1}{a^m}}{x - a}$$

$$= \lim_{x \to a} \frac{a^m - x^m}{x^m \cdot a^m} \times \frac{1}{(x-a)}$$

$$= \lim_{x \to a} \frac{-(x^m - a^m)}{x^m \cdot a^m (x-a)}$$

$$= \lim_{x \to a} \left\{ \left(\frac{-1}{x^m \cdot a^m} \right) \cdot \left(\frac{x^m - a^m}{x - a} \right) \right\}$$

$$= \left(\lim_{x \to a} \frac{-1}{x^m \cdot a^m} \right) \cdot \left(\lim_{x \to a} \frac{x^m - a^m}{x - a} \right)$$

$$= \left(\frac{-1}{a^{2m}} \right) \cdot \left(m \times a^{m-1} \right)$$ (by Case I since m is Positive integer)

$$= -m \, a^{-m-1}$$

$$= n \cdot a^{n-1}$$

Case-IV

Suppose n is rational

Let $n = \frac{p}{q}$, where $p, q \in Z$, and $q \neq 0$

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = \lim_{x \to a} \frac{(x)^{\frac{p}{q}} - a^{\frac{p}{q}}}{x - a}$$

$$= \lim_{x \to a} \frac{\left(x^{\frac{1}{q}}\right)^p - \left(a^{\frac{1}{q}}\right)^p}{x - a}$$

Put $x^{1/q} = y$

& $a^{1/q} = b$

$x \to a \Rightarrow y \to b$

$x = y^q$

$a = b^q$

$b = a^{\frac{1}{q}}$

$$= \lim_{y \to b} \frac{y^p - b^p}{y^q - b^q}$$

$$= \lim_{y \to b} \frac{\left(\frac{y^p - b^p}{y - b}\right)}{\left(\frac{y^q - b^q}{y - b}\right)}$$

$$= \frac{\left(\lim_{y \to b} \frac{y^p - b^p}{y - b}\right)}{\left(\lim_{y \to b} \frac{y^q - b^q}{y - b}\right)}$$

$$= \frac{p\, b^{p-1}}{q\, b^{q-1}} = \frac{p}{q}\, b^{p-q}$$

$$= \frac{p}{q} \cdot \left(a^{\frac{1}{q}}\right)^{p-q}$$

$$= \frac{p}{q} \cdot a^{\frac{p}{q}-1}$$

$$= n \cdot a^{n-1} \quad \text{(Proved)}$$

Note :

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = n \cdot a^{n-1} \quad \text{when } n \in R$$

4. xiii $\lim_{h \to 0} \frac{(x+h)^4 - x^4}{h}$

Put $x+h = y$, $h = y - x$, $h \to 0 \Rightarrow y \to x$

$$= \lim_{y \to x} \frac{y^4 - x^4}{y - x}$$

$$= 4 \cdot x^{4-1} \quad \left(\because \lim_{x \to a} \frac{x^n - a^n}{x - a} = n \cdot a^{n-1} \right)$$

$$= 4x^3$$

Limits of trigonometric function

When we write $\sin x$, $\cos x$, $\tan x$ -- etc we mean that x is an angle measured in radian

(i) $\lim_{x \to 0} \sin x$

$= \sin 0$

$= 0$

(ii) $\lim_{x \to 0} \cos x$

$= \cos 0$

$= 1$

(iii) $\lim_{x \to \frac{\pi}{2}} \sin x$

$= \sin \frac{\pi}{2}$

$= 1$

(iv) $\lim_{x \to \frac{\pi}{2}} \cos x$

$= \cos \frac{\pi}{2}$

$= 0$

Formula : 2

Imp

$$\boxed{\lim_{x \to 0} \frac{\sin x}{x} = 1}$$

Ex:

$$\lim_{y \to 0} \frac{\sin y}{y} = 1$$

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1.$$

dummy variable : The variable which changes but the value remains same.

Problem: $\lim_{x \to 0} \frac{\sin 5x}{x}$

Put $5x = y$

$\Rightarrow x = \frac{y}{5}$

$x \to 0$

$y \to 0$

$$= \lim_{y \to 0} \frac{\sin y}{\frac{y}{5}}$$

$$= \lim_{y \to 0} \frac{5 \cdot \sin y}{y}$$

$$= 5 \lim_{y \to 0} \frac{\sin y}{y}$$

$$= 5 \cdot 1$$

$$= 5$$

Mon : 4 P.M , Wed : 4 P.M, Sun : 11.15 A.M

Formula - 3

$$\lim_{x \to 0} \frac{\tan x}{x} = 1$$

Proof : $\lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \frac{\sin x}{x \cos x}$

$$= \lim_{x \to 0} \frac{\sin x}{x} \cdot \frac{1}{\cos x}$$

$$= \lim_{x \to 0} \frac{\sin x}{x} \times \lim_{x \to 0} \frac{1}{\cos x}$$

$$= 1 \times \frac{1}{1}$$

$$= 1 \quad \text{(Proved)}$$

Formula-4
$$\lim_{x \to 0} \frac{1-\cos x}{x} = 0$$

Proof:
$$\lim_{x \to 0} \frac{1-\cos x}{x}$$

$$= \lim_{x \to 0} \frac{(1-\cos x)(1+\cos x)}{x(1+\cos x)}$$

$$= \lim_{x \to 0} \frac{1-\cos^2 x}{x(1+\cos x)}$$

$$= \lim_{x \to 0} \frac{\sin^2 x}{x(1+\cos x)}$$

$$= \lim_{x \to 0} \frac{\sin x}{x} \cdot \frac{\sin x}{1+\cos x}$$

$$= \lim_{x \to 0} \frac{\sin x}{x} \cdot \lim_{x \to 0} \frac{\sin x}{1+\cos x}$$

$$= 1 \cdot \frac{0}{2}$$

$$= 0 \quad \text{(Proved)}$$

OR

$$\lim_{x \to 0} \frac{1-\cos x}{x}$$

$$= \lim_{x \to 0} \frac{2\sin^2 \frac{x}{2}}{x} \quad \left(\because 1-\cos x = 2\sin^2 \frac{x}{2}\right)$$

$$= \lim_{x\to 0} \frac{2\sin\frac{x}{2}}{x} \cdot \sin\frac{x}{2}$$

$$= \lim_{x\to 0} \frac{\sin\frac{x}{2}}{\frac{x}{2}} \cdot \sin\frac{x}{2}$$

$$= \lim_{x\to 0} \frac{\sin\frac{x}{2}}{\frac{x}{2}} \times \lim_{x\to 0} \sin\frac{x}{2}$$

$$= \lim_{\frac{x}{2}\to 0} \frac{\sin\frac{x}{2}}{\frac{x}{2}} \cdot \lim_{x\to 0} \sin\frac{x}{2}$$

$$= 1 \cdot 0$$

$$= 0 \quad \text{(proved)}$$

Limits of inverse trigonometric functions

formula: 5

$$\lim_{x\to 0} \frac{\sin^{-1}x}{x} = 1$$

proof:

$$\lim_{x\to 0} \frac{\sin^{-1}x}{x}$$

put $\sin^{-1}x = y$

$\Rightarrow x = \sin y$

$x \to 0$

$\Rightarrow y \to 0$

$$= \lim_{y\to 0} \frac{y}{\sin y}$$

$$= \lim_{y\to 0} \frac{1}{\frac{\sin y}{y}}$$

$$= \frac{1}{1}$$

$$= 1$$

∵ Formula ÷ 6 $\lim_{x \to 0} \frac{\tan^{-1}x}{x} = 1$

Proof $\lim_{x \to 0} \frac{\tan^{-1}x}{x}$

Put $\tan^{-1}x = y$

$\Rightarrow x = \tan y$

$x \to 0$

$\Rightarrow y \to 0$

$\Rightarrow \lim_{y \to 0} \frac{y}{\tan y}$

$\Rightarrow \lim_{y \to 0} \frac{1}{\frac{\tan y}{y}}$

$\Rightarrow \frac{1}{1}$

$= 1$

No. 1

(ii) $\lim_{x \to 0} \frac{\sin 3x}{\sin 5x}$

$= \lim_{x \to 0} \frac{\frac{\sin 3x}{3x} \cdot 3x}{\frac{\sin 5x}{5x} \cdot 5x}$

$= \lim_{x \to 0} \frac{\frac{\sin 3x}{3x}}{\frac{\sin 5x}{5x}} \times \frac{3}{5}$

$= \frac{3}{5} \cdot \frac{\lim_{x \to 0} \frac{\sin 3x}{3x}}{\lim_{x \to 0} \frac{\sin 5x}{5x}} = \frac{3}{5} \times \frac{\lim_{3x \to 0} \frac{\sin 3x}{3x}}{\lim_{5x \to 0} \frac{\sin 5x}{5x}}$

$= \frac{3}{5} \cdot \frac{1}{1}$

$= \frac{3}{5}$

(v)

$$\lim_{x\to 0} \frac{1-\cos x}{x^2}$$

$$= \lim_{x\to 0} \frac{(1-\cos x)(1+\cos x)}{x^2(1+\cos x)}$$

$$= \lim_{x\to 0} \frac{1-\cos^2 x}{x^2(1+\cos x)}$$

$$= \lim_{x\to 0} \frac{\sin^2 x}{x^2(1+\cos x)}$$

$$= \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{\sin x}{x(1+\cos x)}$$

$$= \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{\sin x}{x} \lim_{x\to 0} \frac{1}{(1+\cos x)}$$

$$= \lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \frac{1}{(1+\cos x)}$$

$$= 1 \cdot 1 \cdot \frac{1}{2} = \frac{1}{2}$$

$$= \lim_{x\to 0} \frac{2\sin^2 \frac{x}{2}}{4 \cdot \frac{x^2}{4}}$$

$$= \frac{1}{2} \cdot \lim_{x\to 0} \left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2$$

$$= \frac{1}{2}.$$

(vi)

$$\cancel{\lim_{x\to 0} \frac{\sin x^\circ}{x}}$$

$$= \cancel{\lim_{x\to 0} \frac{\sin x}{x}}$$

$$\Rightarrow \lim_{x\to 0} \pi$$

$$\frac{2\sin^2\frac{x}{2}}{4 \cdot \frac{x^2}{4}}$$

$$180^\circ = \pi$$

$$\Rightarrow x^\circ = \frac{\pi}{180^\circ} x$$

$$\lim_{x\to 0} \frac{\sin x^\circ}{x}$$

$$= \lim_{x\to 0} \frac{\sin \frac{\pi x}{180}}{x} = \lim_{x\to 0} \frac{\sin \frac{\pi x}{180}}{\frac{\pi}{180} x} \times \frac{\pi}{180}$$

$a+b-(a-b)$

$$= \frac{\pi}{180} \lim_{x\to 0} \frac{\sin\frac{\pi x}{180}}{\frac{\pi x}{180}}$$

$$= \frac{\pi}{180} \cdot 1$$

$$= \frac{\pi}{180}$$

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q: $\lim_{x\to\frac{\pi}{2}} \frac{\cos x}{x-\frac{\pi}{2}}$

Put $y = x - \frac{\pi}{2}$

$\Rightarrow x = \frac{\pi}{2}+y$

$x \to \frac{\pi}{2}$

$\Rightarrow y \to 0$

$$= \lim_{y\to 0} \frac{\cos(\frac{\pi}{2}+y)}{y}$$

$$= \lim_{y\to 0} \frac{-\sin y}{y}$$

$$= -\lim_{y\to 0} \frac{\sin y}{y}$$

$$= -1 \text{ (ans)}$$

xviii $\lim_{x\to 0} \frac{\cos x - \cos 5x}{\cos 2x - \cos 6x}$

$$= \lim_{x\to 0} \frac{\cancel{2}\sin 3x \cdot \cancel{\sin 2x}}{\cancel{2}\sin 4x \cdot \cancel{\sin 2x}}$$

$$= \lim_{x\to 0} \frac{\sin 3x}{\sin 4x}$$

$$= \lim_{x\to 0} \frac{\frac{\sin 3x}{3x}\cdot 3x}{\frac{\sin 4x}{4x}\cdot 4x}$$

$$= \frac{\lim_{x\to 0} \frac{\sin 3x}{3x} \cdot 3x}{\lim_{x\to 0}\frac{\sin 4x}{4x} \cdot 4x} = \frac{3}{4}$$

xvii ✓ xiii ✓ xii ✓ (x) ✓ xiv ✓ xix ✓ xviii ✓ xvi ✓ xv ✓ xv ✓

$$\lim_{x \to 0} \frac{\sin^2 x}{x^2 (1+\cos x)}$$

$$= \lim_{x \to 0} \frac{\sin^2 x}{x^2} \cdot \frac{1}{1+\cos x}$$

$$= \lim_{x \to 0} \left(\frac{\sin x}{x}\right)^2 \cdot \frac{1}{1+\cos x}$$

$$= \left(\lim_{x \to 0} \frac{\sin x}{x}\right)^2 \operatorname{lt}_{x \to 0} \frac{1}{1+\cos x}$$

