Chapter-2 - Rectangular & Cylindrical Waveguide

A transmission line can be used to guide EM energy from one point (generator) to another (load). A waveguide is another means of achieving the same goal.

In general, a waveguide consists of a hollow metallic tube of a rectangular or circular shape used to guide an electromagnetic wave. Waveguides are used primarily at frequencies in the microwave range. A waveguide can operate only above a certain frequency called the cutoff frequency and therefore acts as a high-pass filter.

Q) Differentiate Transmission Line & Wave guide

Transmission line	Wave guide
1) It is operated on ~~(lower)~~ range of frequency i.e it may operate from d.c (f=0) to a very high frequency.	1) It is used only in very high frequency, It can n't transmit d.c.
2) It acts as one type of Low pass filter	2) It operates after certain cutoff frequency. So it act as a High pass filter.
3) It supports only TEM mode.	3) It does n't support TEM mode but support TE & TM mode
4) Not capable of handling large power.	4) Capable of handling large power.

5) T.L become inefficient as a result of skin effect & dielectric losses.	5) No power loss in radiation. Dielectric loss is negligible, since guides are normally air filled. Small power loss as heat in the walls of guides, but loss is very small.
6) In this metal conductor are used	6) Metal hollow tubes are used to avoid loss.

Skin effect :- [In transmission line]

As the EM (H) wave travels in a conducting medium, because of ohmic losses present, its amplitude is attenuated by a factor $e^{-\alpha z}$. [α = attenuation constant]

The distance 'δ', through which the wave amplitude decrease to a factor e^{-1} (about 37% of the original value) is called skin depth or penetration depth.

$$\boxed{\delta = \frac{1}{\alpha}} = \boxed{\frac{1}{\sqrt{\pi f \mu \sigma}}}$$

$\tan\delta$ is called loss tangent.

→ The skin depth is a measure of depth or to which EM wave can penetrate the medium.

Types of wave guides.

Wave guide can be of different types

(i) Rectangular wave guide

(ii) Cylindrical / circular waveguide.

(iii) Elliptical wave guide

(iv) Parallel Plate waveguide

Rectangular Waveguide

Consider a rectangular waveguide containing lossless dielectric material and having the walls perfectly conducting $(\sigma_c = \infty)$

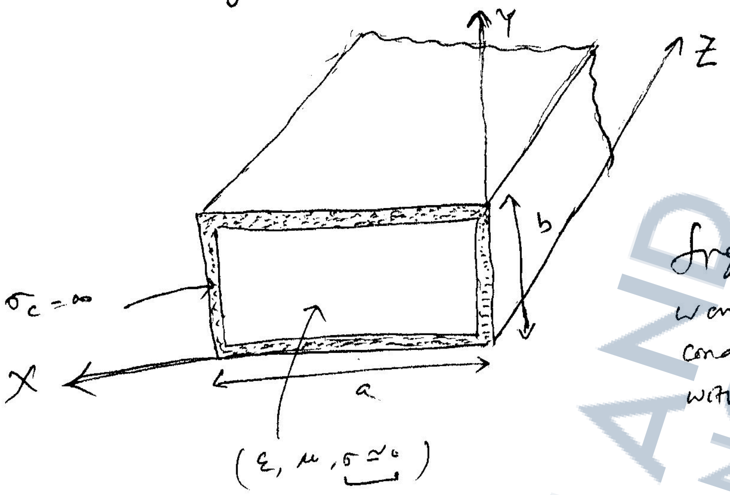


fig : 22 :- A rectangular waveguide with perfectly conducting walls filled with lossless material.

The Maxwell's equations for lossless dielectric medium are

$$\nabla^2 E_s + K^2 E_s = 0 \quad \text{—} \; (1)$$

$$\nabla^2 H_s + K^2 H_s = 0 \quad \text{—} \; (2)$$

where $K = \omega\sqrt{\mu\epsilon}$ —

K = wave number.

ω = Angular freq.

μ = Permeability of the medium

ϵ = Permittivity of the medium

Derivation Not required

$\because \; \nabla \times E = -\frac{\partial B}{\partial t}$ — (i)

$\nabla \times H = \frac{\partial D}{\partial t}$ — (ii)

$\nabla \times (\nabla \times \vec{E}) = \nabla \times \left(-\frac{\partial B}{\partial t}\right)$

$= \nabla \times (-j\omega\mu H)$

$= -j\omega\mu (\nabla \times H)$

$= -j\omega\mu \left(\frac{\partial D}{\partial t}\right)$

$= (-j\omega\mu)(j\omega)(\epsilon E)$

$= \omega^2 \mu \epsilon E = K^2 E$

$\Rightarrow \nabla(\nabla \cdot \vec{E}) - \nabla^2 E = K^2 E$

$\Rightarrow 0 - \nabla^2 E = K^2 E$

$(\because \nabla \cdot \vec{E} = 0$ in a source-free region$)$

$\Rightarrow \nabla^2 E + K^2 E = 0$

expanding eqn (1),

$$\left(\frac{\partial^2 E_s}{\partial x^2} + \frac{\partial^2 E_s}{\partial y^2} + \frac{\partial^2 E_s}{\partial z^2}\right) + K^2 E_s = 0 \quad \text{—} \; (3)$$

If we let

$$E_s = (E_{xs}, E_{ys}, E_{zs}) \quad \text{and} \quad H_s = (H_{xs}, H_{ys}, H_{zs}) \quad \text{—} \; (4)$$

For Z-component, eqn (3) becomes

$$\frac{\partial^2 E_{zs}}{\partial x^2} + \frac{\partial^2 E_{zs}}{\partial y^2} + \frac{\partial^2 E_{zs}}{\partial z^2} + k^2 E_{zs} = 0 \quad \text{——— (5)}$$

Which is a partial differential equation. It can be solved by separation of variables. So let the solution to the above eqn is given as,

$$E_{zs}(x,y,z) = X(x)\, Y(y)\, Z(z) \quad \text{——— (6)}$$

Where $X(x)$, $Y(y)$ and $Z(z)$ are function of x, y, z respectively. Substituting eqn (6) in eqn (5) and dividing by XYZ, we have

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = -k^2 \quad \text{——— (7)}$$

$\therefore$

$$\frac{\partial^2}{\partial x^2} XYZ + \frac{\partial^2}{\partial y^2} XYZ + \frac{\partial^2}{\partial z^2} XYZ + k^2 XYZ = 0$$

$$YZ \frac{\partial^2 X}{\partial x^2} + XZ \frac{\partial^2 Y}{\partial y^2} + XY \frac{\partial^2 Z}{\partial z^2} + k^2 XYZ = 0$$

Dividing by XYZ both the sides

$$\Rightarrow \frac{1}{X} \cdot X'' + \frac{1}{Y} \cdot Y'' + \frac{1}{Z} \cdot Z'' + k^2 = 0$$

Since ~~the~~ k has 3 components along x, y, z direction,

$$\Rightarrow \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = -k_x^2 - k_y^2 - k_z^2 \quad \text{——— (8)}$$

Equating the coefficients,

$$\frac{X''}{X} = -k_x^2 \quad \text{——— (9)}$$

$$\frac{Y''}{Y} = -k_y^2 \quad \text{——— (10)}$$

$$\frac{Z''}{Z} = -k_z^2 \quad \text{——— (11)}$$

Since the guided wave propagates along the Z-direction the solution along Z-axis is given as

$$Z(z) = e^{-\gamma z}$$

$$Z'' = \gamma^2 e^{-\gamma z} = \gamma^2 Z$$

$$\Rightarrow \frac{Z''}{Z} = \gamma^2 \quad \text{—— (12)}$$

$\because Z = e^{-\gamma z}$

$\frac{\partial Z}{\partial z} = -\gamma \cdot e^{-\gamma z}$

$\frac{\partial^2 Z}{\partial z} = \gamma^2 e^{-\gamma z}$

$\therefore$ Putting eqn (12) in eqn (11), we have

$$\gamma^2 = -K_z^2 \quad \text{—— (13)}$$

So eqn (11) becomes,

$$\frac{Z''}{Z} = \gamma^2 \quad \text{—— (14)}$$

From eqn (9), (10) & (14), we can write

$$X'' + X K_x^2 = 0 \quad \text{—— (15)}$$

$$Y'' + Y K_y^2 = 0 \quad \text{—— (16)}$$

$$Z'' - Z\gamma^2 = 0 \quad \text{—— (17)}$$

Solution to (15), (16) & (17) are in the form

$$X(x) = C_1 \cos K_x x + C_2 \sin K_x x \quad \text{—— (18)}$$

$$Y(y) = C_3 \cos K_y y + C_4 \sin K_y y \quad \text{—— (19)}$$

$$Z(z) = C_5 e^{\gamma z} + C_6 e^{-\gamma z} \quad \text{—— (20)}$$

we have assumed that wave propagates along waveguide in the +z direction, the multiplicative

Constant $C_5 = 0$, Eq^n (20) becomes

$$Z(z) = C_6 e^{-\gamma z} \quad \text{—— (21)}$$

Putting eq^n (18), (19), (21) in eq^n (6), we have

$$E_{zs}(x,y,z) = (C_1 \cos k_x x + C_2 \sin k_x x)(C_3 \cos k_y y + C_4 \sin k_y y) C_6 e^{-\gamma z}$$

$$= (C_1 C_6 \cos k_x x + C_2 C_6 \sin k_x x)\left[C_3 \cos k_y y + C_4 \sin k_y y\right] e^{-\gamma z}$$

$$\Rightarrow \boxed{E_{zs}(x,y,z) = (A_1 \cos k_x x + A_2 \sin k_x x)(A_3 \cos k_y y + A_4 \sin k_y y) e^{-\gamma z}} \quad \text{—— (22)}$$

where $A_1 = C_1 C_6$, $A_2 = C_2 C_6$, $A_3 = C_3$, $A_4 = C_4$

By taking the similar steps, we get the solution of Z- component of eq^n (2) as,

$$\boxed{H_{zs}(x,y,z) = (B_1 \cos k_x x + B_2 \sin k_x x)(B_3 \cos k_y y + B_4 \sin k_y y) e^{-\gamma z}} \quad \text{—— (23)}$$

→ Instead of solving for other field components $E_{xs}, E_{ys}, H_{xs}, H_{ys}$ in eq^n (1) & (2) in the same manner, it is more convenient to use Maxwell eq^n to determine them from E_{zs} and H_{zs}.