Problem:

9. (i) $$\lim_{x \to \alpha} \frac{x \sin \alpha - \alpha \sin x}{x - \alpha}$$

$$= \lim_{x \to \alpha} \frac{x \sin \alpha - \alpha \sin \alpha + \alpha \sin \alpha - \alpha \sin x}{x - \alpha}$$

$$= \lim_{x \to \alpha} \frac{\sin \alpha (x - \alpha) + \alpha (\sin \alpha - \sin x)}{(x - \alpha)}$$

$$= \lim_{x \to \alpha} \left\{ \frac{\sin \alpha (x - \alpha)}{(x - \alpha)} + \frac{\alpha (\sin \alpha - \sin x)}{(x - \alpha)} \right\}$$

$$= \lim_{x \to \alpha} \left\{ \sin \alpha + \frac{\alpha (\sin \alpha - \sin x)}{x - \alpha} \right\}$$

α = constant
x = variable

$$= \lim_{x \to \alpha} \sin \alpha + \lim_{x \to \alpha} \frac{\alpha (\sin \alpha - \sin x)}{x - \alpha}$$

$$= \sin \alpha + \lim_{x \to \alpha} \alpha \frac{\left\{ 2 \cos\left(\frac{\alpha + x}{2}\right) \sin\left(\frac{\alpha - x}{2}\right) \right\}}{x - \alpha}$$

$$= \sin \alpha + 2\alpha \lim_{x \to \alpha} \frac{\cos\left(\frac{x+\alpha}{2}\right) \left\{ -\sin \frac{x - \alpha}{2} \right\}}{x - \alpha}$$

$$= \sin\alpha - \alpha \lim_{x\to\alpha} \frac{\cos\left(\frac{x+\alpha}{2}\right) \cdot \sin\frac{(x-\alpha)}{2}}{\left(\frac{x-\alpha}{2}\right)}$$

$$= \sin\alpha - \alpha \cdot \left\{\lim_{x\to\alpha} \cos\left(\frac{x+\alpha}{2}\right)\right\}\left\{\lim_{x\to\alpha} \frac{\sin\frac{(x-\alpha)}{2}}{\left(\frac{x-\alpha}{2}\right)}\right\}$$

$$= \sin\alpha - \alpha \cdot \{\cos\alpha\}\{1\}$$

$$= \sin\alpha - \alpha\cos\alpha \quad \text{(ans)}$$

Imp Question

Evaluate $\lim_{x\to 1} (1-x)\left(\tan\frac{\pi x}{2}\right)$

Ans:- $\lim_{x\to 1} (1-x)\left(\tan\frac{\pi x}{2}\right)$

Put

$y = x-1$

$\Rightarrow x = 1+y$

$x\to 1 \Rightarrow y\to 0$

$$= \lim_{y\to 0} -y \cdot \tan\frac{\pi}{2}(y+1)$$

$$= \lim_{y\to 0} (-y) \cdot \tan\left(\frac{\pi}{2} + \frac{\pi y}{2}\right)$$

$$= \lim_{y\to 0} (-y)\left(-\cot\frac{\pi y}{2}\right)$$

$$= \lim_{y\to 0} y \cdot \cot\frac{\pi y}{2}$$

$$= \lim_{y\to 0} \frac{y}{\tan\frac{\pi y}{2}}$$

$$= \lim_{y\to 0} \frac{1}{\frac{\tan\frac{\pi y}{2}}{y}} = \lim_{y\to 0} \frac{1}{\frac{\tan\frac{\pi y}{2}}{\frac{\pi y}{2}} \times \frac{\pi}{2}}$$

$$= \frac{2}{\pi} \lim_{y\to 0} \frac{1}{\frac{\tan\pi y}{\pi y}} = \frac{2}{\pi} \cdot 1 = \frac{2}{\pi} \quad \text{(ans)}$$

Infinite limit:

$x \to \infty$ means x is very large

$\therefore \lim_{x\to\infty} \frac{1}{x} = 0$

$\lim_{x\to\infty} \frac{1}{x^2} = 0$

Problem

Evaluate $\lim_{x\to\infty} \frac{x^2-5x+7}{3x^2+2x-3}$

$= \lim_{x\to\infty} \frac{1-\frac{5}{x}+\frac{7}{x^2}}{3+\frac{2}{x}-\frac{3}{x^2}}$ (dividing numarater D.r by x^2)

$= \frac{\lim_{x\to\infty} 1 - \lim_{x\to\infty} \frac{5}{x} + \lim_{x\to\infty} \frac{7}{x^2}}{\lim_{x\to\infty} 3 + \lim_{x\to\infty} \frac{2}{x} - \lim_{x\to\infty} \frac{3}{x^2}}$

$= \frac{1-0+0}{3+0-0}$

$= \frac{1}{3}$ (ans)

Rule: In case of infinite limits divide the numarater and denominator by the term x^n where n is the highest digree in n.r and d.r.

Problem :- Evaluate [surds]

$$\lim_{x\to\infty} x\cdot(x-\sqrt{x^2+1})$$

$$= \lim_{x\to\infty} \frac{x(x-\sqrt{x^2+1})(x+\sqrt{x^2+1})}{(x+\sqrt{x^2+1})}$$

$$= \lim_{x\to\infty} \frac{x(\cancel{x^2}-\cancel{x^2}-1)}{x+\sqrt{x^2+1}}$$

$$= \lim_{x\to\infty} \frac{-x}{(x+\sqrt{x^2+1})}$$

$$= \lim_{x\to\infty} \frac{\frac{-x}{x}}{\frac{x}{x}+\frac{\sqrt{x^2+1}}{x}}$$

$$= \lim_{x\to\infty} \frac{-1}{1+\sqrt{1+\frac{1}{x^2}}} \quad \left(\because \frac{\sqrt{x^2+1}}{x} = \sqrt{\frac{x^2+1}{x^2}} = 1+\frac{1}{x^2}\right)$$

squaring and taking $\sqrt{}$

$$= \frac{-1}{1+\sqrt{1}} = \frac{-1}{2} \quad \text{(ans)}$$

Problem 1. $\lim\limits_{n \to \infty} \dfrac{1+2+3 \cdots\cdots n}{n^2}$

$= \lim\limits_{n \to \infty} \dfrac{\frac{n(n+1)}{2}}{n^2}$

$= \lim\limits_{n \to \infty} \dfrac{\cancel{n}(n+1)}{2n^{\cancel{2}}}$

$= \lim\limits_{n \to \infty} \dfrac{n+1}{2n}$

$= \lim\limits_{n \to \infty} \dfrac{1+\frac{1}{n}}{2}$

$= \dfrac{1+0}{2}$

$= \dfrac{1}{2}$

2. Note

1. $1^2+2^2+3^2 \cdots\cdots +n^2$

$= \dfrac{n(n+1)(2n+1)}{6}$

2. $1^3+2^3+3^3 \cdots\cdots +n^3$

$= \dfrac{n(n+1)}{2}^2 \cancel{n\left(\frac{n+1}{2}\right)^2}$

$= \left\{\dfrac{n(n+1)}{2}\right\}^2$

Pro $\lim\limits_{n \to \infty} \dfrac{1^2+2^2+3^2 \cdots\cdots +n^2}{n^3}$

$= \lim\limits_{n \to \infty} \dfrac{\cancel{n}(n+1)(2n+1)}{6n^2}$

$= \lim\limits_{n \to \infty} \dfrac{(n+1)(2n+1)}{6n^2}$

$$= \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}{6}$$

$$= \lim_{n\to\infty} \frac{(1+0)(2+0)}{6}$$

$$= \frac{1\times 2}{6}$$

$$= \frac{1}{3} \quad \text{(ans)}$$

(iii) $$\lim_{n\to\infty} \frac{1^3+2^3+3^3 - - - n^3}{n^4}$$

$$= \lim_{n\to\infty} \frac{\left(\frac{n(n+1)}{2}\right)^2}{n^4}$$

$$= \lim_{n\to\infty} \frac{n^2 (n+1)^2}{4n^4}$$

$$= \lim_{n\to\infty} \frac{(n+1)^2}{4n^2}$$

$$= \lim_{n\to\infty} \frac{n^2+2n+1}{4n^2}$$

$$= \lim_{n\to\infty} \frac{1+\frac{2}{n}+\frac{1}{n^2}}{4}$$

$$= \frac{1+0+0}{4}$$

$$= \frac{1}{4} \quad \text{(ans)}$$

Note: $a + a+d + a+2d + - - - a+(n-1)d$ are in A.P

$$= \frac{n}{2}\left\{2a + (n-1)d\right\}$$

2) $a + ar + ar^2$ ———— ar^{n-1} are called G.P

a = first term

r = common ratio

$$= \frac{a(r^n - 1)}{r - 1}$$

(3) $a + ar + ar^2$ ————

$$= \frac{a}{1-r} \quad (\text{provided } |r| < 1)$$

(∵ for $|r| < 1$, $r^n \to 0$, $\frac{a(0-1)}{r-1} = \frac{a}{1-r}$)

4. $a + ar^2 + ar^4$ ---- $= \frac{a}{1-r^2}$ (Providing $|r| < 1$)

(3) a_1, a_2 ---------- a_n

are in H.P (Harmonic progression)

if $\frac{1}{a_1}, \frac{1}{a_2}$ ———— $\frac{1}{a_n}$ are

in A.P

(4) Factorial of n

n factorial

It is denoted by $\lfloor n$ or $n!$

It is defined by the product of first n natural numbers.

i.e $\lfloor n$ or $n! = 1 \times 2 \times 3 \times 4 ---- \times n$

Ex

~~Factorial 1 is~~

$\lfloor 1 = 1$

$\lfloor 2 = 1 \times 2 = 2$

$\lfloor 3 = 1 \times 2 \times 3 = 6$

$\lfloor 4 = 1 \times 2 \times 3 \times 4 = 24$

$\lfloor 0 = 1$

Factorial of Negative can not defined.

$\lfloor n-1 \times n = \lfloor n$

~~$\lfloor n \times (n+1)$~~

$\lfloor n \,(n+1) = \lfloor n+1$

5. (xiii)

$$\lim_{n\to\infty} \frac{1+\frac{1}{2}+\frac{1}{2^2}+ \cdots +\frac{1}{2^n}}{1+\frac{1}{3}+\frac{1}{3^2} \cdots +\frac{1}{3^n}}$$

$$= \cancel{\lim_{n\to\infty} \frac{1\times\left[\left(\frac{1}{2}\right)^{n+1}-1\right]}{}}$$

$$= \lim_{n\to\infty} \frac{\frac{1\times\left[1-\left(\frac{1}{2}\right)^{n+1}\right]}{\left(1-\frac{1}{2}\right)}}{\frac{1\times\left[1-\left(\frac{1}{3}\right)^{n+1}\right]}{\left(1-\frac{1}{3}\right)}}$$

$$= \lim_{n\to\infty} \frac{\frac{1-\frac{1}{2^{n+1}}}{\frac{1}{2}}}{1-\frac{1}{3^{n+1}}}$$

$$\lim_{n\to\infty} \frac{1-\frac{1}{2^{n+1}} \times 2}{1-\frac{1}{3^{n+1}} \times \frac{3}{2}}$$

$$\frac{2\left(1-\frac{1}{2^{n+1}}\right)2}{\left(1-\frac{1}{3^{n+1}}\right)3} = \frac{4}{3}$$

$2/3$

$$= \cancel{\lim_{n\to\infty} \frac{2\left(1-\frac{1}{2^{n+1}}\right)}{3}} = \lim_{n\to\infty} \frac{\frac{4}{3}\left(1-\frac{1}{2^{n+1}}\right)}{\left(1-\frac{1}{3^{n+1}}\right)}$$

$$= \frac{\frac{4}{3}(1-0)}{1-0} = \frac{4}{3} \text{ (ans)}$$

$a + ar + ar^2 + \cdots = \frac{a}{1-r}$ provided $|r|<1$

here $r = \frac{1}{2}$, $r = \frac{1}{3}$.

$$\lim_{n\to\infty} \frac{1+\frac{1}{2}+\cdots+\frac{1}{2^n}}{1+\frac{1}{3}+\cdots+\frac{1}{3^n}} = \frac{\frac{1}{1-\frac{1}{2}}}{\frac{1}{1-\frac{1}{3}}}$$

$= 2 \div \frac{2}{3} = \frac{4}{3}$ (Ans)

xiv) $$\lim_{n\to\infty} \frac{\lfloor n}{\lfloor (n+1) - \lfloor n}$$

$$= \lim_{n\to\infty} \frac{\lfloor n}{\lfloor n \times (n+1) - \lfloor n}$$

$$= \lim_{n\to\infty} \frac{\lfloor n \cdot 1}{\lfloor n\,(n+1-1)}$$

$$= \lim_{n\to\infty} \frac{1}{n}$$

$$= 0$$

Limits of ~~Exponential~~ exponential and logarithmic functions

1. We define $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = e$

It can be proved by using L'Hospital's rule

($e = 2.718$ Approximately)

2. $$\lim_{x\to 0} (1+x)^{\frac{1}{x}} = e$$

Proof

$$\lim_{x\to 0} (1+x)^{\frac{1}{x}}$$

$$= \lim_{y\to\infty} \left(1+\frac{1}{y}\right)^y$$

$$= e \quad \square$$

Put $\frac{1}{x} = y$

$\Rightarrow x = \frac{1}{y}$

$x \to 0 \Rightarrow y \to \infty$

3. $$\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$$

Proof :-

$$\lim_{x\to 0} \frac{\log(1+x)}{x}$$

$$= \lim_{x\to 0} \frac{1}{x} \{\log(1+x)\}$$

My class

$$= \lim_{x\to 0} \log(1+x)^{\frac{1}{x}}$$

$$\left\{ \because \lim \log g(x) = \log \lim g(x) \right\}$$

$$= \log\left\{ \lim_{x\to 0} (1+x)^{\frac{1}{x}} \right\}$$

$$= \log e$$

$$= 1$$

4. $\lim_{x\to 0} \frac{a^{\alpha x}-1}{x} = \alpha \log a$

Proof: $\lim_{x\to 0} \frac{a^{\alpha x}-1}{x}$

Put

$a^{\alpha x}-1 = y$

$\Rightarrow a^{\alpha x} = 1+y$

$\Rightarrow \log a^{\alpha x} = \log(1+y)$

$\Rightarrow \alpha x \log a = \log(1+y)$

$\Rightarrow x = \frac{\log(1+y)}{\alpha \log a}$

$x \to 0 \Rightarrow y \to 0$

$$= \lim_{y\to 0} \frac{y}{\frac{\log(1+y)}{\alpha \log a}}$$

$$= \lim_{y\to 0} \frac{y(\alpha \log a)}{\log(1+y)}$$

$$= \alpha \log a \lim_{y\to 0} \frac{y}{\log(1+y)}$$

$$= \alpha \log a \lim_{y\to 0} \frac{1}{\frac{\log(1+y)}{y}}$$

$$= \alpha \log a \cdot (1)$$

$$= \alpha \log a$$

5. $\lim_{x\to 0} \frac{e^{\alpha x}-1}{x} = \alpha$

Proof:

$$\lim_{x\to 0} \frac{e^{\alpha x}-1}{x}$$

Put $e^{\alpha x}-1 = y$

$\Rightarrow e^{\alpha x} = y+1$

$\Rightarrow \log e^{\alpha x} = \log(y+1)$

$\Rightarrow \alpha x \log e = \log(y+1)$

$\Rightarrow \alpha x = \log(y+1)$

$\Rightarrow x = \dfrac{\log(y+1)}{\alpha}$

$x \to 0 \Rightarrow y \to 0$

$$= \lim_{y\to 0} \frac{y}{\frac{\log(y+1)}{\alpha}}$$

$$= \lim_{y\to 0} \frac{\alpha y}{\log(y+1)}$$

$$= \alpha \cdot \lim_{y\to 0} \frac{y}{\log(y+1)}$$

$$= \alpha \cdot \lim_{y\to 0} \frac{1}{\frac{\log(y+1)}{y}}$$

$$= \alpha \cdot 1$$

$$= \alpha \quad \text{(proved)}$$

6. $\displaystyle \lim_{x\to 0} \frac{a^x-1}{x} = \log a$

imp

Proof: $\displaystyle \lim_{x\to 0} \frac{a^x-1}{x}$

Put $a^x - 1 = y$

$\Rightarrow a^x = 1+y$

$\Rightarrow \log a^x = \log(1+y)$

$\Rightarrow x \log a = \log(1+y)$

$\Rightarrow x = \dfrac{\log(1+y)}{\log a}$

$x \to 0 \Rightarrow y \to 0$

$$= \lim_{y\to 0} \frac{y}{\frac{\log(1+y)}{\log a}}$$

$$= \lim_{y\to 0} \frac{y \log a}{\log(1+y)}$$

$$= \log a \lim_{y\to 0} \frac{y}{\log(1+y)}$$

$= \cancel{\frac{\log a \cdot \{1\}}{\log a}} = \log a \cdot \lim_{y \to 0} \frac{1}{\frac{\log(1+y)}{y}}$

$= \log a \cdot 1 = \log a$

7. $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$

Proof $\Rightarrow$ $\lim_{x \to 0} \frac{e^x - 1}{x}$

$= \lim_{y \to 0} \frac{y}{\log(1+y)}$

$= \lim_{y \to 0} \frac{\log(1+y)}{y}$

$= 1$ (proved)

Put $e^x - 1 = y$

$\Rightarrow e^x = 1 + y$

$\Rightarrow \log e^x = \log(1+y)$

$\Rightarrow x \log e = \log(1+y)$

$\Rightarrow x \cdot (1) = \log(1+y)$

$\Rightarrow x = \log(1+y)$

$x \to 0 \Rightarrow y \to 0$

8. $\lim_{x \to \infty} e^{x} = \infty$

$e^{\cos x} \cdot x$

9. $\lim_{x \to \infty} e^{-x} = 0$

Note :
$$\boxed{\begin{aligned} e^{\log x} &= x \\ \log_e e^{x} &= x \end{aligned}}$$

10-(c) 3,4,5

3. (i) $\lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$

$= \lim_{h \to 0} \frac{2 \cos \frac{x+h+x}{2} \cdot \sin \frac{x+h-x}{2}}{h}$

$= \lim_{h \to 0} \frac{2 \cos \frac{2x+h}{2} \cdot \sin \frac{h}{2}}{h}$

$= \lim_{h \to 0} \frac{\cos \frac{2x+h}{2} \cdot \sin \frac{h}{2}}{\frac{h}{2}}$

$= \lim_{h \to 0} \cos\left(\frac{2x+h}{2}\right) \cdot \lim_{h \to 0} \frac{\sin \frac{h}{2}}{\frac{h}{2}}$

$= \cos x \cdot (1)$

$= \cos x$ (ans)

(iii) $\lim_{h \to 0} \frac{\tan(x+h) - \tan x}{h}$

$$= \lim_{h\to 0} \left(\frac{\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x}}{h} \right)$$

$$= \lim_{h\to 0} \left(\frac{\cos x \cdot \sin(x+h) - \sin x \cdot \cos(x+h)}{\cos(x+h)\cos x \cdot h} \right)$$

$$= \lim_{h\to 0} \left(\frac{\sin(\cancel{x}+h-\cancel{x})}{h \cos x \cdot \cos(x+h)} \right)$$

$$= \lim_{h\to 0} \frac{\sin h}{h} \cdot \frac{1}{\cos x \cdot \cos(x+h)}$$

$$= \lim_{h\to 0} \frac{\sin h}{h} \cdot \lim_{h\to 0} \frac{1}{\cos x \cos(x+h)}$$

$$= 1 \cdot \frac{1}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x \text{ (ans)}$$

x

$$\lim_{h\to 0} \frac{a^{x+h} - a^x}{h}$$

$$= \lim_{h\to 0} \frac{a^x \cdot a^h - a^x}{h}$$

$$= \lim_{h\to 0} \frac{a^x(a^h - 1)}{h}$$

$$= a^x \cdot \lim_{h\to 0} \left(\frac{a^h - 1}{h} \right) \quad \left(\because \lim_{x\to 0} \frac{a^x - 1}{x} = \log a \right)$$

$$= a^x \cdot \log a \text{ (ans)}$$

4. (i) $\lim_{x \to 0} \frac{\log\left(1+\frac{x}{2}\right)}{x}$

$= \lim_{x \to 0} \frac{\log\left(1+\frac{x}{2}\right)}{\frac{x}{2} \cdot 2}$

$= \frac{1}{2} \lim_{\frac{x}{2} \to 0} \frac{\log\left(1+\frac{x}{2}\right)}{\frac{x}{2}}$

$= \frac{1}{2} \cdot 1 \qquad \left(\because \lim_{x \to 0} \frac{\log(1+x)}{x} = 1\right)$

$= \frac{1}{2}$ (an)

(iii) $\lim_{x \to 1} \frac{\log(2x-1)}{x-1}$

Put $x-1 = y$
$\Rightarrow x = 1+y$
$x \to 1 \Rightarrow y \to 0$

$= \lim_{y \to 0} \frac{\log\{2.(1+y)-1\}}{(1+y)-1}$

$= \lim_{y \to 0} \frac{\log(2y+1)}{y}$

$= \lim_{y \to 0} \frac{2\log(1+2y)}{2y}$

$= 2 \lim_{2y \to 0} \frac{\log(1+2y)}{2y}$

$= 2 \cdot 1$

$= 2$

(v) $\lim_{x \to 2} \frac{\log(x-1)}{x^2-3x+2}$

$$= \lim_{x \to 2} \frac{\log(x-1)}{x^2-2x-x+2}$$

$$= \lim_{x \to 2} \frac{\log(x-1)}{x(x-2)-1(x-2)}$$

$$= \lim_{x \to 2} \frac{\log(x-1)}{(x-2)(x-1)}$$

Put $x-2 = y$

$\Rightarrow x = 2+y$

$x \to 2 \Rightarrow y \to 0$

$$= \lim_{y \to 0} \frac{\log\{2+y-1\}}{y \cdot \{2+y-1\}}$$

$$= \lim_{y \to 0} \frac{\log(1+y)}{y\,(y+1)}$$

$$= \lim_{y \to 0} \frac{\log(1+y)}{y} \cdot \lim_{y \to 0} \frac{1}{y+1}$$

$$= 1 \cdot \frac{1}{0+1}$$

$$= 1 \cdot 1$$

$$= 1$$

xiv

$$\lim_{x \to 0} \frac{3^x - 2^x}{4^x - 3^x}$$

$$= \lim_{x \to 0} \frac{3^x - 1 + 1 - 2^x}{4^x - 1 + 1 - 3^x}$$

$$= \lim_{x \to 0} \frac{(3^x - 1) - (2^x - 1)}{(4^x - 1) - (3^x - 1)}$$

$$= \lim_{x \to 0} \left\{ \frac{\left(\frac{3^x - 1}{x}\right) - \left(\frac{2^x - 1}{x}\right)}{\left(\frac{4^x - 1}{x}\right) - \left(\frac{3^x - 1}{x}\right)} \right\}$$

$$= \frac{\lim\limits_{x\to 0} \frac{3^x-1}{x} - \lim\limits_{x\to 0} \frac{2^x-1}{x}}{\lim\limits_{x\to 0} \frac{4^x-1}{x} - \lim\limits_{x\to 0} \frac{3^x-1}{x}}$$

$$= \frac{\log 3 - \log 2}{\log 4 - \log 3}$$

$$= \frac{\log \frac{3}{2}}{\log \frac{4}{3}}$$

3. viii $\dfrac{\lim\limits_{h\to 0}\left(\dfrac{\log_a(x+h)}{\log_a x}\right)}{\lim\limits_{h\to 0} h} = \dfrac{\lim\limits_{h\to 0} \dfrac{\log_a x}{\cancel{\log_a x}}}{h} = \dfrac{1}{0} = \infty$

Co-ordinate Geometry → Two point form.