Determination of $E_{xs}, E_{ys}, H_{xs}, H_{ys}$

From Maxwell's eq^n

$$\nabla \times E_s = -j\omega\mu H_s \quad \text{—— (24)}$$

$$\nabla \times H_s = j\omega \epsilon E_s \quad \text{—— (25)}$$

From eq^n (24), expanding the curl

$$\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_{xs} & E_{ys} & E_{zs} \end{vmatrix} = -j\omega\mu\left[H_{xs}\hat{a}_x + H_{ys}\hat{a}_y + H_{zs}\hat{a}_z\right]$$

∵ $\nabla \times E = -\frac{\partial B}{\partial t}$

$= -(j\omega) \cdot B$

$= -(j\omega)(\mu H)$

$= -j\omega\mu H$

∵ $\frac{\partial}{\partial t} = j\omega$

$B = \mu H$

Similarly

$\nabla \times H = \frac{\partial D}{\partial t}$

$= j\omega \cdot (\epsilon E)$

$= j\omega \epsilon E$

$$\Rightarrow \hat{a}_x \left[\frac{\partial}{\partial y} E_{zs} - \frac{\partial}{\partial z} E_{ys} \right] + \hat{a}_y \left[\frac{\partial}{\partial x} E_{zs} - \frac{\partial}{\partial z} E_{xs} \right]$$

$$+ \hat{a}_z \left[\frac{\partial}{\partial x} E_{ys} - \frac{\partial}{\partial y} E_{xs} \right]$$

$$= -j\omega\mu H_{xs}\, \hat{a}_x - j\omega\mu H_{ys}\, \hat{a}_y - j\omega\mu H_{zs}\, \hat{a}_z$$

Comparing the Coefficients of unit vector, both the sides.

$$\frac{\partial}{\partial y} E_{zs} - \frac{\partial}{\partial z} E_{ys} = -j\omega\mu H_{xs} \quad \text{—— (26)}$$

$$\frac{\partial}{\partial z} E_{xs} - \frac{\partial}{\partial x} E_{zs} = -j\omega\mu H_{ys} \quad \text{—— (27)}$$

$$\frac{\partial}{\partial x} E_{ys} - \frac{\partial}{\partial y} E_{xs} = -j\omega\mu H_{zs} \quad \text{—— (28)}$$

Similarly from eqn (25)

$$\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_{xs} & H_{ys} & H_{zs} \end{vmatrix} = j\omega\epsilon \left[E_{xs}\, \hat{a}_x + E_{ys}\, \hat{a}_y + E_{zs}\, \hat{a}_z \right]$$

Expanding & equating the Coefficients, we have.

$$\frac{\partial}{\partial y} H_{zs} - \frac{\partial}{\partial z} H_{ys} = j\omega\epsilon E_{xs} \quad \text{—— (29)}$$

$$\frac{\partial}{\partial z} H_{xs} - \frac{\partial}{\partial x} H_{zs} = j\omega\epsilon E_{ys} \quad \text{—— (30)}$$

$$\frac{\partial}{\partial x} H_{ys} - \frac{\partial}{\partial y} H_{xs} = j\omega\epsilon E_{zs} \quad \text{—— (31)}$$

We will now express E_{xs}, E_{ys}, H_{xs} & H_{ys} in terms of E_{zs} and H_{zs}.

~~Putting eqn~~ From eqn (29), we have

$$j\omega\epsilon E_{xs} = \frac{\partial}{\partial y} H_{zs} - \frac{\partial}{\partial z} H_{ys}$$

$$= \frac{\partial}{\partial y} H_{zs} - \frac{\partial}{\partial z}\left[\frac{-1}{j\omega\mu}\left(\frac{\partial}{\partial z} E_{xs} - \frac{\partial}{\partial x} E_{zs}\right)\right]$$

[using eqn 27]

$$\Rightarrow j\omega\epsilon E_{xs} = \frac{\partial}{\partial y} H_{zs} + \frac{1}{j\omega\mu}\frac{\partial}{\partial z}\left[\frac{\partial}{\partial z} E_{xs} - \frac{\partial}{\partial x} E_{zs}\right]$$

$$\Rightarrow j\omega\epsilon E_{xs} = \frac{\partial}{\partial y} H_{zs} + \frac{1}{j\omega\mu}\left(\frac{\partial^2 E_{xs}}{\partial z^2} - \frac{\partial^2 E_{zs}}{\partial x \partial z}\right) \quad \text{—— (32)}$$

From eqn (22) & (23), it is clear that all field components vary with z according to $e^{-\gamma z}$,

Let $E_{zs} = E_p e^{-\gamma z}$

$$\frac{\partial E_{zs}}{\partial z} = (-\gamma) \cdot E_p e^{-\gamma z} = (-\gamma) \cdot E_{zs}$$

$$\therefore \frac{\partial E_{zs}}{\partial z} = -\gamma E_{zs} \quad \text{—— (33)}$$

$$\frac{\partial^2 E_{zs}}{\partial z^2} = \frac{\partial}{\partial z} \cdot \frac{\partial}{\partial z}\left[E_p \cdot e^{-\gamma z}\right] = \frac{\partial}{\partial z} \cdot \left[-\gamma \cdot E_p e^{-\gamma z}\right]$$

$$= (-\gamma) \cdot (-\gamma) \cdot E_p e^{-\gamma z}$$

$$\frac{\partial^2 E_{zs}}{\partial z^2} = \gamma^2 E_{zs} \quad \text{—— (34)}$$

Similarly

$$\frac{\partial^2 E_{xs}}{\partial z^2} = \gamma^2 E_{xs} \quad \text{--- (35)}$$

So from eqn (32),

$$\Rightarrow j\omega\epsilon E_{xs} = \frac{\partial}{\partial y} H_{zs} + \frac{1}{j\omega\mu}\left[\frac{\partial^2}{\partial z^2} E_{xs} - \frac{\partial}{\partial x}\left(\frac{\partial E_{zs}}{\partial z}\right)\right]$$

Using eqn (33) & (35), we have

$$\Rightarrow j\omega\epsilon E_{xs} = \frac{\partial}{\partial y} H_{zs} + \frac{1}{j\omega\mu}\left[\gamma^2 E_{xs} - \frac{\partial}{\partial x}(-\gamma E_{zs})\right]$$

$$\Rightarrow j\omega\epsilon E_{xs} = \frac{\partial}{\partial y} H_{zs} + \frac{1}{j\omega\mu}\left[\gamma^2 E_{xs} + \gamma \frac{\partial E_{zs}}{\partial x}\right]$$

$$\Rightarrow j\omega\epsilon E_{xs} - \frac{\gamma^2 E_{xs}}{j\omega\mu} = \frac{\gamma}{j\omega\mu}\frac{\partial}{\partial x} E_{zs} + \frac{\partial}{\partial y} H_{zs}$$

$$\Rightarrow \frac{-1}{j\omega\mu}\left[\gamma^2 + \omega^2\mu\epsilon\right] E_{xs} = \frac{\gamma}{j\omega\mu}\frac{\partial}{\partial x} E_{zs} + \frac{\partial}{\partial y} H_{zs}$$

Let $h^2 = \gamma^2 + \omega^2\mu\epsilon = \gamma^2 + k^2$ --- 35(a) $\quad$ ($\because k = \omega\sqrt{\mu\epsilon}$ in eqn ① & ②)

Then above eqn becomes,

$$\Rightarrow \frac{-1}{j\omega\mu}(h^2) \cdot E_{xs} = \frac{\gamma}{j\omega\mu}\frac{\partial}{\partial x} E_{zs} + \frac{\partial}{\partial y} H_{zs}$$

$$\Rightarrow E_{xs} = \left(\frac{\cancel{j\omega\mu}}{-h^2}\right) \times \left(\frac{\gamma}{\cancel{j\omega\mu}}\right)\frac{\partial}{\partial x} E_{zs} - \frac{j\omega\mu}{h^2}\frac{\partial}{\partial y} H_{zs}$$

$$\Rightarrow \boxed{E_{xs} = -\frac{\gamma}{h^2}\frac{\partial E_{zs}}{\partial x} - \frac{j\omega\mu}{h^2}\frac{\partial}{\partial y} H_{zs}} \quad \text{--- (36)}$$

$\therefore$ E_{xs} is now expressed in terms of E_{zs} & H_{zs}.