One sided limits:

$x \to a+$ means x assumes the values greater than a and x is close to 'a'.

$x \to a-$ means x assumes the values less than a and x is close to 'a'

Right Hand Limit (R.H.L)

The ~~R.H.L of~~ Right hand limit of $f(x) = l$ as $x \to a$ is denoted by

$$\lim_{x \to a+} f(x) = l$$

It means that $f(x)$ is close to l whenever x is ~~close~~ close to a and x assumes the values more than a. Here in the limit we take $x = a+h$ where $h \to 0$

<u>Left Hand Limit (L.H.L)</u>

The ~~Lef~~ Left Hand Limit ~~of~~ of $f(x)$ is l as $x \to a$ is denoted by

$$\lim_{x \to a-} f(x) = l$$

It means that $f(x)$ is close to l whenever x is close to a and x assumes the value less than a. Here in the limit we take $x = a-h$ where $h \to 0$

<u>Note</u>: If $\lim_{x \to a+} f(x) = \lim_{x \to a-} f(x) = l$

Then we say that $\lim_{x \to a} f(x)$ ~~exists~~ exists

and $\lim_{x \to a} f(x) = l$

If $\lim_{x \to a+} f(x) \neq \lim_{x \to a-} f(x)$,

then we say that $\lim_{x \to a} f(x)$ does not exists.

Problem

1\. Prove that $\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = 4$

Proof :

R.H.L $= \lim_{x \to 2+} \frac{x^2 - 4}{x - 2}$

$= \lim_{h \to 0} \frac{(2+h)^2 - 4}{(2+h) - 2}$ (As $x \to 2+$ we take $x = 2+h$ and $h \to 0$)

$= \lim_{h \to 0} \frac{\cancel{4} + h^2 + 4h - \cancel{4}}{h}$

$= \lim_{h \to 0} \frac{\cancel{h}(h+4)}{\cancel{h}}$

$= \lim_{h \to 0} h + 4$

$= 0 + 4$

$= 4$

L.H.L $= \lim_{x \to 2-} \frac{x^2 - 4}{x - 2}$

$$= \lim_{h \to 0} \frac{(2-h)^2 - 4}{(2-h) - 2}$$ (As $x \to 2-$ we take $x = 2-h$ and $h \to 0$)

$$= \lim_{h \to 0} \frac{\cancel{4} + h^2 - 4h - \cancel{4}}{\cancel{2} - h - \cancel{2}}$$

$$= \lim_{h \to 0} \frac{h(h-4)}{-h}$$

L'Hospital $\Rightarrow$ = Lopital

$$= \lim_{h \to 0} \frac{-h(4-h)}{-h}$$

$$= \lim_{h \to 0} 4 - h$$

$$= 4 - 0$$

$$= 4$$

$\therefore$ R.H.L = L.H.L = 4

$\therefore \lim_{x \to 2} \frac{x^2-4}{x-2}$ exists and

$$\lim_{x \to 2} \frac{x^2-4}{x-2} = 4 .$$ (proved)

Ex: Find $\lim_{x \to 0} |x|$

Solution: R.H.L $= \lim_{x \to 0+} |x|$

$= \lim_{x \to 0+} x$ ($\because x \to 0+$ means x is +ve $\therefore |x| = x$)

$= 0$

$$L.H.L = \lim_{x \to 0-} |x|$$

$$= \lim_{x \to 0-} -x \quad \left(\because x \to 0- \text{ mean } x \text{ is negative. } \therefore |x| = -x \right)$$

$$= -0$$

$$= 0$$

$\therefore$ $R.H.L = L.H.L = 0$

$\therefore$ $\lim_{x \to 0} |x|$ exists and $\lim_{x \to 0} |x| = 0$

3. Find $\lim_{x \to 0} \frac{|x|}{x}$

Solution

$$R.H.L = \lim_{x \to 0+} \frac{|x|}{x}$$

$$= \lim_{x \to 0+} \frac{x}{x} \quad \left(\because x \to 0+ \text{ means } x \text{ is positive } \therefore |x| = x \right)$$

$$= \lim_{x \to 0+} 1$$

$$= 1$$

$$L.H.L = \lim_{x \to 0-} \frac{|x|}{x}$$

$$= \lim_{x \to 0-} \frac{-x}{x} \quad \left(\because x \to 0- \text{ means } x \text{ is negative } \therefore |x| = -x \right)$$

$= \lim_{x \to 0-} -1$

$= -1$

$\therefore$ R.H.L $\neq$ L.H.L

$\therefore \lim_{x \to 0} \frac{|x|}{x}$ does exist.

4. Find $\lim_{x \to 0} \frac{1}{x}$

<u>Solution</u> $R.H.L = \lim_{x \to 0+} \frac{1}{x}$

$= \infty$

$\Rightarrow$ $L.H.L = \lim_{x \to 0-} \frac{1}{x}$

$= -\infty$

$\therefore$ R.H.L $\neq$ L.H.L

$\therefore \lim_{x \to 0} \frac{1}{x}$ does not exist.

Find limit

5. $\lim_{x \to \frac{\pi}{2}} \tan x$

<u>Solution</u> R.H.L. $\lim_{x \to \frac{\pi}{2}+} \tan x$

$= \lim_{h \to 0} \tan\left(\frac{\pi}{2} + h\right)$ (As $x \to \frac{\pi}{2}+$ we take $x = \frac{\pi}{2} + h$ and $h \to 0$

$$= \lim_{h \to 0} -\coth$$

$$= -\infty$$

<u>L.H.L</u>

$$\lim_{x \to \frac{\pi}{2}^-} \tan x$$

$$= \lim_{h \to 0} \tan\left(\frac{\pi}{2} - h\right)$$

$$= \lim_{h \to 0} \cot h$$

$$= \infty$$

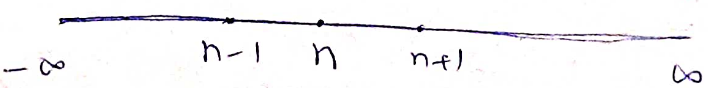
$\therefore$ R.H.L $\neq$ L.H.L

$\therefore$ $\lim_{x \to \frac{\pi}{2}} \tan x$ does not exist.

Ilmly Find

6. $\lim_{x \to n} [x]$ where n is integer.

<u>Soln</u>



<u>R.H.L</u> $= \lim_{x \to n^+} [x]$

$$= \lim_{x \to n^+} n \quad \left(\because x \to n^+ \; \therefore n < x < n+1 \; \therefore [x] = n\right)$$

$= n$

L.H.L

$$\lim_{x \to n^-} [x]$$

$$= \lim_{x \to n^-} n-1 \quad \left(\because x \to n^- \ \therefore n-1 < x < n \ \therefore [x] = n-1 \right)$$

$$= n-1$$

$\therefore$ R.H.L $\neq$ L.H.L

$\therefore \lim_{x \to n} [x]$ does not exist.

Note : $\lim_{x \to n} [x]$ (where n is a fraction) exists.

Ex : $\lim_{x \to 5.5} [x]$

$$\text{R.H.L} = \lim_{x \to 5.5^+} [x]$$

$$= \lim_{x \to 5.5^+} [5] = 5$$

L.H.L

$$\lim_{x \to 5.5^-} [x]$$

$$= \lim_{x \to 5.5^-} [5]$$

$$= 5$$

L.H.L = R.H.L.

$\therefore \lim_{x \to 5.5} [x]$ exists. and $\lim_{x \to 5.5} [x] = 5$

~~7 $f(x) = x^2 - 1$ if $x \geq 1$~~

7 $f(x) = \begin{cases} x^2 - 1 & \text{if } x \geq 1 \\ 2x+1 & \text{if } x < 1 \end{cases}$

Find $\lim_{x \to 1} f(x)$

Soln

$f(x) = \begin{cases} x^2 - 1 & \text{if } x \geq 1 \\ 2x+1 & \text{if } x < 1 \end{cases}$

R.H.L $= \lim_{x \to 1+} f(x)$

$= \lim_{x \to 1+} x^2 - 1$

$= 1 - 1$

$= 0$

L.H.L $\lim_{x \to 1-} f(x)$

$= \lim_{x \to 1-} 2x+1$

$= 2+1$

$= 3$

$\therefore$ L.H.L $\neq$ R.H.L

$\therefore \lim_{x \to 1} f(x)$ does not exist.

5. $f(x) = \begin{cases} \dfrac{\log x}{1+x} & \text{if } x > 1 \\ 1 & \text{if } x \leq 1 \end{cases}$

Find $\lim_{x \to 1} f(x)$

Soln.

$f(x) = \begin{cases} \dfrac{\log x}{1+x} & \text{if } x > 1 \\ 1 & \text{if } x \leq 1 \end{cases}$

R.H.L

$\lim_{x \to 1+} f(x)$

$= \lim_{x \to 1} \dfrac{\log x}{1+x}$

$= \dfrac{0}{1+1}$

$= 0$

R.H.L $\lim_{x \to 1-} f(x)$

$= \lim_{x \to 1-} 1$

$= 1$

L.H.L $\neq$ R.H.L

$\therefore \lim_{x \to 1} f(x)$ does not exist

Sandwich theorem : or Squeezing theorem, Pinching theorem

Calculus: Johnson, Quo Chemistten

Statement : If $f(x) \leq g(x) \leq h(x)$

and if $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = l$

then $\lim_{x \to a} g(x) = l$.

Problems

1. Prove that $\lim_{x \to 0} x \sin \frac{1}{x} = 0$

gnp

Proof : we know

$$0 \leq \left|\sin \frac{1}{x}\right| \leq 1$$

$$\Rightarrow 0 \times |x| \leq |x| \cdot \left|\sin \frac{1}{x}\right| \leq 1 \cdot |x|$$

$$\Rightarrow 0 \leq \left|x \cdot \sin \frac{1}{x}\right| \leq |x|$$

But $\lim_{x \to 0} 0 = 0$ and

$\lim_{x \to 0} |x| = 0$

$\therefore \lim_{x \to 0} \left|x \sin \frac{1}{x}\right| = 0$ by Sandwich theorem.

$\Rightarrow \lim_{x \to 0} x \sin \frac{1}{x} = 0$

(Proved)

2. Prove $\lim_{x \to 0} x \cos\frac{1}{x} = 0$

Proof : We know that

$$0 \le \left|\cos\frac{1}{x}\right| \le 1$$

$$\Rightarrow 0\,|x| \le |x|.\left|\cos\frac{1}{x}\right| \le 1.|x|$$

$$\Rightarrow 0 \le \left|x\cos\frac{1}{x}\right| \le |x|$$

We know that

$$\lim_{x \to 0} 0 = 0 \text{ an}$$

$$\lim_{x \to 0} |x| = 0$$

$$\therefore \lim_{x \to 0} \left|x \cos\frac{1}{x}\right| = 0$$ by sandwich theorem.

$$\therefore \lim_{x \to 0} x \cos\frac{1}{x} = 0$$ (Proved)

3. $\lim_{x \to 0} x^2 \sin\frac{1}{x} = 0$

We know that $0 \le \left|\sin\frac{1}{x}\right| \le 1$

$$\Rightarrow 0.|x^2| \le |x^2|\left|\sin\frac{1}{x}\right| \le 1\,|x^2|$$

$$\Rightarrow 0 \le \left|x^2 \sin\frac{1}{x}\right| \le |x^2|$$

$$\lim_{x \to 0} 0 = 0$$

$$\lim_{x \to 0} |x^2| = 0$$

$$\therefore \lim_{x \to 0} \left|x^2 \sin\frac{1}{x}\right| = 0$$ by sandwich theorem

$$\therefore \lim_{x \to 0} x^2 \sin\frac{1}{x} = 0$$ (proved)

4. Prove $\lim_{x \to 0} x^5 \sin\frac{1}{x} = 0$

We know that $0 \leq \left|\sin\frac{1}{x}\right| \leq 1$

$$\Rightarrow 0 \cdot |x^5| \leq |x^5| \left|\sin\frac{1}{x}\right| \leq 1 \cdot |x^5|$$

$$\Rightarrow 0 \leq \left|x^5 \sin\frac{1}{x}\right| \leq |x^5|$$

$\lim_{x \to 0} 0 = 0$ and $\lim_{x \to 0} x^5 = 0$

$\therefore \lim_{x \to 0} \left|x^5 \sin\frac{1}{x}\right| = 0$ by Sandwich theorem.

$\therefore \lim_{x \to 0} x^5 \sin\frac{1}{x} = 0$ (Proved)

5. Prove $\lim_{x \to 0} x^8 \cos\frac{1}{x} = 0$

We know that

$$0 \leq \left|\cos\frac{1}{x}\right| \leq 1$$

$$\Rightarrow 0 \cdot |x^8| \leq |x^8| \cdot \left|\cos\frac{1}{x}\right| \leq 1 \cdot |x^8|$$

$$\Rightarrow 0 \leq \left|x^8 \cos\frac{1}{x}\right| \leq |x^8|$$

$\lim_{x \to 0} 0 = 0$

$\lim_{x \to 0} x^8 = 0$

$\therefore \lim_{x \to 0} \left|x^8 \cos\frac{1}{x}\right| = 0$ by sandwich theorem

$\therefore \lim_{x \to 0} x^8 \cos\frac{1}{x} = 0$ (Proved)

6. Prove $\lim_{x\to\infty} \frac{\sin x}{x} = 0$

Proof: We know $0 \leq |\sin x| \leq 1$

$$\Rightarrow 0 \leq \frac{|\sin x|}{|x|} \leq \frac{1}{|x|}$$

$$\Rightarrow 0 \leq \left|\frac{\sin x}{x}\right| \leq \frac{1}{|x|}$$

But $\lim_{x\to\infty} 0 = 0$

and $\lim_{x\to\infty} \frac{1}{|x|} = 0$

$\therefore \lim_{x\to\infty} \left|\frac{\sin x}{x}\right| = 0$ by sandwich theorem.

$\Rightarrow \lim_{x\to\infty} \frac{\sin x}{x} = 0$ (Proved)

Note: (a) $\lim_{x\to 0} \sin\frac{1}{x}$ does not exist because $\sin\frac{1}{x}$ oscillates between -1 and 1.

(b) $\lim_{x\to\infty} \sin x$ does not exist because $\sin x$ oscillates between -1 and 1.

(c) When ever limit exists the value of the limit is unique.

(d) $\lim_{x \to \infty} x^p = \infty$ (if $p > 0$)

$= 1$ if $p = 0$

$= 0$ if $p < 0$

$\lim_{n \to \infty} r^n = 0$ if $|r| < 1$

$= 1$ if $r = 1$

$= \infty$ if $r > 1$

$=$ does not exist if $r \leq -1$

Q. Find $\lim_{x \to \infty} \left(\frac{a}{b}\right)^x$ where $0 < a < b$

Soln Since $a < b$

$\therefore \frac{a}{b} < 1$

Also $a > 0$ & $b > 0$

$\therefore \frac{a}{b} > 0$

$\therefore \left|\frac{a}{b}\right| = \frac{a}{b} < 1$

$$\therefore \lim_{x\to\infty}\left(\frac{a}{b}\right)^{x} = 0 \quad \left(\because \lim_{n\to\infty} r^{n} = 0 \text{ if } |r|<1\right)$$

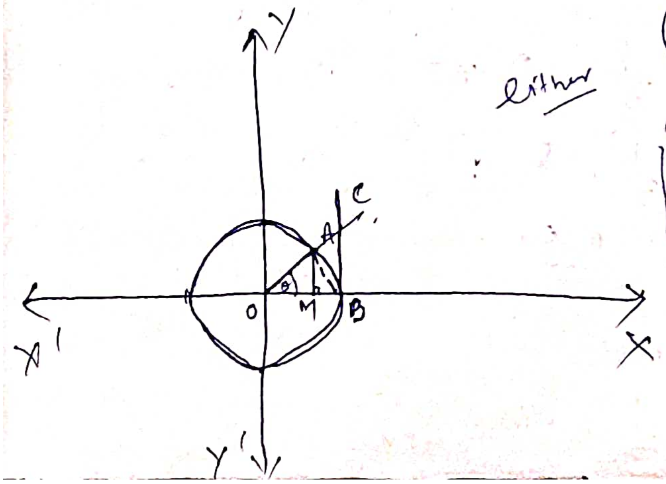
—— x ——

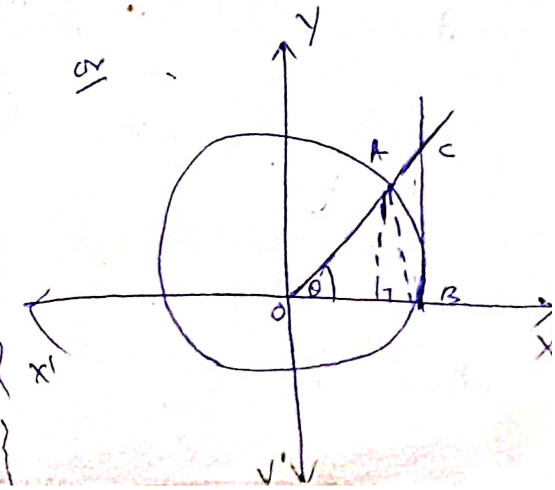
Prove that

1. $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$

Proof: ~~root~~ $\theta \to 0 \Rightarrow \theta \to 0^{+}$ or $\theta \to 0^{-}$

Consider $\theta \to 0^{+}$ i.e θ is +ve and θ is close to 0. i.e θ lies in first quadrant.





Draw a unit circle with center at Origin "O".

Let $\angle AOB = \theta$. Draw a $\perp$ $\overline{AM}$ on x-axis. Join $\overline{AB}$. Draw a $\perp$ $\overline{BC}$ which meets the extended $\overline{OA}$ at "C".

Now $\overline{OA} = \overline{OB} = 1$

From $\triangle$ AMO we have

$$\sin\theta = \frac{AM}{OA} = \frac{AM}{1} = AM$$

$$\therefore AM = \sin\theta$$

From $\triangle$ OBC

$$\tan\theta = \frac{BC}{OB} = \frac{BC}{1} = BC$$

$$\therefore BC = \tan\theta$$

Area of the $\triangle$ AOB

$$= \frac{1}{2} \cdot OB \cdot AM$$

$$= \frac{1}{2} \cdot 1 \cdot \sin\theta$$

$$= \frac{1}{2}\sin\theta$$

Area of the Sector AOB

$$= \frac{1}{2} \cdot \theta \cdot r^2$$

$$= \frac{1}{2} \cdot \theta \cdot (1)^2$$

$$= \frac{1}{2}\theta$$

Area of $\triangle$ OBC

$$= \frac{1}{2} \cdot OB \cdot BC$$

$$= \frac{1}{2} \cdot 1 \cdot \tan\theta$$

$$= \frac{1}{2}\tan\theta$$

Area of Δ AOB $\leq$ Area of the sector AOB $\leq$ Area of the Δ OBC.

$$\Rightarrow \cancel{\tfrac{1}{2}}\sin\theta \leq \cancel{\tfrac{1}{2}}\theta \leq \cancel{\tfrac{1}{2}}\tan\theta$$

$$\Rightarrow \sin\theta \leq \theta \leq \tan\theta$$

$$\Rightarrow 1 \leq \frac{\theta}{\sin\theta} \leq \frac{1}{\cos\theta}$$ ($\because$ θ is in 1st quadrant $\sin\theta$ is +ve)

$$\Rightarrow 1 \geq \frac{\sin\theta}{\theta} \geq \cos\theta$$

$$\Rightarrow \cos\theta \leq \frac{\sin\theta}{\theta} \leq 1$$ $\rightarrow \left(\begin{array}{c} a < b \\ \frac{1}{a} > b \end{array} \right)$

But $\lim\limits_{\theta \to 0+} \cos\theta = 1$

and $\lim\limits_{\theta \to 0+} 1 = 1$

$\therefore \lim\limits_{\theta \to 0+} \frac{\sin\theta}{\theta} = 1$ (by Sandwich theorem)

Consider $\theta \to 0-$ i.e θ is close to "0" and θ is negative.

Let $\theta = -\phi$ where $\phi \to 0+$

$$\therefore \lim_{\theta \to 0-} \frac{\sin\theta}{\theta} = \lim_{\phi \to 0+} \frac{\sin(-\phi)}{-\phi}$$

$$= \lim_{\phi \to 0+} \frac{\sin\phi}{\phi} = 1$$ (by previous case)

$$\therefore \lim_{\theta \to 0+} \frac{\sin\theta}{\theta} = \lim_{\theta \to 0-} \frac{\sin\theta}{\theta} = 1$$

∴ The $\lim\limits_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$.

<u>Notes</u>:- 1. $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$

(Provided $|x| < 1$)

2. $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$

(Provided $|x| < 1$)

3. $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

(Provided $|x| < 1$)

4. $e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \cdots$

(For all $x \in R$)

5. $\sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \frac{x^7}{\lfloor 7} + \cdots$

($\forall x \in R$)

6. $\cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \frac{x^6}{\lfloor 6} + \cdots$

($\forall x \in R$)

<u>Ex</u>: Prove that $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$

<u>Proof</u>:- $\lim\limits_{x \to 0} \frac{\sin x}{x} = \lim\limits_{x \to 0} \frac{x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \frac{x^7}{\lfloor 7} + \cdots}{x}$

$= \lim\limits_{x \to 0} \left(1 - \frac{x^2}{\lfloor 3} + \frac{x^4}{\lfloor 5} - \cdots\right)$

$= 1 - 0 + 0 + \cdots$

$= 1$

Problem:

1. $\lim_{x \to \infty} \cos x$

$= \lim_{x \to \infty} \sin\left(\frac{\pi}{2} + x\right)$

$= \lim_{y \to \infty} \sin y$

$= \lim_{z \to 0} \sin \frac{1}{z}$

Put $\frac{\pi}{2} + x = y$

$x \to \infty \Rightarrow y \to \infty$

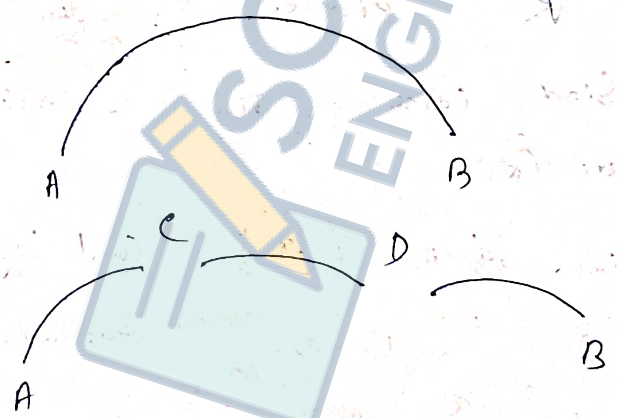
Put $y = \frac{1}{z}$

$\therefore y \to \infty \Rightarrow z \to 0$

which does not exist

because $\sin \frac{1}{z}$ oscillates between -1 and 1.

Continuity of a function at a point



Defn: A function $f(x)$ is said to be continuous at a point $x = a$ if

(i) f is defined at $x = a$, i.e. $f(a)$ exists

(ii) $\lim_{x \to a} f(x)$ exists.