Similar manipulation of eqⁿs 26 to 31, yield expression for E_{ys}, H_{xs} and H_{ys} in terms of E_{zs} & H_{zs}. Thus

$$\boxed{\begin{aligned} E_{xs} &= \frac{-\gamma}{h^2}\frac{\partial E_{zs}}{\partial x} - \frac{j\omega\mu}{h^2}\frac{\partial H_{zs}}{\partial y} && \text{— 37(a)} \\ E_{ys} &= \frac{-\gamma}{h^2}\frac{\partial E_{zs}}{\partial y} + \frac{j\omega\mu}{h^2}\frac{\partial H_{zs}}{\partial x} && \text{— 37(b)} \\ H_{xs} &= \frac{j\omega\epsilon}{h^2}\frac{\partial E_{zs}}{\partial y} - \frac{\gamma}{h^2}\frac{\partial H_{zs}}{\partial x} && \text{— 37(c)} \\ H_{ys} &= \frac{-j\omega\epsilon}{h^2}\frac{\partial E_{zs}}{\partial x} - \frac{\gamma}{h^2}\frac{\partial H_{zs}}{\partial y} && \text{— 37(d)} \end{aligned}}$$

37

where $h^2 = \gamma^2 + k^2 = \gamma^2 + k_x^2 + k_y^2 + k_z^2 = -\cancel{k_z^2} + k_x^2 + k_y^2 + \cancel{k_z^2}$

(From eqⁿ 13)

$\therefore$ $h^2 = \gamma^2 + k^2$ or $h^2 = k_x^2 + k_y^2$

From eqⁿ 22, 23 & 37, we notice that the field pattern or configuration comes in different types. Each of these distinct field patterns is called a mode. Four different mode categories can exist. Namely:

1) $E_{zs} = 0 = H_{zs}$ (TEM mode) :- In the Transverse Electromagnetic mode, both E & H fields are

transverse to the direction of wave propagation.

From eqn (37) [All the field Compents E_{xs}, E_{ys}, H_{xs}, H_{ys} vanishes because E_{zs} & $H_{zs} = 0$]. So, we conclude that a rectangular waveguide can n't support TEM mode. Imp BPUT exam

2) TE Mode ($E_{zs} = 0$, $H_{zs} \neq 0$)

For this case, the remaining components (E_{xs} & E_{ys}) of electric field are transverse to the direction of propagation a_z. Under this condition, fields are said to be in transverse electric (TE) mode.

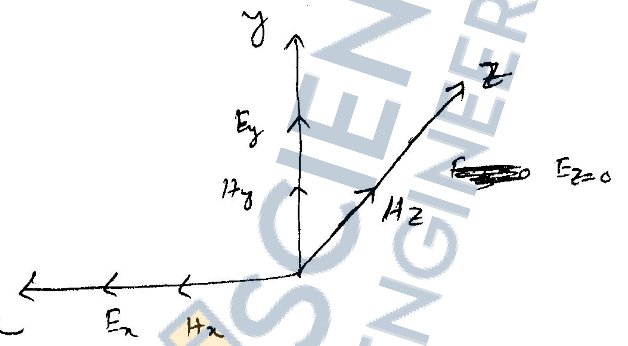


fig-23:- Component of EM fields on a rectangular waveguide: (a) TE mode ($E_z = 0$)

3) TM mode ($E_{zs} \neq 0$, $H_{zs} = 0$)

In this case, the H field is transverse to the direction of wave propagation. Thus we have transverse magnetic (TM) mode. [See fig 24]

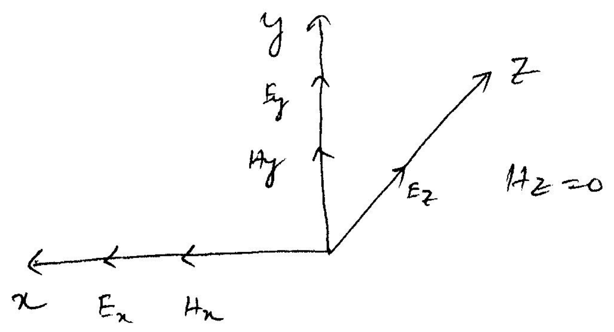


fig :- 24 :- Components of EM fields in a rectangular wave guide : (b) TM mode, $H_z = 0$

4(3) Hybrid mode: (HE mode) : ($E_{zs} \neq 0$, $H_{zs} \neq 0$)

In this case neither the E nor the H field is transverse to the direction of wave propagation. Sometimes these modes are referred to as hybrid modes.

Imp - BPUT

Transverse Magnetic (TM) Modes in Rectangular waveguide

For TM mode, the magnetic field has it's components transverse (or normal) to the direction of wave propagation.

$\therefore$ H_z (Component along Z direction) $= 0$.

At the walls (perfect conductor) of the waveguide, the tangential components of E field must be continuous.

i.e $E_{zs} = 0$ at $y = 0$ (bottom wall) — 38(a).

$E_{zs} = 0$ at $y = b$ (top wall) — 38(b)

$E_{zs} = 0$ at $x = 0$ (Left wall) — 38(c)

$E_{zs} = 0$ at $x = a$ (Right wall) — 38(d)

[Refer figure 22.]

From eqⁿ (22),

$$E_{zs} = (A_1 \cos k_x x + A_2 \sin k_x x)(A_3 \cos k_y y + A_4 \sin k_y y) \cdot e^{-\gamma z} \quad (22)$$

Putting the boundary condition, at $y=0$, $E_z = 0$

$$\Rightarrow 0 = (A_1 \cos k_x x + A_2 \sin k_x x)(A_3)$$

$$\Rightarrow \boxed{A_3 = 0} \quad \text{———— (39)}$$

At $x=0$, $E_z = 0$

$$0 = (A_1)(A_3 \cos k_y y + A_4 \sin k_y y)$$

$$\Rightarrow \boxed{A_1 = 0} \quad \text{———— (40)}$$

Putting eqⁿ (39) & (40) in eqⁿ (22)

$$E_{zs} = A_2 \sin k_x x \cdot A_4 \sin k_y y \cdot e^{-\gamma z}$$

$$E_{zs} = A_2 A_4 \sin k_x x \sin k_y y \; e^{-\gamma z}$$

$$E_{zs} = E_0 \sin k_x x \sin k_y y \; e^{-\gamma z} \quad \text{———— (41)}$$

where $E_0 = A_2 A_4$.

Again putting the boundary condⁿ, in eqⁿ (41)

At $y = b$, $E_z = 0$

$$0 = E_0 \sin k_x x \cdot \sin k_y b \; e^{-\gamma z}$$

$$\therefore \sin k_y b = 0$$

$$\Rightarrow k_y b = n\pi \quad , n = 1, 2, 3 \, ---$$

$$k_y = \frac{n\pi}{b} \quad , \quad n = 1, 2, 3 \; --- \qquad (42)$$

Similarly, At $x = a$, $E_z = 0$, using eqn (41)

$$0 = E_0 \cdot \sin k_x a \; \sin k_y b \cdot e^{-\gamma z}$$

$$\Rightarrow \sin k_x a = 0$$

$$\Rightarrow k_x a = m\pi \quad , \quad m = 1, 2, 3 \; ---$$

$$\Rightarrow k_x = \frac{m\pi}{a} \quad , \quad m = 1, 2, 3 \; --- \qquad (43)$$

Using Eqn (42) & (43), in eqn (41), we have

$$\boxed{E_{zs} = E_0 \sin\left(\frac{m\pi}{a}\right)x \; \sin\left(\frac{n\pi}{b}\right)y \; \cdot e^{-\gamma z}} \qquad (44)$$

where m & n denotes number of half cycles along x-axis & y-axis respectively.

The other field components can be obtained, using eqn (37)

$$E_{xs} = \frac{-\gamma}{h^2} \frac{\partial}{\partial x} E_{zs} \qquad \left[\because H_{zs} = 0 \text{ for TM mode} \right]$$

$$= \frac{-\gamma}{h^2} \cdot \frac{\partial}{\partial x} \left[E_0 \sin\left(\frac{m\pi}{a}\right)x \; \sin\left(\frac{n\pi}{b}\right)y \cdot e^{-\gamma z} \right]$$

$$= \left(\frac{-\gamma}{h^2} E_0 \sin\left(\frac{n\pi}{b}\right)y \cdot e^{-\gamma z} \right) \left(\cos\left(\frac{m\pi}{a}\right)x \right) \cdot \left(\frac{m\pi}{a}\right)$$

$$\boxed{E_{xs} = \frac{-\gamma}{h^2} \left(\frac{m\pi}{a}\right) E_0 \cos\left(\frac{m\pi}{a}\right)x \; \sin\left(\frac{n\pi}{b}\right)y \; \cdot e^{-\gamma z}} \qquad (45)$$

Similarly, Putting $H_{zs} = 0$ and taking the derivative we have,

$$E_{ys} = \frac{-\gamma}{h^2} \cdot \frac{\partial}{\partial y}\left[E_0 \sin\left(\frac{m\pi}{a}\right)x \sin\left(\frac{n\pi}{b}\right)y\, e^{-\gamma z}\right]$$

$$\boxed{E_{ys} = \frac{-\gamma}{h^2}\left(\frac{n\pi}{b}\right) E_0 \sin\left(\frac{m\pi}{a}\right)x \cdot \cos\left(\frac{n\pi}{b}\right)y\, e^{-\gamma z}} \quad \text{--- (46)}$$

$$H_{xs} = \frac{j\omega\epsilon}{h^2} \frac{\partial}{\partial y}\left[E_0 \sin\left(\frac{m\pi}{a}\right)x \sin\left(\frac{n\pi}{b}\right)y\, e^{-\gamma z}\right]$$

$$\boxed{H_{xs} = \frac{j\omega\epsilon}{h^2} \cdot \left(\frac{n\pi}{b}\right) \cdot E_0 \sin\left(\frac{m\pi}{a}\right)x \cdot \cos\left(\frac{n\pi}{b}\right)y\, e^{-\gamma z}} \quad \text{--- (47)}$$

$$H_{ys} = \frac{-j\omega\epsilon}{h^2} \frac{\partial}{\partial x} \cdot \left[E_0 \sin\left(\frac{m\pi}{a}\right)x \sin\left(\frac{n\pi}{b}\right)y \cdot e^{-\gamma z}\right]$$

$$\Rightarrow \boxed{H_{ys} = \frac{-j\omega\epsilon}{h^2} \left(\frac{m\pi}{a}\right) \cdot E_0 \cos\left(\frac{m\pi}{a}\right)x \sin\left(\frac{n\pi}{b}\right)y \cdot e^{-\gamma z}} \quad \text{--- (48)}$$

Where $h^2 = K_x^2 + K_y^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$ --- (49)

From eqn 35 (a), $h^2 = \gamma^2 + k^2$

$$\Rightarrow \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 = \gamma^2 + k^2$$

$$\Rightarrow \gamma = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2} \quad \text{--- (50)}$$

Note :-

1) For each set of integers m & n gives a different field pattern or mode, referred to as TM_{mn} mode, in the waveguide.