(iii) $\lim_{x \to a} f(x) = f(a)$

If atleast one of the above Conditions ~~both~~ ~~both~~ fails to hold then the function is called discontinuous (not continuous) at the point.

Removable discontinuity
jump discontinuity

Continuity of a function on a set

If a function is continuous at every point ~~one~~ on a set A then the function is said to be Continuous on the set A

$\log e = 1$
$\log_e e = 1$
2. $\log(1+x)$ 2 series and cosx series relation
3. $\tan x$ 2 series or $\tan^{-1}x$ 2 series
4. Continuous on a set or underform

Notes:-

(i) All polynomial functions are Continuous on $\mathbb{R}$

(ii) All rational functions are continuous on $\mathbb{R}$ provide the denominator $\neq 0$

(iii) $\sin x$, $\cos x$ are continuous every where. on $(\mathbb{R})$

(iv) Exponential function (e^x) and logarithm function are continuous everywhere.

$y = e^x$, a^x are continuous everywhere where $a > 0$

$y = \log x$ is continuous provided $x > 0$

(v) $|x|$ continuous every where

(vi) $[x]$ is discontinuous at all integers and is continuous at other points

7. Characteristic function is continuous no where that is discontinuous everywhere

8 If conditions 1 or 3 is not satisfied then function is ~~called~~ said to have removable discontinuity.

9. If limit does not exist i.e. R.H.L $\neq$ L.H.L then discontinuity is called jump discontinuity.

Problems:- Discuss the continuity of

$$f(x) = \begin{cases} \dfrac{\sin \alpha x}{x} & \text{when } x \neq 0 \\ \alpha & \text{when } x = 0 \end{cases}$$

at $x = 0$

Soln:-

$$f(x) = \begin{cases} \dfrac{\sin \alpha x}{x} & \text{when } x \neq 0 \\ \alpha & \text{when } x = 0 \end{cases}$$

To test the continuity of f at $x = 0$

(i) At $x = 0$, $f(x) = \alpha$

i.e. $f(0) = \alpha$ which exists.

(ii) Consider $\lim_{x\to 0} f(x)$

$$= \lim_{x\to 0} \frac{\sin\alpha x}{x}$$

$$= \cancel{\lim_{x\to 0} \alpha \cdot \sin\alpha x}$$

$$= \lim_{x\to 0} \frac{\alpha\sin\alpha x}{\alpha x}$$

$$= \alpha \lim_{x\to 0} \frac{\sin\alpha x}{\alpha x} = \alpha \lim_{\alpha x\to 0} \frac{\sin\alpha x}{\alpha x}$$

$$= \alpha \cdot 1$$

$= \alpha$ which exists

(iii) $\lim_{x\to 0} f(x) = \alpha = f(0)$

$\therefore$ f is continuous at $x=0$

(2) Discuss the continuity of

$$f(x) = \begin{cases} 2x-1 & \text{when } x<3 \\ 5 & \text{when } x=3 \\ 8-x & \text{when } x>3 \end{cases}$$

at $x=3$.

Soln

$$f(x) = \begin{cases} 2x-1 & \text{when } x<3 \\ 5 & \text{when } x=3 \\ 8-x & \text{when } x>3 \end{cases}$$

To test the continuity of f

at $x = 3$

(i) At $x = 3$ $f(x) = 5$ i.e

$f(3) = 5$ which exists.

(ii) Consider $\lim_{x \to 3} f(x)$

L.H.L $= \lim_{x \to 3-} f(x) = \lim_{x \to 3-} (2x-1)$

$= 6-1 = 5$

R.H.L $= \lim_{x \to 3+} f(x) = \lim_{x \to 3+} 8-x$

$= 8-3 = 5$

$\therefore$ L.H.L = R.H.L = 5

$\therefore$ $\lim_{x \to 3} f(x)$ exists and $\lim_{x \to 3} f(x) = 5$

(iii) $\lim_{x \to 3} f(x) = 5 = f(3)$

$\therefore$ f is continuous at $x = 3$

2. Discuss the continuity of

$$f(x) = \begin{cases} \frac{\log x}{1+x} & \text{when } x > 1 \\ 1 & \text{when } x \leq 1 \end{cases}$$

at $x = 1$

Sum

$$f(x) = \begin{cases} \dfrac{\log x}{1+x} & \text{when } x > 1 \\ 1 & \text{when } x \le 1 \end{cases}$$

To find the continuity of f at $x = 1$

(i) At $x = 1$, $f(x) = 1$

i.e $f(1) = 1$ which exists

(ii) Consider $\lim_{x \to 1} f(x)$

R.H.L

$$\lim_{x \to 1+} f(x)$$

$$= \lim_{x \to 1+} \frac{\log x}{1+x}$$

$$= \frac{0}{2} = 0$$

L.H.L

$$\lim_{x \to 1-} f(x)$$

$$= \lim_{x \to 1-} 1$$

$$= 1$$

L.H.L $\neq$ R.H.L $\therefore \lim_{x \to 1} f(x)$ does not exist.

$\therefore$ f is discontinuous at $x=1$.

4. Examine the continuity of ~~fn~~

$$f(x) = \frac{x^3 - 6x^2 + 11x - 6}{x^2 + x + 1}$$

at $x=0$

~~An~~ Soln

$$f(x) = \frac{x^3 - 6x^2 + 11x - 6}{x^2 + x + 1}$$

To test the continuity of f at $x=0$

(i) ~~At x=0~~, $f(0) = \cancel{(0)^3} \; \frac{-6}{1} = -6$ which exists.

(ii) Consider $\lim_{x \to 0} f(x)$

$$= \lim_{x \to 0} \frac{x^3 - 6x^2 + 11x - 6}{x^2 + x + 1}$$

$= -6$ which exists.

(iii) $\lim_{x \to 0} f(x) = -6 = f(0)$

$\therefore$ f is continuous at $x=0$.

5. Discuss the continuity of $f(x) = \begin{cases} x^2 \sin\frac{1}{x} & \text{when } x \neq 0 \\ 0 & \text{when } x = 0 \end{cases}$

at $x=0$

~~To test the Con~~

$$f(x) = \begin{cases} x^2 \sin\frac{1}{x} & \text{when } x \neq 0 \\ 0 & \text{when } x = 0 \end{cases}$$

To find the continuity of f at $x = 0$

(i) At $x = 0$, $f(x) = 0$ i.e. $f(0) = 0$ which exists

(ii) Consider $\lim\limits_{x \to 0} f(x)$

$$= \lim_{x \to 0} x^2 \sin\frac{1}{x}$$

We know

$$0 \leq \left|\sin\frac{1}{x}\right| \leq 1$$

$$\Rightarrow 0 \cdot |x^2| \leq |x^2| \left|\sin\frac{1}{x}\right| \leq |x^2| \cdot 1$$

$$\Rightarrow 0 \leq \left|x^2 \sin\frac{1}{x}\right| \leq x^2$$

$$\cancel{\Rightarrow 0 \leq x^2 \sin\frac{1}{x} \leq x^2}$$

but $\lim\limits_{x \to 0} 0 = 0$

$\lim\limits_{x \to 0} x^2 = 0$

$\therefore \lim\limits_{x \to 0} \left|x^2 \sin\frac{1}{x}\right| = 0$ by sandwich theorem.

$$\Rightarrow \lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$$

$$\Rightarrow \lim_{x\to 0} f(x) = 0 \quad \text{which exists}$$

$$\lim_{x\to 0} f(x) = 0 = f(0)$$

$\therefore$ f is is continuous at $x = 0$

6. Find the points of discontinuity of $f(x) = \begin{cases} x^2 & \text{when } x \le 1 \\ x^2 - \frac{1}{2} & \text{when } x > 1 \end{cases}$

Soln

$$f(x) = \begin{cases} x^2 & \text{when } x \le 1 \\ x^2 - \frac{1}{2} & \text{when } x > 1 \end{cases}$$

$-\infty$ ——— 1 ——— ∞

Consider $x \in (-\infty, 1)$ i.e $x < 1$

then $f(x) = x^2$ which is a polynomial and is continuous

~~Consider~~ Consider $x \in (1, \infty)$ i.e $x > 1$

then $f(x) = x^2 - \frac{1}{2}$ which is a polynomial and is continuous

Consider $x = 1$

(i) At $x = 1$, $f(x) = x^2$

$f(1) = x^2 = 1$ which exists

(ii) Consider $\lim_{x \to 1} f(x)$

~~$= \lim_{x \to 1}$~~

R.H.L

$\lim_{x \to 1^+} f(x)$

$= \lim_{x \to 1^+} \left(x^2 - \frac{1}{2}\right)$

$= 1 - \frac{1}{2}$

$= \frac{1}{2}$

L.H.L

$\lim_{x \to 1^-} f(x)$

$= \lim_{x \to 1^-} (x^2)$

$= 1$

$\therefore$ R.H.L $\neq$ L.H.L

$\therefore$ $\lim_{x \to 1} f(x)$ does not exists.

$\therefore$ $f(x)$ is discontinuous at $x = 1$

$\therefore$ $x = 1$ is the only possible old's continuity ch 6

9, 10, 11, 12, 13 (10 (c))

$2^{3}\sqrt{\ldots}$ $\sqrt{81}$ $\sqrt[3]{9^3}$

10-(b) 6. (i), vii, viii, x, (xi), xiii, xiv, xv, 5. viii

10-(c) 3. viii, (x), xii

4. (i),

5. (ix), xii, xiii, xiv, xv

6. iv, v, vi, (ix), x, xii, xiii, xv, xvi

7. (i), iv, v, vi,

8. (i), (viii)

10-B

6. (xiii) $\lim_{x\to\infty} \sin x$ | let $x = \frac{1}{y}$, $x\to\infty \Rightarrow y\to 0$

$= \lim_{y\to 0} \sin\frac{1}{y}$

which does not exist because $\sin\frac{1}{y}$ oscillates between -1 and 1

xiv $\lim_{x\to 0} \cos\frac{1}{x} = \lim_{x\to 0} \sin\left(\frac{\pi}{2}+\frac{1}{x}\right)$

$= \lim_{y\to\infty} \sin y$

$= \lim_{z\to 0} \sin\frac{1}{z}$

which does not exist because $\sin\frac{1}{z}$ oscillates between -1 and 1

Let $\frac{\pi}{2}+\frac{1}{x} = y$

$x\to 0$ $y\to\infty$

Put $y = \frac{1}{z}$

$y\to\infty$, $z\to 0$

4)

$$\lim_{x \to \frac{1}{2}} \frac{|2x-1|}{2x-1}$$

Put $2x-1 = y$

$x \to \frac{1}{2}$, $y \to 0$.

$$= \lim_{y \to 0} \frac{|y|}{y}$$

R.H.L $= \lim_{y \to 0+} \frac{|y|}{y} = \lim_{y \to 0+} \frac{y}{y}$ ($\because y \to 0+$ $\therefore y$ is +ve, $|y| = y$)

$$= \lim_{y \to 0+} 1 = 1$$

L.H.L $\lim_{y \to 0-} \frac{|y|}{y}$

$= \lim_{y \to 0-} \frac{-y}{y}$ ($\because y \to 0-$ means y is negative. $\therefore |y| = -y$)

$= \lim_{y \to 0-} -1$

$= -1$

$|x| = x$ if $x > 0$

$|x| = -x$ if $x < 0$

R.H.L $\neq$ L.H.L

$\therefore \lim_{y \to 0} \frac{|y|}{y}$ does not exist.

i.e. $\lim_{x \to \frac{1}{2}} \frac{|2x-1|}{2x-1}$ does not exist.

viii) $\lim_{x \to 1} [2x+3]$

Put $2x+3 = y$

$x \to 1 \quad y \to 5$

$= \lim_{y \to 5} [y]$

$= \cancel{\lim_{y \to 5} 5 = 5}$

R.H.L $= \lim_{y \to 5^+} [y]$

$= \lim_{y \to 5^+} 5$

$= 5$

($\because y \to 5^+$ means $5 < y < 6$ $\therefore [y] = 5$.)