2) Integer 'm' equals to number of half-cycle variation in the x-direction and integer 'n' is the number of half-cycle variation in the y-direction.

Case-I (Cutoff) $\left[k^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \right]$

Since waveguide is a HPF we have to determine cutoff freq (f_c), it is the minm freq after which propagation occurs inside waveguide.

From eqn (50), we have

$$\gamma = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}$$

At $f = f_c$, $\gamma = 0$ [No propagation at this freq, more than f_c propagation takes place]

$[\text{i.e. } \alpha = 0, \beta = 0]$

Since $\gamma = 0$

$$\Rightarrow \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 = k^2$$

$$\Rightarrow \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 = \omega_c^2 \mu \epsilon \qquad \left(\because k = \omega\sqrt{\mu\epsilon}\right)$$

$$\Rightarrow \omega_c^2 = \frac{1}{\mu\epsilon}\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]$$

$$\Rightarrow \boxed{\omega_c = \frac{1}{\sqrt{\mu\epsilon}}\sqrt{\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]}} \quad \text{--- (51)}$$

$$\Rightarrow 2\pi f_c = \frac{1}{\sqrt{\mu\epsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

$$\Rightarrow f_c = \frac{1}{\sqrt{\mu\epsilon}} \cdot \frac{1}{2\cancel{\pi}} \cdot \cancel{\pi} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

$$\Rightarrow f_c = \left(\frac{\frac{1}{\sqrt{\mu\epsilon}}}{2}\right) \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

$$\Rightarrow \boxed{f_c = \frac{u'}{2} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}} \quad \text{--- (52)}$$

where $u' = \frac{1}{\sqrt{\mu\epsilon}}$ = Phase velocity of EM wave in lossless dielectric medium $(\sigma = 0, \mu, \epsilon)$ in absence of [illegible] wave guide.

Imp

$$\therefore \boxed{f_c = \frac{1}{2\sqrt{\mu\epsilon}} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}} \quad \text{--- (53)}$$

~~The~~ $\therefore$ The cutoff freq is the operating freq below which attenuation occures and above which propagation takes place.

Cutoff Wavelength (λ_c)

$\lambda_c = \frac{u'}{f_c}$, From eq^n (52),

$$\boxed{\lambda_c = \frac{2}{\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}}} \quad \text{--- (54)}$$

Case-2 (Evanescent)

If $k^2 < \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$

i.e $\omega^2\mu\epsilon < \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$

$$\boxed{\gamma = \alpha}, \ \beta = 0$$

→ No wave propagation at all. < Attenuation occurs >

→ It is called Evanescent or attenuating mode or non-propagation mode due to $\gamma = \alpha$ (attenuation const).

∵ $\gamma = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}$

∴ Since $k^2 < \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$

γ = +ve real number

No complex part

$\gamma = \alpha + j.(0)$

∴ $\beta = 0$

Case-3 (Propagation)

$$k^2 > \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$

$\gamma = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}$

∴ $\gamma = \sqrt{-ve}$

$\gamma = j\beta$

$$\boxed{\gamma = j\beta} \quad (\alpha = 0, \text{ loss less medium})$$

∴ $\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2} = j\beta$

$\Rightarrow \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2 = -\beta^2$

$\Rightarrow \beta^2 = k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2$

$$\Rightarrow \boxed{\beta = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}} \quad \text{--- (55)}$$

This is the only case in which propagation takes place because all field components will have the factor $e^{-\gamma z} = e^{-j\beta z}$.

$$\beta = \sqrt{K^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

$$= \sqrt{K^2 \left(1 - \frac{1}{K^2}\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]\right)}$$

$$= K \sqrt{1 - \frac{1}{\omega^2 \mu \epsilon}\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]}$$

$$= K \sqrt{1 - \frac{1}{\omega^2} \cdot \omega_c^2} \quad \left[\because \omega_c = \frac{1}{\sqrt{\mu\epsilon}} \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \right.$$

$$\beta = \omega \sqrt{\mu\epsilon} \sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2} \quad \left. eq^n (51) \right]$$

gmr

$$\boxed{\beta = \beta' \sqrt{1 - \left(\frac{f_c}{f}\right)^2}} \quad (56)$$

$$\beta' = \omega\sqrt{\mu\epsilon} = 2\pi f' \cdot \frac{1}{U'} = \frac{2\pi}{\left(\frac{U'}{f'}\right)} = \frac{2\pi}{\lambda'}$$

Where

β' = Phase constant of EM wave in dielectric medium in absence of wave guide.

$U', \beta' \rightarrow$ Absence of wave guide

$U, \beta \rightarrow$ Presence of wave guide.

$\beta \rightarrow$ Phase Const. in Presence of wave guide.

$f_c \rightarrow$ Cutoff freq

$f \rightarrow$ operating freq.

Intrinsic Wave Impedance (η_{TM}) [The impedance offered by wave guide in TE/TM mode]

$$\eta_{TM} = \frac{E_x}{H_y} = \frac{-E_y}{H_x} \quad (57)$$

(See eq^n 45, 46, 47, 48)

From eqⁿ (45) & (48)

$$\eta_{TM} = \frac{E_{xs}}{H_{ys}} = \frac{\gamma}{j\omega\epsilon} = \frac{j\beta}{j\omega\epsilon} = \frac{\beta'\sqrt{1-\left(\frac{f_c}{f}\right)^2}}{\omega\epsilon}$$

$$= \frac{\cancel{\omega}\sqrt{\mu\epsilon}}{\cancel{\omega}\epsilon}\sqrt{1-\left(\frac{f_c}{f}\right)^2}$$

$$\eta_{TM} = \sqrt{\frac{\mu}{\epsilon}}\sqrt{1-\left(\frac{f_c}{f}\right)^2}$$

Imp.

$$\boxed{\eta_{TM} = \eta'\sqrt{1-\left(\frac{f_c}{f}\right)^2}} \quad \text{—— (58)}$$

where $\eta' = \sqrt{\frac{\mu}{\epsilon}}$ = intrinsic wave impedance in dielectric medium without waveguide

(Phase) Velocity inside waveguide :- (U_p or u)

$$U_p/u = \frac{\omega}{\beta} = \frac{\omega}{\beta'\sqrt{1-\left(\frac{f_c}{f}\right)^2}}$$

$\frac{\omega}{\beta} = \frac{2\pi f}{\frac{2\pi}{\lambda}} = \frac{2\pi f \times \lambda}{2\pi} = f\lambda = u$

from eqⁿ (56)

$$U_p/u = \frac{\cancel{\omega}}{\cancel{\omega}\sqrt{\mu\epsilon}\sqrt{1-\left(\frac{f_c}{f}\right)^2}}$$

$$U_p/u = \frac{\frac{1}{\sqrt{\mu\epsilon}}}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}}$$

Imp

$$\Rightarrow \boxed{U_p/u = \frac{u'}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}}} \quad \text{—— (59)}$$

where $u' = \frac{1}{\sqrt{\mu\epsilon}}$ = velocity in dielectric medium in absence of waveguide

If it is free space
$u' = c$

$$u = \frac{c}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}}$$

f_c is Cutoff freq.

f " operating freq.

$f > f_c$.

$\Rightarrow f_c < f$

$\Rightarrow \frac{f_c}{f} < 1$

$\Rightarrow \left(\frac{f_c}{f}\right)^2 < 1$

$\therefore$ The denominator is a fraction.

$\boxed{u > u'}$

$\therefore$ Velocity of wave inside waveguide > Velocity of light.

Imp wavelength in the guide

$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\beta' \sqrt{1-\left(\frac{f_c}{f}\right)^2}}$ [from eqⁿ 56]

$\lambda = \frac{\left(\frac{2\pi}{\beta'}\right)}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}}$

Imp $\boxed{\lambda = \frac{\lambda'}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}}}$ — 60

$\lambda > \lambda'$

$\because \lambda'$ = wavelength in dielectric medᵐ in absence of waveguide.

Dominant mode :-

From eqⁿ 44, 45, 46, 47, 48,

If $m = 0$, $n = 0$, All field component vanish.

$m = 0$, $n = n$, " " " [Either $\sin\left(\frac{m\pi}{a}\right)x = 0$ or ~~is Const.~~ Constant $\left(\frac{m\pi}{a}\right)$ i.e multiplied makes them zero]

$m = m$, $n = 0$, All field component vanish. [$\sin\left(\frac{n\pi}{b}\right)y = 0$ or $\left(\frac{n\pi}{b}\right) = 0$]

$\therefore$

~~TM_{00}, TM_{mo}, TM_{om} i.e TM_{00}, T~~

i.e TM_{00}, TM_{mo}, TM_{on} e.g TM_{00} TM_{10} TM_{01} --- etc don't exist.

Lowest value of m, n for which TM_{mn} exist ~~for~~ is $m = 1, n = 1$

∴ The mode in which lowest cutoff freq (or largest cutoff wavelength) occurs is known as dominant mode.

∴ $\boxed{TM_{11}}$ is the dominant mode for TM.