L.H.L $\lim_{y \to 5^-} [y]$

$= \lim_{y \to 5^-} 4$

$= 4$

($\because y \to 5^-$ means $4 < y < 5$ $\therefore [y] = 4$)

$\therefore$ R.H.L $\neq$ L.H.L

$\therefore \lim_{y \to 5} [y]$ doesn't exist

$\Rightarrow \lim_{x \to 1} [2x+3]$ doesn't exist

Q. (i) $\lim_{x \to \sqrt{3}} [x]$.

R.H.L $= \lim_{x \to \sqrt{3}^+} [x] = \lim_{x \to \sqrt{3}^+} 1$ ($\because x \to \sqrt{3}^+$ $\therefore 1 < x < 2$ $\therefore [x] = 1$)

$= 1$

R.H.L

$$\lim_{x \to \sqrt{3}-} [x] = \lim_{x \to \sqrt{3}-} 1 \quad \left(\begin{array}{l} \because x \to \sqrt{3}-1 \\ \therefore 1 < x < 2 \\ \therefore [x] = 1 \end{array} \right)$$

$= \underline{1}$

$\therefore$ L.H.L = R.H.L. $\lim_{x \to \sqrt{3}} [x]$ exists.

and $\lim_{x \to \sqrt{3}} = 1$.

10 - c

1. xii)

$$\lim_{x \to 0} \frac{(1-\cos x)^2}{\tan^3 x - \sin^3 x}$$

$$= \lim_{x \to 0} \frac{(1-\cos x)^2}{\frac{\sin^3 x}{\cos^3 x} - \sin^3 x}$$

$$= \lim_{x \to 0} \frac{(1-\cos^3 x)^2}{\cancel{\frac{\sin^3 x - \sin^3}{\cos^3 x}}\ \sin^3 x \left(\frac{1}{\cos^3 x} \frac{1-\cos^3 x}{\sin^3 \cos^3 x} \right)}$$

$$= \lim_{x \to 0} \frac{(1-\cos x)\ \cancel{(1-\cos x)}\ \cos^3 x}{\sin^3 x\ \cancel{(1-\cos x)}(1+\cos^2 x+\cos x)}$$

$$= \lim_{x \to 0} \left(\frac{1-\cos x)\ \cos^3 x}{\sin^3 x\ (1+\cos x+\cos^2 x)} = \lim_{x \to 0} \frac{1-\cos x}{\sin^3 x} \cdot \lim_{x \to 0} \frac{\cos^3 x}{1+\cos x+\cos^2 x}$$

$$= \lim_{x \to 0} \frac{1-\cos x}{\sin^3 x} \times \lim_{x \to 0} \frac{\cos^3 x}{1+\cos^2 x + \cos x}$$

$$= \lim_{x \to 0} \frac{1-\cos x}{\sin^3 x} \times \frac{1}{3}$$

$$= \frac{1}{3} \lim_{x \to 0} \frac{(1-\cos x)(1+\cos x)}{(\sin^3 x)(1+\cos x)}$$

$$= \frac{1}{3} \lim_{x \to 0} \frac{1-\cos^2 x}{\sin^3 x\,(1+\cos x)}$$

$$= \frac{1}{3} \lim_{x \to 0} \frac{\cancel{\sin^2 x}}{\cancel{\sin^3 x}_{\sin x}\,(1+\cos x)}$$

$$= \frac{1}{3} \lim_{x \to 0} \frac{1}{\sin x\,(1+\cos x)} = \cancel{\frac{1}{3} \lim_{x \to 0} \frac{1}{\sin x} \cdot \lim_{x \to 0} \frac{1}{1+\cos x}}$$

R.H.L $= \frac{1}{3} \lim_{x \to 0^+} \frac{1}{\sin x\,(1+\cos x)}$

$= \infty$

L.H.L $\frac{1}{3} \lim_{x \to 0^-} \frac{1}{\sin x\,(1+\cos x)}$

$= -\infty$

$\therefore$ limit does not exist.

10(b)

6. $\lim_{x\to 1} \frac{|x^2-3x+1|}{x^2-3x+2}$

$= \lim_{x\to 1} \frac{|x^2-3x+1|}{x^2-2x-x+2} = \lim_{x\to 1} \frac{|x^2-3x+1|}{x(x-2)-1(x-2)}$

$= \lim_{x\to 1} \frac{|x^2-3x+1|}{(x-2)(x-1)}$

Some exam problems

2. $\lim_{x\to 0} \frac{\cos ax - 1}{\cos bx - 1}$

$= \lim_{x\to 0} \frac{1-\cos ax}{1-\cos bx}$

$= \lim_{x\to 0} \frac{2\sin^2 \frac{ax}{2}}{2\sin^2 \frac{bx}{2}}$

$= \lim_{x\to 0} \left(\frac{\sin \frac{ax}{2}}{\sin \frac{bx}{2}}\right)^2 = \lim_{x\to 0} \left(\frac{\frac{\sin \frac{ax}{2}}{\frac{ax}{2}} \times \frac{ax}{2}}{\frac{\sin \frac{bx}{2}}{\frac{bx}{2}} \times \frac{bx}{2}}\right)^2$

$= \lim_{x\to 0} \frac{\left(\frac{\sin \frac{ax}{2}}{\frac{ax}{2}}\right)^2}{\left(\frac{\sin \frac{bx}{2}}{\frac{bx}{2}}\right)^2} \times \frac{\left(\frac{a}{2}\right)^2}{\left(\frac{b}{2}\right)^2}$

$$= \frac{\left(\frac{a}{2}\right)^2}{\left(\frac{b}{2}\right)^2} = \frac{a^2}{b^2}$$

$$= \left(\frac{a}{2}\right)^2 \frac{\lim_{x\to 0}\left(\frac{\sin ax}{\frac{ax}{2}}\right)^2}{\left(\frac{b}{2}\right)^2 \lim_{x\to 0}\left(\frac{\sin bx}{\frac{bx}{2}}\right)^2} = \frac{a^2}{b^2} \cdot \frac{\lim_{x\to 0}\left(\frac{\sin\frac{ax}{2}}{\frac{ax}{2}}\right)\left(\frac{\sin\frac{ax}{2}}{\frac{ax}{2}}\right)}{\lim_{x\to 0}\left(\frac{\sin\frac{bx}{2}}{\frac{bx}{2}}\right)\left(\frac{\sin\frac{bx}{2}}{\frac{bx}{2}}\right)}$$

$$= \frac{a^2}{b^2} \cdot \frac{\lim_{x\to 0}\frac{\sin\frac{ax}{2}}{\frac{ax}{2}} \cdot \lim_{x\to 0}\frac{\sin\frac{ax}{2}}{\frac{ax}{2}}}{\lim_{x\to 0}\frac{\sin\frac{bx}{2}}{\frac{bx}{2}} \cdot \lim_{x\to 0}\frac{\sin\frac{bx}{2}}{\frac{bx}{2}}}$$

$$= \frac{a^2}{b^2}$$

12. $\lim_{x\to 0} \frac{|\sin x|}{x}$

R.H.L $\lim_{x\to 0+} \frac{|\sin x|}{x}$

$= \lim_{x\to 0+} \frac{\sin x}{x}$

$= 1$

$\because x \to 0+$

$\therefore x$ is in the first quadrant

$\therefore \sin x$ is +ve

$\therefore |\sin x| = \sin x$

L.H.L $\lim_{x\to 0-} \frac{|\sin x|}{x}$

$= \lim_{x\to 0-} -\frac{\sin x}{x}$

$= -\lim_{x\to 0} \frac{\sin x}{x}$

$= -1$

$\because x \to 0-$

x is negative and x lies in 4th quadrant

$\therefore \sin x$ is −ve

$\therefore |\sin x| = -\sin x$

$\therefore$ L.H.L $\neq$ R.H.L

$\lim_{x\to 0} \frac{|\sin x|}{x}$ does not exist.

13. $$\lim_{x\to 0} \frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}$$

<u>R.H.L.</u> $$\lim_{x\to 0+} \frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}$$

$$= \lim_{x\to 0} \frac{1-e^{-\frac{1}{x}}}{1+e^{-\frac{1}{x}}} \quad \left(\text{dividing n.r and d.r by } e^{\frac{1}{x}}\right)$$

$$= \frac{1}{1} = 1 \quad \left(\because \lim_{x\to 0+} e^{-\frac{1}{x}} = 0\right)$$

<u>LHL</u>

$$\lim_{x\to 0-} \frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}$$

$$= \frac{0-1}{0+1} \quad \left(\because \lim_{x\to 0-} e^{\frac{1}{x}} = 0\right)$$

$$= -1 \text{ (ans)}$$

R.H.L $\neq$ L.H.L

$\lim_{x\to 0} \frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}$ does not exist.

$$\lim_{x\to 0-} e^{\frac{1}{x}}$$

$$= e^{\left(\lim_{x\to 0-} \frac{1}{x}\right)} = e^{-\infty} = 0 \quad \left(\because \lim_{x\to 0-} \frac{1}{x} = -\infty\right)$$

$\cancel{\lim f(g(x))}$

$\cancel{= f\left(\lim g(x)\right)}$

$\cancel{\lim (\tan 5x)}$

$$\lim_{x\to 0+} e^{\frac{1}{x}} = e^{\lim_{x\to 0+} \frac{1}{x}}$$

$$= e^{\infty} \quad \left(\because \lim_{x\to 0+} \frac{1}{x} = \infty\right)$$

$$= \infty$$

$$\lim f(g(x)) = f\left(\lim g(x)\right)$$

Ex: $\lim_{x\to 0} e^{\frac{1}{x}} = e^{\lim_{x\to 0} \frac{1}{x}}$

$$\lim_{x \to 0} \frac{e^{\frac{1}{x}}}{\cancel{g(x)}}$$

$$\lim f(g(x))$$

$$= \cancel{\lim} f\left(\lim g(x)\right)$$

$f(x) = e^x$

$g(x) = \frac{1}{x}$

$f(g(x)) = f\left(\frac{1}{x}\right)$

$= e^{1/x}$

$$\lim_{x \to 0} \sin(2x^2 + 3x + 5)$$

$= \sin\left(\lim \ldots\right)$

$=$

$$\lim_{x \to \frac{\pi}{2}} (1 + \cos x)^{3\sec x}$$

$$= \lim_{x \to \frac{\pi}{2}} (1 + \cos x)^{\frac{3}{\cos x}}$$

Put $\cos x = y$

$x \to \frac{\pi}{2} \Rightarrow y \to 0$

$$= \lim_{y \to 0} (1 + y)^{\frac{3}{y}}$$

$$= \lim_{y \to 0} \left\{(1 + y)^{\frac{1}{y}}\right\}^3$$

$$= \cancel{\lim} \; e^3$$

$$\lim_{x \to \frac{\pi}{2}}$$

$$= \left\{\lim_{y \to 0} (1 + y)^{\frac{1}{y}}\right\}^3$$

$$= e^3$$

| Proved

$$2 \quad \lim_{x \to \infty} \left(\frac{2x+3}{2x+1}\right)^{2x+5}$$

$$= \cancel{\lim_{x \to \infty} \left(1 + \frac{1}{2x+1}\right)}$$

Put $2x+1 = y$

$x \to \infty \quad y \to \infty$

$2x+3 = y+2$

$2x+5 = y+4$

$$= \lim_{y \to \infty} \left(\frac{y+2}{y}\right)^{y+4}$$

$$= \lim_{x\to\infty} \left(1+\frac{2}{y}\right)^{y+4}$$

$$= \lim_{x\to\infty} \left(1+\frac{1}{y/2}\right)^{y+4}$$

Put

$y/2 = z$

$y = 2z$

$y \to \infty,\ z \to \infty$

$$= \lim_{z\to\infty} \left(1+\frac{1}{z}\right)^{2z+4}$$

$$= \lim_{z\to\infty} \left(1+\frac{1}{z}\right)^{2z} \cdot \left(1+\frac{1}{z}\right)^{4}$$

$$= \lim_{z\to\infty} \left\{\left(1+\frac{1}{z}\right)^{z}\right\}^{2} \lim_{z\to\infty}\left(1+\frac{1}{z}\right)^{4}$$

$$= \left\{\lim_{z\to\infty} \left(1+\frac{1}{z}\right)^{z}\right\}^{2} \cdot \lim_{x\to\infty}\left(1+\frac{1}{z}\right)^{4}$$