So in dominant mode

$$f_c = \frac{u'}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} \quad (61)$$

Imp

$$\boxed{f_c = \frac{u'}{2}\sqrt{\frac{1}{a^2} + \frac{1}{b^2}} \quad (62) \qquad \lambda_c = \frac{2}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}} \quad (63)}$$

where $u' = \frac{1}{\sqrt{\mu\epsilon}}$

Degenerate modes:-

→ Whenever two or more modes have the same cutoff freq, they are said to be degenerate modes.

→ In rectangular guide the corresponding TE_{mn} and TM_{mn} modes are always degenerate.

e.g. TE_{11} and TM_{11}, → both have same cutoff freq.

$$f_c = \frac{u'}{2}\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}$$

Similarly $\{TE_{21}, TM_{21}\}$, $\{TE_{12} + TM_{12}\}$

Q) Prove that

$$\frac{1}{\lambda_g^2} = \frac{1}{\lambda_0^2} - \frac{1}{\lambda_c^2}$$

or $$\frac{1}{\lambda^2} = \frac{1}{\lambda_0^2} - \frac{1}{\lambda_c^2}$$ or $$\frac{1}{\lambda^2} = \frac{1}{\lambda'^2} - \frac{1}{\lambda_c^2}$$

$(\lambda' = \lambda_0)$

$\lambda_g / \lambda \rightarrow$ wavelength in the guide

$\lambda' / \lambda_0 \rightarrow$ " " free space / dielectric medium.

$\lambda_c \rightarrow$ cutoff " wavelength.

Ans:

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}}$$ [using eqn 55]

$$\lambda = \frac{2\pi}{\sqrt{\omega^2\mu\epsilon - \omega_c^2\mu\epsilon}}$$

$\because$ $k^2 = \omega^2\mu\epsilon$ and $\omega_c = \frac{1}{\sqrt{\mu\epsilon}}\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]$ [from eqn (51)]

$$\lambda = \frac{2\pi}{\sqrt{\mu\epsilon}\sqrt{\omega^2 - \omega_c^2}}$$

$$\lambda = c \times \frac{\cancel{2\pi}}{\cancel{2\pi}\sqrt{f^2 - f_c^2}}$$

$$\lambda = \frac{c}{f\sqrt{1 - \left(\frac{f_c}{f}\right)^2}}$$

$$\Rightarrow \lambda = \frac{\lambda_0}{\sqrt{1 - \left(\frac{f_c}{f}\right)^2}}$$ [Same as derived in eqn (60)]

$$\Rightarrow \lambda = \frac{\lambda_0}{\sqrt{1 - \left(\frac{c/\lambda_c}{c/\lambda_0}\right)^2}} = \frac{\lambda_0}{\sqrt{1 - \left(\frac{\lambda_0}{\lambda_c}\right)^2}}$$

Squaring both the sides,

$$\lambda^2 = \frac{\lambda_0^2}{\left[1 - \left(\frac{\lambda_0}{\lambda_c}\right)^2\right]}$$

$$\Rightarrow \lambda^2 - \lambda^2 \cdot \left(\frac{\lambda_0}{\lambda_c}\right)^2 = \lambda_0^2$$

$$\Rightarrow \lambda^2 = \lambda_0^2 \left[1 + \frac{\lambda^2}{\lambda_c^2}\right]$$

Dividing both the sides by $\lambda^2 \cdot \lambda_c^2 \cdot \lambda_0^2$

$$\Rightarrow \frac{\cancel{\lambda^2}}{\cancel{\lambda^2}\,\cancel{\lambda_c^2}\,\lambda_0^2} = \frac{\cancel{\lambda_0^2}}{\lambda^2 \cdot \cancel{\lambda_c^2}\,\cancel{\lambda_0^2}} \left[1 + \frac{\lambda^2}{\lambda_c^2}\right]$$

$$\Rightarrow \frac{1}{\lambda_0^2} = \frac{1}{\lambda^2}\left[1 + \frac{\lambda^2}{\lambda_c^2}\right] = \frac{1}{\lambda^2} + \frac{1}{\lambda_c^2}$$

$$\Rightarrow \boxed{\frac{1}{\lambda^2} = \frac{1}{\lambda_0^2} - \frac{1}{\lambda_c^2}} \quad \text{——} (64)$$

(Proved)

Imp- RPUT

Transverse Electric mode (TE mode) in Rectangular wave guide

In TE mode, the electric field is transverse (or normal) to the direction of wave propagation

For TE mode, $E_{zs} = 0$

So other field components E_x, H_x, E_y, H_y and H_z

Can be determined using eqn (23) & eqn (37) and putting the boundary conditions.

From eqn (23),

$$H_{zs}(x,y,z) = (B_1 \cos k_x x + B_2 \sin k_x x)(B_3 \cos k_y y + B_4 \sin k_y y) \cdot e^{-\gamma z} \quad \text{--- (65)}$$

From eqn 37(a)

$$E_{xs} = \frac{-\gamma}{h^2} \frac{\partial}{\partial x} E_{zs} - \frac{j\omega\mu}{h^2} \frac{\partial}{\partial y} H_{zs}$$

For TE mode, $E_{zs} = 0$,

$$\Rightarrow \quad E_{xs} = -\frac{j\omega\mu}{h^2} \frac{\partial}{\partial y} H_{zs} \quad \text{--- (66)}$$

The boundary conditions are obtained from the requirement that tangential components of the electric field be continuous at the walls (perfect conductors) of the waveguides;

$$\left.\begin{array}{lll} E_{xs} = 0, & \text{at } y = 0 & \text{(bottom wall)} \quad (a) \\ E_{xs} = 0, & \text{at } y = b & \text{(top wall)} \quad (b) \\ E_{ys} = 0, & \text{at } x = 0 & \text{(Left wall)} \quad (c) \\ E_{ys} = 0, & \text{at } x = a & \text{(Right wall)} \quad (d) \end{array}\right\} \text{--- (67)}$$

At $y = 0$, $E_{xs} = 0$, putting this condn in eqn (66), we have

$$\frac{\partial}{\partial y} H_{zs} = 0$$

$$\therefore \frac{\partial}{\partial y} H_{zs} = 0, \quad \text{at } y = 0 \quad \text{--- (68)}$$

From eqⁿ (65), we have $\frac{\partial}{\partial y} H_{zs}$

$$= \frac{\partial}{\partial y}\left[(B_1 \cos k_x x + B_2 \sin k_x x)(B_3 \cos k_y y + B_4 \sin k_y y)\, e^{-\gamma z}\right]$$

$$= \left[B_1 \cos k_x x + B_2 \sin k_x x\right]\left[B_3 (-\sin k_y y)\cdot k_y + B_4 (\cos k_y y)\cdot k_y\right] e^{-\gamma z}$$

At $y = 0$, $\frac{\partial}{\partial y} H_{zs} = 0$

$$\Rightarrow \left[B_1 \cos k_x x + B_2 \sin k_x x\right]\left[0 + B_4 k_y\right] e^{-\gamma z} = 0$$

$$\Rightarrow \boxed{B_4 = 0} \qquad (69)$$

[Note:- $k_y \neq 0$, otherwise H_{zs} will be always zero]

Similarly $\frac{\partial}{\partial y} H_{zs} = 0$, for $y = b$

$\Rightarrow$ At $y = b$, $\frac{\partial}{\partial y} H_{zs} = 0$

$$\Rightarrow \left[B_1 \cos k_x x + B_2 \sin k_x x\right]\left[-B_3 \sin k_y b + \cancel{B_4} (\cos k_y b)\, k_y\right] e^{-\gamma z} = 0$$

$B_4 = 0 \rightarrow$ eqn (69)

(Note:- $B_3 \neq 0$, if B_3 is zero & $B_4 = 0$ H_{zs} will be always zero)

$$\Rightarrow \sin k_y b = 0$$

$$\Rightarrow k_y b = n\pi$$

$$\Rightarrow \boxed{k_y = \frac{n\pi}{b}} \qquad n = 0, 1, 2, 3 \text{ ----} \qquad (70)$$

Similarly From eqⁿ 37 (b)

$$E_{ys} = \frac{j\omega\mu}{h^2} \frac{\partial}{\partial x} H_{zs} \quad \text{for TE mode, As } E_z = 0$$

and

$E_{ys} = 0$ for $x = 0$ & a [From eqⁿ 67]

$$\Rightarrow \left.\frac{\partial}{\partial x} H_{zs}\right]_{x=0\ \&\ a} = 0$$

$\therefore \frac{\partial}{\partial x}[H_{zs}]$

$$= \frac{\partial}{\partial x}\left[(B_1 \cos k_x x + B_2 \sin k_x x)(B_3 \cos k_y y + B_4 \sin k_y y)\, e^{-\gamma z}\right]$$

$$= (B_3 \cos k_y y + B_4 \sin k_y y) \cdot e^{-\gamma z} \left[-B_1 (\sin k_x x) \cdot k_x + B_2 (\cos k_x x) \cdot k_x\right]$$

At $x = 0$, $\frac{\partial}{\partial x} H_{zs} = 0$

$$\Rightarrow (B_3 \cos k_y y + B_4 \sin k_y y)(e^{-\gamma z})(B_2 k_x) = 0$$

$$\Rightarrow \boxed{B_2 = 0} \quad \text{—— (71)}$$

(Note: - k_x can't be zero, otherwise H_{zs} will be always zero)

At $x = a$, $\frac{\partial}{\partial x} H_{zs} = 0$

$$\Rightarrow \frac{\partial}{\partial x}\left[(B_1 \cos k_x x + \cancel{B_2 \sin k_x x})(B_3 \cos k_y y + B_4 \sin k_y y)\, e^{-\gamma z}\right] = 0 \qquad B_2 = 0$$

$$\Rightarrow B_1 \cdot -(\sin k_x a) \cdot k_x (B_3 \cos k_y y + B_4 \sin k_y y) \cdot e^{-\gamma z} = 0.$$

$$\Rightarrow \sin k_x a = 0$$

$$\Rightarrow k_x a = m\pi, \quad m = 0, 1, 2, 3 \text{ ---}$$

$$\Rightarrow k_x = \frac{m\pi}{a} \quad \text{—— (72)}$$

Combining eqn $\boxed{69, 70, 71, 72}$, we have

$B_4 = 0$, $B_2 = 0$, $K_x = \frac{m\pi}{a}$, $K_y = \frac{n\pi}{b}$

$$H_{zs} = (B_1 \cos k_x x + B_2 \sin k_x x)(B_3 \cos k_y y + B_4 \sin k_y y)\, e^{-\gamma z}$$

Putting the above conditions, we have

$$H_{zs} = \left(B_1 \cos \frac{m\pi}{a} x\right)\left(B_3 \cos\left(\frac{n\pi}{b}\right) y\right) e^{-\gamma z}$$

$$H_{zs} = B_1 B_3 \cos\left(\frac{m\pi}{a}\right)x \cos\left(\frac{n\pi}{b}\right)y \; e^{-\gamma z}$$

$$\boxed{H_{zs} = H_0 \cos\left(\frac{m\pi}{a}\right)x \cos\left(\frac{n\pi}{b}\right)y \; e^{-\gamma z}} \quad \text{--- (73)}$$

where $H_0 = B_1 B_3$, and m & n denote number of half cycles along x & y-axis respectively.

Other components can be found out using eqn (37)

$$E_{xs} = \frac{-j\omega\mu}{h^2} \frac{\partial H_{zs}}{\partial y} \quad \left[\text{From } 37(a)\right] \left[\because E_{zs} = 0 \text{ for TE mode}\right]$$

$$= \frac{-j\omega\mu}{h^2} \frac{\partial}{\partial y}\left[H_0 \cos\left(\frac{m\pi}{a}\right)x \cos\left(\frac{n\pi}{b}\right)y \cdot e^{-\gamma z}\right]$$

$$= \frac{-j\omega\mu}{h^2} H_0 \cos\left(\frac{m\pi}{a}\right)x \cdot e^{-\gamma z}\left[-\sin\left(\frac{n\pi}{b}\right)y \cdot \frac{n\pi}{b}\right]$$

$$\therefore \boxed{E_{xs} = \frac{j\omega\mu}{h^2}\left(\frac{n\pi}{b}\right) H_0 \cos\left(\frac{m\pi}{a}\right)x \sin\left(\frac{n\pi}{b}\right)y \; e^{-\gamma z}} \quad \text{--- (74)}$$

From 37 (b),

$$E_{ys} = \frac{j\omega\mu}{h^2} \frac{\partial H_{zs}}{\partial x} \quad \left[\because E_{zs} = 0\right]$$

$$\Rightarrow E_{ys} = \frac{j\omega\mu}{h^2} \frac{\partial}{\partial x}\left[H_o \cos\left(\frac{m\pi}{a}\right)x \cos\left(\frac{n\pi}{b}\right)y \, e^{-\gamma z}\right]$$

$$\Rightarrow E_{ys} = \frac{j\omega\mu}{h^2} \cdot H_o \cdot \cos\left(\frac{n\pi}{b}\right)y \cdot e^{-\gamma z} \cdot \left[-\sin\left(\frac{m\pi}{a}\right)x \cdot \frac{m\pi}{a}\right]$$

$$\Rightarrow \boxed{E_{ys} = \frac{-j\omega\mu}{h^2} \cdot \left(\frac{m\pi}{a}\right) \cdot H_o \cdot \sin\left(\frac{m\pi}{a}\right)x \cos\left(\frac{n\pi}{b}\right)y \cdot e^{-\gamma z}} \quad (75)$$

From eqn 37 (c)

$$H_{xs} = -\frac{\gamma}{h^2} \frac{\partial H_{zs}}{\partial x} \qquad \left[\because E_{zs} = 0\right]$$

$$= \frac{-\gamma}{h^2} \frac{\partial}{\partial x}\left[H_o \cos\left(\frac{m\pi}{a}\right)x \cos\left(\frac{n\pi}{b}\right)y \, e^{-\gamma z}\right]$$

$$= \frac{-\gamma}{h^2} \cdot H_o \cdot \cos\left(\frac{n\pi}{b}\right)y \cdot e^{-\gamma z} \cdot \frac{\partial}{\partial x}\left[\cos\left(\frac{m\pi}{a}\right)x\right]$$

$$H_{xs} = -\frac{\gamma}{h^2} H_o \cdot \cos\left(\frac{n\pi}{b}\right)y \cdot e^{-\gamma z}\left[-\sin\left(\frac{m\pi}{a}\right)x \cdot \frac{m\pi}{a}\right].$$

$$\Rightarrow \boxed{H_{xs} = \frac{\gamma}{h^2}\left(\frac{m\pi}{a}\right) \cdot H_o \sin\left(\frac{m\pi}{a}\right)x \cos\left(\frac{n\pi}{b}\right)y \, e^{-\gamma z}} \quad (76)$$

From eqn 37 (d),

$$H_{ys} = -\frac{\gamma}{h^2} \frac{\partial}{\partial y} H_{zs} \qquad \left[\because E_{zs} = 0\right]$$

$$\Rightarrow H_{ys} = -\frac{\gamma}{h^2} \frac{\partial}{\partial y}\left[H_o \cos\left(\frac{m\pi}{a}\right)x \cos\left(\frac{n\pi}{b}\right)y \, e^{-\gamma z}\right]$$

$$\Rightarrow \boxed{H_{ys} = \frac{\gamma}{h^2}\left(\frac{n\pi}{b}\right) . H_0 \cos\left(\frac{m\pi}{a}\right)x . \sin\left(\frac{n\pi}{b}\right)y \, e^{-\gamma z}} \quad (77)$$

where $h^2 = K_x^2 + K_y^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$ [As defined for TM mode.]

and $\gamma = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - K^2}$

The value of f_c, λ_c, β is same as that for TM mode, i.e

Imp

$$f_c = \frac{1}{2\sqrt{\mu\epsilon}} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} \quad (78)$$

or

$$f_c = \frac{u'}{2} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} \quad (79)$$

Imp

$$\lambda_c = \frac{2}{\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}} \quad (80)$$

Imp

$$\beta = \beta' \sqrt{1 - \left(\frac{f_c}{f}\right)^2} \quad (81)$$

where $\beta' = \omega\sqrt{\mu\epsilon}$

But value of intrinsic wave impedance of TE mode differs from that of TM mode.

$$\eta_{TE} = \frac{E_x}{H_y} = \frac{j\omega\mu}{\gamma} = \frac{-E_y}{H_x} \qquad \left[\begin{array}{l} \text{eqn } (74) \ \& \ (77) \\ \text{eqn } (75) \ \& \ (76) \end{array}\right]$$

$$\eta_{TE} = \frac{j\omega\mu}{\gamma} = \frac{\cancel{j}\omega\mu}{\cancel{j}\beta} \qquad \left[\because \ \gamma = j\beta \text{ for Propagation mode } \alpha = 0\right]$$

$$\eta_{TE} = \frac{\omega\mu}{\beta} = \frac{\omega\mu}{\beta' \sqrt{1-\left(\frac{f_c}{f}\right)^2}} = \frac{\omega\mu}{\cancel{\omega}\sqrt{\mu\epsilon}\sqrt{1-\left(\frac{f_c}{f}\right)^2}}$$

$$\eta_{TE} = \sqrt{\frac{\mu}{\epsilon}} \left[\frac{1}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}}\right]$$

Imp

$$\boxed{\eta_{TE} = \eta' \left[\frac{1}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}}\right]} \quad \text{——— (82)}$$

η_{TE} = In Presence of waveguide.

η' = In Absence of waveguide.

From eqn (58) & (82)

Imp

$$\boxed{\eta_{TE} \cdot \eta_{TM} = (\eta')^2} \quad \text{——— (83)}$$

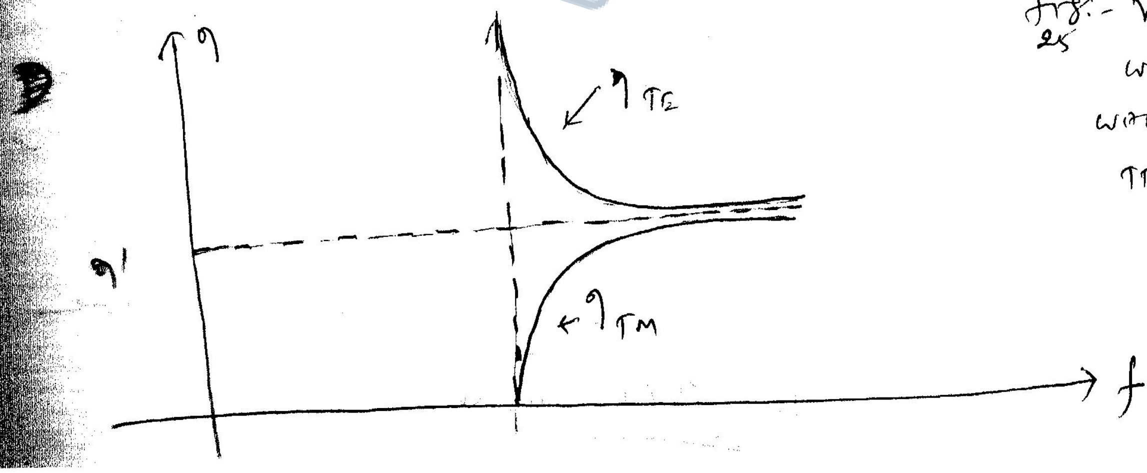


fig. 25 - Variation of wave impedance with freq for TE & TM mode.