$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} = e$

$\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e$

$$= e^{2} \cdot 1$$

$$= e^{2}$$

7. $$\lim_{x\to 0} \left(1+3\tan^{2}x\right)^{\cot^{2}x}$$

Put

$\tan x = y$

$x \to 0,\ y \to 0$

$$= \lim_{y\to 0} \left(1+3y^{2}\right)^{\frac{1}{y^2}}$$

Put

$3y^2 = z$

$\Rightarrow y^2 = \frac{z}{3}$

$y \to 0,\ z \to 0$

$$= \lim_{z\to 0} (1+z)^{\frac{3}{z}}$$

$$= \left\{\lim_{z\to 0} (1+z)^{\frac{1}{z}}\right\}^{3}$$

$$= e^{3}$$

4. $\lim_{x\to\infty}\left(\frac{x}{1+x}\right)^{x}$

$= \lim_{x\to} \left(\frac{\cancel{x}}{1+\frac{1}{x}}\right)^{x}$

$= \lim_{y\to\infty}\left(\frac{y-1}{y}\right)^{y-1}$

Put
$1+x=y$
$\Rightarrow x=y-1$
$x\to\infty \Rightarrow y\to\infty$

$= \lim_{y\to\infty}\left(1-\frac{1}{y}\right)^{y-1}$

$= \frac{\lim_{y\to\infty}\left(1-\frac{1}{y}\right)^{y}}{\lim_{y\to\infty}\left(1-\cancel{\frac{1}{y}}\right)}$

$= \lim_{y\to\infty}\left(1-\frac{1}{y}\right)^{y}$

Put
$-\frac{1}{y}=z$
$\Rightarrow y=-\frac{1}{z}$
as $y\to\infty$
$z\to 0$

$= \lim_{z\to 0}(1+z)^{-\frac{1}{z}}$

$= \left\{\lim_{z\to 0}(1+z)^{\frac{1}{z}}\right\}^{-1}$

$= e^{-1}$

$= \frac{1}{e}$

$-\frac{1}{z} = 1+\cancel{x}$

$x = -1-\frac{1}{z}$

10-(C), 9, 10, 11, 12, 13.

10-(C)

9. $f(x) = \frac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1}$ if $x \neq 0$

$f(x) = 0$ if $x = 0$

Neighbourhood of a point

(Nbd or Nhd)

Suppose $x = c$ is a point in $\mathbb{R}$

The neighbourhood of c is $(c-\delta, c+\delta)$.

Neighbourhoods are always open interval

$-\infty \quad c-\delta \quad c \quad c+\delta \quad \infty$

It is called δ-Nbd of c

Deleted neighbourhood or ~~Punctcture~~ **punctured Nbd**

The deleted Nbd of c

$= (c-\delta, c) \cup (c, c+\delta)$

Note: 1. Usually we denote δ, ϵ, η for small quantities.

2. $|x| < c \iff -c < x < c$

3. $|x| \leq c \iff -c \leq x \leq c$

4. Limit of a function (ϵ - δ def^n)

1. $\lim_{x \to a} f(x) = l$

It means that $f(x)$ is close to l whenever x is close to a but $x \neq a$

i.e ~~[illegible]~~ (There exists)

for given $\epsilon > 0$ $\exists$ a $\delta > 0$ (depending upon ϵ) we have $f(x) \in (l-\epsilon, l+\epsilon)$ whenever $x \in (a-\delta, a+\delta)$ and $x \neq a$

i.e for given $\epsilon > 0$ $\exists$ a $\delta > 0$ (depending upon ϵ) we have

$$l - \epsilon < f(x) < l + \epsilon$$

whenever $a - \delta < x < a + \delta$ and $x \neq a$.

i.e for given $\epsilon > 0$ $\exists$ a $\delta > 0$ (depending upon ϵ) we have

$$-\epsilon < f(x) - l < \epsilon$$

when ever $-\delta < x - a < \delta$

and $x \neq a$

$$\boxed{\begin{array}{l}\text{i.e for given } \epsilon > 0 \ \exists \text{ a } \delta > 0 \\ \text{(depending upon } \epsilon \text{) we have} \\ \qquad |f(x) - l| < \epsilon \\ \text{when ever } |x-a| < \delta \text{ and } x \neq a \\ \text{i.e } 0 < |x-a| < \delta\end{array}}$$

Problem :- 1. $\lim_{x \to 3} (x+4)$

We expect that $\lim_{x \to 3} (x+4) = 7$

We will ~~do~~ prove it by $\epsilon - \delta$ definition.

Here $f(x) = x+4$

Let $\epsilon > 0$ be given

Now $|f(x) - 7| = |x+4-7| = |x-3|$ ... $|x-3| < \epsilon$

~~and $|x-3| < \epsilon$~~

So we need to choose

$$\delta = \epsilon$$

$\therefore$ for given $\epsilon > 0$ $\exists$

$\delta = \epsilon > 0$ such that

$$|f(x) - 7| < \epsilon$$

whenever $|x-3| < \delta$ and $x \neq 3$

$\therefore \lim_{x \to 3} f(x) = 7$ (Proved)

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1. (i) $\lim_{x \to 0} (2x+3) = 3$

Here $f(x) = 2x+3$

Let $\epsilon > 0$ be given

Now $|f(x) - 3| = |2x+3-3| = |2x|$

$$= 2|x| < \epsilon$$

i.e $|x| < \frac{\epsilon}{2}$

So need to choose $\delta = \frac{\epsilon}{2} > 0$

$\therefore$ For given $\epsilon > 0$ $\exists$ $\delta = \frac{\epsilon}{2} > 0$

Such that

$$|f(x)-3| < \epsilon$$

whenever $|x-0| < \delta$ and $x \neq 0$

$\therefore \lim_{x \to 0} f(x) = 3$ (proved)

(iii) $\lim_{x \to -2} (3x+8) = 2$

Here $f(x) = 3x+8$

Let $\epsilon > 0$ be given

Now $|f(x)-2| = |3x+8-2| = |3x+6|$

$= |3(x+2)| < \epsilon$

$= 3|x+2| < \epsilon$

i.e. $|x+2| < \frac{\epsilon}{3}$

So we need to choose $\delta = \frac{\epsilon}{3} > 0$

$\therefore$ For given $\epsilon > 0$ $\exists$ $\delta = \frac{\epsilon}{3} > 0$

such that $|f(x)-2| < \epsilon$

whenever $|x+2| < \delta$ and $x \neq -2$

$\therefore \lim_{x \to -2} f(x) = 2$ (proved)

vi) $\lim_{x \to a} \sqrt{x} = \sqrt{a}$, $\sqrt{a} > 0$

Here $f(x) = \sqrt{x}$

Let $\epsilon > 0$ be given

Now $|f(x) - \sqrt{a}| = |\sqrt{x} - \sqrt{a}|$

$$= \left| \frac{x-a}{\sqrt{x}+\sqrt{a}} \right| = \frac{|x-a|}{\sqrt{x}+\sqrt{a}}$$

Here $\sqrt{x}$ is to be defined

we have $x \geq 0$

Now $|x-a| < \delta$

$$-\delta < x-a < \delta$$

$$\Rightarrow a-\delta < x < a+\delta$$

also $a-\delta \geq 0$

$$\Rightarrow \delta \leq a$$

Now $\sqrt{x} + \sqrt{a} \geq \sqrt{a}$

$$\Rightarrow \frac{1}{\sqrt{x}+\sqrt{a}} < \frac{1}{\sqrt{a}}$$

$\therefore$ $f(x) - \sqrt{a} = \frac{|x-a|}{\sqrt{x}+\sqrt{a}} < \frac{\delta}{\sqrt{a}} < \epsilon$

$\therefore$ $\delta < \epsilon\sqrt{a}$

Choose $\delta = \min^m\{a, \epsilon\sqrt{a}\}$

$\therefore$ for given $\epsilon > 0 \ \exists \ \delta$

$= \min^m\{a, \epsilon\sqrt{a}\} > 0$

such that $|f(x) - \sqrt{a}| < \epsilon$

whenever $|x-a| < \delta$ and $x \neq a$

$\therefore \lim\limits_{x \to a} f(x) = \sqrt{a}$.

Problem

1. $\lim\limits_{x\to\infty}\left(\dfrac{x+3}{x-1}\right)^{x+3}$

Put $x-1=y$

$x \to \infty \Rightarrow y \to \infty$

$x = y+1$

$= \lim\limits_{y\to\infty}\left(\dfrac{y+4}{y}\right)^{y+4}$

$= \lim\limits_{y\to\infty}\left(1+\dfrac{4}{y}\right)^{y+4}$

$= \lim\limits_{y\to\infty}\left(1+\dfrac{4}{y}\right)^{y} \times \left(1+\dfrac{4}{y}\right)^{4}$

$= \lim\limits_{y\to\infty}\left(1+\dfrac{4}{y}\right)^{y} \cdot \lim\limits_{y\to\infty}\left(1+\dfrac{4}{y}\right)^{4}$

$= \lim\limits_{y\to\infty}\left(1+\dfrac{4}{y}\right)^{y} \times 1$

Put $\dfrac{4}{y} = z$

$y \to \infty \Rightarrow z \to 0$

$y = \dfrac{4}{z}$

$= \lim\limits_{z\to 0}(1+z)^{\frac{4}{z}}$

$$= \lim_{z \to \infty} \left\{ \left(1+\frac{1}{z}\right)^{z} \right\}^{4}$$

$$= \left\{ \lim_{z \to \infty} \left(1+\frac{1}{z}\right)^{z} \right\}^{4} = e^{4} \quad \text{(Ans)}$$

Q: Find the $\lim_{x \to \infty} \sqrt{x+\sqrt{x+\sqrt{x}}} - \sqrt{x}$

Ans

$$\lim_{x \to \infty} \frac{\left(\sqrt{x+\sqrt{x+\sqrt{x}}} - \sqrt{x}\right)\left(\sqrt{x+\sqrt{x+\sqrt{x}}} + \sqrt{x}\right)}{\sqrt{x+\sqrt{x+\sqrt{x}}} + \sqrt{x}}$$

$$= \lim_{x \to \infty} \frac{\cancel{x} + \sqrt{x+\sqrt{x}} - \cancel{x}}{\sqrt{x+\sqrt{x+\sqrt{x}}} + \sqrt{x}}$$

$$= \lim_{x \to \infty} \frac{\frac{\sqrt{x+\sqrt{x}}}{\sqrt{x}}}{\frac{\sqrt{x+\sqrt{x+\sqrt{x}}}}{\sqrt{x}} + 1}$$

$$= \lim_{x \to \infty} \frac{\sqrt{\frac{x+\sqrt{x}}{x}}}{\sqrt{\frac{x+\sqrt{x+\sqrt{x}}}{x}} + 1}$$

$$= \lim_{x \to \infty} \frac{\sqrt{1+\frac{1}{\sqrt{x}}}}{\sqrt{1+\frac{\sqrt{x+\sqrt{x}}}{x}} + 1}$$

$$= \lim_{x \to \infty} \frac{\sqrt{1 + \frac{1}{\sqrt{x}}}}{\sqrt{1 + \frac{\sqrt{x + \sqrt{x}}}{\sqrt{(x^2)}}} + 1}$$

$$= \lim_{x \to \infty} \frac{\sqrt{1 + \frac{1}{\sqrt{x}}}}{\sqrt{1 + \sqrt{\frac{x + \sqrt{x}}{x^2}}} + 1}$$

$$= \lim_{x \to \infty} \frac{1 + \frac{1}{\sqrt{x}}}{1 + \sqrt{\frac{1}{x} + \frac{1}{x\sqrt{x}}} + 1}$$

$$= \cancel{\lim} \frac{1 + 0}{1 + \sqrt{0 + 0} + 1}$$

$$= \frac{1}{2} \quad (\text{Ans})$$