Dominant Mode :-

The dominant mode is the mode with lowest cutoff frequency (or longest cutoff wavelength),

For TE mode, (m, n) may be (0,1) or (1,0) but not (0,0). Both m & n cannot be zero at the same time because this will force the field components to vanish.

For $m=1, n=0$

$$f_{c_{10}} = \frac{u'}{2}\sqrt{\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2} = \frac{u'}{2}\times\frac{1}{a} = \frac{u'}{2a}$$

For $m=0, n=1$

$$f_{c_{01}} = \frac{u'}{2b}$$

Generally for rectangular wave guide $a > b$,

$$\frac{u'}{2a} < \frac{u'}{2b}$$

$$\Rightarrow f_{c_{10}} < f_{c_{01}}$$

$\therefore$ $f_{c_{10}}$ is the lowest cut off freq.

$\therefore$ $\boxed{TE_{10}}$ is the dominant mode for $\boxed{TE}$ mode.

We have defined, $\boxed{TM_{11}}$ is dominant mode for $\boxed{TM}$ mode.

Overall for a rectangular waveguide

$$(f_c)_{TE_{10}} = \frac{u'}{2a}.$$

$$(f_c)_{TM_{11}} = \frac{u'}{2}\sqrt{\frac{1}{a^2}+\frac{1}{b^2}} = \frac{u'}{2}\sqrt{\frac{a^2+b^2}{a^2b^2}}$$

$$= \frac{u'}{2a}\cdot\sqrt{\frac{a^2+b^2}{b^2}}$$

$$= \frac{u'}{2a}\sqrt{1+\frac{a^2}{b^2}}$$

$$(f_c)_{TM_{11}} > (f_c)_{TE_{10}}$$

$\therefore$ $(f_c)_{TE_{10}}$ is lowest.

$\therefore$ For a rectangular waveguide $\boxed{TE_{10}}$ is the dominant mode.

Note :- For TE mode phase velocity & wave length is also same as that of TM mode.

Imp

$$\boxed{U_p = \frac{u'}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}}} \quad \text{(84)}$$

where $u' = \frac{1}{\sqrt{\mu\epsilon}}$

Imp

$$\boxed{\lambda = \frac{\lambda'}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}}} \quad \text{(85)}$$

$\lambda' = \frac{2\pi}{\beta'}$ = wave length, in absence of waveguide.

Cutoff freq & wavelength at dominant mode

At TE_{10} mode

$m=1, n=0.$

$$\boxed{f_c = \frac{u'}{2a}} \quad \text{---} \quad (86)$$

$$\lambda_c = \frac{2}{\sqrt{\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2}} = \frac{2}{\sqrt{\frac{1}{a^2}}} = 2a$$

$$\boxed{\lambda_c = 2a} \quad \text{---} \quad (87)$$

Phase velocity / Vs Group velocity :-

Phase velocity (U_p) is the velocity at which loci of constant phase are propagated down the guide.

$$u_p = \frac{u'}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}} \quad \text{[As derived earlier]}$$

Group velocity (U_g) is the velocity with which the resultant repeated reflected waves are travelling down the guide. It is the energy propagation velocity in the guide and is given by

$$u_g = \frac{\partial \omega}{\partial \beta}$$

$$\beta = \beta' \sqrt{1-\left(\frac{f_c}{f}\right)^2} = \omega\sqrt{\mu\epsilon}\sqrt{1-\left(\frac{f_c}{f}\right)^2}$$

$$\therefore \beta = \omega\sqrt{\mu\epsilon}\sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2} = \cancel{\omega}\sqrt{\mu\epsilon}\frac{\sqrt{\omega^2-\omega_c^2}}{\cancel{\omega}}$$

$$\Rightarrow \cancel{\frac{\partial \omega}{\partial \beta} = \frac{1}{\sqrt{\mu\epsilon}} \cdot \frac{1}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}}}$$

$$\cancel{\Rightarrow \frac{\partial \omega}{\partial \beta} = \frac{u'}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}}} \quad \left(\because u' = \frac{1}{\sqrt{\mu\epsilon}} \right)$$

$$\cancel{\Rightarrow \boxed{u_g = \frac{u'}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}}}} \text{ --- } (88)$$

$$\cancel{u_p \cdot u_g = \frac{u'}{\left(\sqrt{1-\left(\frac{f_c}{f}\right)^2}\right)} \times \frac{u'}{}}$$

$$\because \beta = \sqrt{\mu\epsilon}\,\sqrt{\omega^2-\omega_c^2} \qquad \because \frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$$

$$\Rightarrow \frac{\partial \beta}{\partial \omega} = \sqrt{\mu\epsilon}\,\frac{\cancel{2}\omega}{\cancel{2} \times \sqrt{\omega^2-\omega_c^2}}$$

$$= \sqrt{\mu\epsilon}\,\frac{\cancel{\omega}}{\cancel{\omega}\sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2}}$$

$$= \sqrt{\mu\epsilon} \cdot \frac{1}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}}$$

$$\Rightarrow \frac{\partial \omega}{\partial \beta} = \frac{1}{\sqrt{\mu\epsilon}} \cdot \sqrt{1-\left(\frac{f_c}{f}\right)^2}$$

$$\Rightarrow \frac{\partial \omega}{\partial \beta} = u'\sqrt{1-\left(\frac{f_c}{f}\right)^2} \quad \left(\because u' = \frac{1}{\sqrt{\mu\epsilon}},\ u' = c \text{ for freespace} \right)$$

$$\Rightarrow \boxed{u_g = u'\sqrt{1-\left(\frac{f_c}{f}\right)^2}} \text{ --- } (88)$$

$$\therefore \boxed{\begin{array}{l} U_p \cdot U_g = (u')^2 \quad \text{—— (89)} \\ U_p \cdot U_g = c^2 \quad \text{for free space} \end{array}}$$

Power transmission inside a waveguide :-

To determine power flow on the waveguide, we first find the average Poynting vector

$$P_{avg} = \frac{1}{2} \, Re \left(E_s \times H_s^* \right) \quad \text{—— (90)}$$

In this case, the Poynting vector is along the Z-direction, so

$P_{avg} \big|_{Z\text{-direction}}$ can be found as follows.

$$P_{avg} = \frac{1}{2} \, Re \begin{vmatrix} a_x & a_y & a_z \\ E_{xs} & E_{ys} & E_{zs} \\ H_{xs}^* & H_{ys}^* & H_{zs}^* \end{vmatrix}$$

$$P_{avg} \big|_{Z\text{-direction}} = \frac{1}{2} \, Re \left[E_{xs} H_{ys}^* - E_{ys} H_{xs}^* \right] \hat{a}_z$$

$$= \frac{1}{2} \, Re \left[E_{xs} \cdot \frac{E_{xs}^*}{\eta^*} - E_{ys} \cdot \left(-\frac{E_{ys}^*}{\eta^*} \right) \right] \hat{a}_z$$

$$\left(\because \frac{E_x}{H_y} = \eta = -\frac{E_y}{H_x} \; ; \text{ or } \; \frac{E_x^*}{H_y^*} = \eta^* = -\frac{E_y^*}{H_x^*} \right)$$

$$\Rightarrow \quad P_{avg}\Big|_{z\text{-direction}} = \frac{1}{2} Re\left[\frac{|E_{xs}|^2}{\eta^*} + \frac{|E_{ys}|^2}{\eta^*} \right] a_z$$

91

$$\Rightarrow \quad P_{avg}\Big|_{z\text{-direction}} = \frac{|E_{xs}|^2 + |E_{ys}|^2}{2\eta} a_z \left(\because \eta^* = \eta = real \right)$$

Total avg power transmitted across the cross section of the waveguide ds

$$P_{avg} = \int P_{avg} \cdot ds$$

$$\boxed{P_{avg} = \int_{x=0}^{a} \int_{y=0}^{b} \left\{ \frac{|E_{xs}|^2 + |E_{ys}|^2}{2\eta} \right\} dy\, dx \quad \text{watt}}$$

For TE mode

$$\boxed{P_{avg} = \int_{x=0}^{a} \int_{y=0}^{b} \left(\frac{|E_{xs}|^2 + |E_{ys}|^2}{2\eta_{TE}} \right) dy\, dx \quad \text{watt}}$$

where $\eta_{TE} = \dfrac{\eta'}{\sqrt{1 - \left(\frac{f_c}{f}\right)^2}}$

For TM mode

$$\boxed{P_{avg} = \int_{x=0}^{a} \int_{y=0}^{b} \left(\frac{|E_{xs}|^2 + |E_{ys}|^2}{2\eta_{TM}} \right) dy\, dx \quad \text{watt}}$$

where

$$\eta_{TM} = \eta' \sqrt{1 - \left(\frac{f_c}{f}\right)^2}$$

where $\eta' = \sqrt{\dfrac{\mu}{\epsilon}}$ = Intrinsic wave impedance, in absence of wave guide. in dielectric medium

Attenuation in a lossy waveguide

Practically all waveguides are lossy in nature so that there is a loss of power along the waveguide as the wave propagates.

→ We have assumed lossless waveguides ($\sigma = 0$, $\sigma_c = \infty$) for which $\alpha = 0$, $\gamma = j\beta$.

→ When dielectric medium is lossy ($\sigma \neq 0$) and the guide walls are not perfectly conducting ($\sigma_c \neq \infty$), there is a continuous loss of power along the guide as a wave propagates.

∴ The loss occurs in the dielectric medium or in the conducting walls.

Assuming that wave propagate along Z-axis,

$$E \propto e^{-\gamma z}$$

$$H \propto e^{-\gamma z}$$

$$P_{avg} \propto e^{-2\gamma z} \qquad [\because P_{avg} \propto E \times H]$$

$$\Rightarrow P_{avg} = P_0 e^{-2\gamma z}$$

$$\Rightarrow P_{avg} = P_0 \cdot e^{-2(\alpha + j\beta)z}$$

$$\Rightarrow P_{avg} = P_0 \cdot e^{-2\alpha z} \cdot e^{-2j\beta z}$$

$$\Rightarrow |P_{avg}| = P_0 \cdot e^{-2\alpha z} \quad \left(\because |e^{2j\beta z}| = 1 \right)$$

$$(\because |\cos 2\beta z \mp j \sin 2\beta z| = 1)$$

In general

$\alpha = \alpha_c + \alpha_d$ (~~Total loss~~)

where α_c ~~and~~ α_d are attenuation constants due to ohmic or conduction (α_c) losses [Since $\sigma_c \neq \infty$] and dielectric (α_d) losses $(\sigma \neq 0)$, respectively.

Imp

$$\boxed{\alpha_d = \frac{K^2 \tan\theta}{2\beta} \text{ Np/m for a rectangular waveguide}}$$

Note:-
Np = Neper/m
1 Np = 8.686 dB

where $K = \omega\sqrt{\mu\epsilon}$

$$\beta = \sqrt{K^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2} = \sqrt{\omega^2\mu\epsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2} = \beta'\sqrt{1-\left(\frac{f_c}{f}\right)^2}$$

$\tan\theta = \frac{\sigma}{\omega\epsilon}$ = Loss tangent = Ratio of conduction current to displacement current $= \frac{J_c}{J_d}$

For lossless waveguide, $\alpha_d = 0$ because $\sigma = 0$.

Similarly, expression for α_c can be derived as

$$\alpha_c = \frac{R_s}{a^3 b \beta K \eta}\left(2b\pi^2 + a^3K^2\right) \text{ Np/m}$$

where $R_s = \sqrt{\frac{\omega\mu}{2\sigma}}$ = Surface resistance of the conductor

a & b are the dimension of waveguide.

K and β are same as expressed for 'α_d'.

$$\boxed{\text{Total attenuation } (\alpha) = \alpha_c + \alpha_d}$$

Imp Important eqn for TE & TM modes :-

TM mode	TE Mode
$H_{zs} = 0$	$E_{zs} = 0$
$\eta_{TM} = \eta' \sqrt{1-\left(\frac{f_c}{f}\right)^2}$	$\eta_{TE} = \dfrac{\eta'}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}}$

Imp Common eqn

$$f_c = \frac{\omega}{2} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

where $\eta' = \sqrt{\frac{\mu}{\epsilon}}$

$$\lambda_c = \frac{2}{\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}}$$

$$\beta = \beta' \sqrt{1 - \left(\frac{f_c}{f}\right)^2} \quad , \quad \beta' = \omega\sqrt{\mu\epsilon} = K$$

$$u_p = \frac{u'}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}} \quad \text{where } u' = \frac{1}{\sqrt{\mu\epsilon}}$$

$$\lambda = \frac{\lambda'}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}}$$

$$K = \omega\sqrt{\mu\epsilon} \quad , \quad h^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$

Problems

1) BPUT 2020 The phase constant of the TE_{10} mode of an air-filled waveguide with $b = 1$ cm is 102.65 rad/m. If the operating freq of the waveguide is 12 GHz. and the only mode of propagation is TE_{10}. Calculate

(a) The length 'a' of the waveguide

(b) Wave impedance.

Ans ÷ Given

$b = 1$ cm, $\beta = 102.65$ rad/m, $f = 12$ GHz.

$a = ?$, $\eta_{TE} = ?$

We know

$$\beta = \beta' \sqrt{1-\left(\frac{f_c}{f}\right)^2}$$

$$\Rightarrow \beta = \omega\sqrt{\mu\epsilon}\sqrt{1-\left(\frac{f_c}{f}\right)^2}$$

$$\Rightarrow \beta = \frac{\omega}{c}\sqrt{1-\left(\frac{f_c}{f}\right)^2}$$

$$\Rightarrow \beta = \frac{2\pi f}{3\times10^8}\sqrt{1-\frac{(f_c)^2}{(12\times10^9)^2}}$$

$$c = \frac{1}{\sqrt{\mu\epsilon}} \Rightarrow \sqrt{\mu\epsilon} = \frac{1}{c}$$

$$\Rightarrow 102.65 = \frac{2\pi \times 12 \times 10^{9}/10}{3 \times 10^{8}} \sqrt{1 - \frac{f_c^2}{144 \times 10^{18}}}$$

$$\Rightarrow \sqrt{1 - \frac{f_c^2}{144 \times 10^{18}}} = \frac{102.65}{80\pi} = 0.408$$

$$\Rightarrow 1 - \frac{f_c^2}{144 \times 10^{18}} = 0.1668$$

$$\Rightarrow 1 - 0.1668 = \frac{f_c^2}{144 \times 10^{18}}$$

$$\Rightarrow f_c^2 = 119.97 \times 10^{18}$$

$$\Rightarrow \boxed{f_c = 10.9536147}$$

For TE_{10} mode,

$$f_c = \frac{u'}{2a} = \frac{3 \times 10^8}{2a}$$

$$\Rightarrow 10.953 \times 10^{9}/10 = \frac{3 \times 10^8}{2a}$$

$$\Rightarrow 2a = \frac{3}{109.53}$$

$$\Rightarrow a = 0.01369 \text{ meter}.$$

(a) $\Rightarrow \boxed{a = 1.369 \text{ Cm}}$

(b) Wave impedance $\eta_{TE} = \dfrac{\eta'}{\sqrt{1 - \left(\frac{f_c}{f}\right)^2}}$

where $\eta' = \sqrt{\frac{\mu}{\epsilon}} = 377$ ohm for freespace / air.

$$\Rightarrow \eta_{TE} = \frac{377}{\sqrt{1 - \left(\frac{10.953 \times 10^9}{12 \times 10^9}\right)^2}}$$

$$\Rightarrow \boxed{\eta_{TE} = 922.847\ \Omega}$$

BPUT-2010

2) An air-filled rectangular waveguide with dimensions $a = 3cm$, $b = 2cm$, it is excited with TE mode at 6 GHz. The loss tangent in air is 0.001, and $\sigma = 5.8 \times 10^7$ S/m.

Calculate

(i) Cut off freq

(ii) Phase constant

(iii) Skin depth

(iv) attenuation constant α_d & α_c.

Ans : Given

$a = 3cm$, $b = 2cm$, $f = 6 \times 10^9$, $\tan\delta = 0.001$

$\sigma = 5.8 \times 10^7$ S/m. For TE_{10} mode,

$$f_c = \frac{u'}{2a} = \frac{\cancel{3} \times 10^8}{2 \times \cancel{3} \times 10^{-2}} = \frac{10^8 \times \cancel{10}^{5} \times 10}{\cancel{2}} = 5\ GHz.$$

$$\text{Phase constant } (\beta) = \beta' \sqrt{1 - \left(\frac{f_c}{f}\right)^2} = \omega\sqrt{\mu\epsilon}\sqrt{1 - \left(\frac{f_c}{f}\right)^2}$$

$$\boxed{\frac{1}{\sqrt{\mu\epsilon}} \Rightarrow \sqrt{\mu\epsilon} = \frac{1}{c}} \Rightarrow \beta = \frac{\omega}{c}\sqrt{1 - \left(\frac{f_c}{f}\right)^2}$$

$$\Rightarrow \beta = \frac{2\pi f}{3\times10^8}\sqrt{1-\left(\frac{f_c}{f}\right)^2}$$

$$= \frac{2\pi \times \cancel{6}^{2}\times10^{9}/10}{\cancel{3}\times\cancel{10^8}}\sqrt{1-\left(\frac{5}{6}\right)^2}$$

$$\Rightarrow \beta = 40\pi\sqrt{\frac{36-25}{36}} = 40\pi \times \sqrt{\frac{11}{36}}$$

$$\Rightarrow \beta = 69.46 \ \frac{rad}{m}.$$

(iii) Skin depth $(\delta) = \dfrac{1}{\sqrt{\pi f \mu \sigma}}$

$$\therefore \delta = \frac{1}{\sqrt{\pi \times 6\times10^9 \times 4\pi\times10^{-7}\times 5.8\times10^7}}$$

Note
$\mu_0 = 4\pi\times10^{-7}$
$\epsilon_0 = \frac{10^{-9}}{36\pi}$

$$\delta = 8.53\times10^{-7} \text{ meter.}$$

(iv) Attenuation Constant

$$\alpha_d = \frac{k^2 \tan\theta}{2\beta}$$

$$k = \omega\sqrt{\mu\epsilon} = \frac{\omega}{c} = \frac{2\pi f}{c} = \frac{2\pi\times\cancel{6}^{2}\times10^9}{\cancel{3}\times10^8} = 4\pi\times10 = 40\pi = 125.66$$

$\tan\theta =$ loss tangent $= 0.001$ (given)

$\beta = 69.46 \ \frac{rad}{m}$ (derived in (ii) bit)

$$\therefore \alpha_d = \frac{(125.66)^2\times0.001}{2\times69.46} \simeq 0.1136 \ Np/m$$

$$= 0.9867 \ dB/m$$

$$\alpha_c = \frac{R_s}{a^3 b \beta K \eta}\left(2b\pi^2 + a^3 K^2\right)$$

$$R_s = \sqrt{\frac{\omega\mu}{2\sigma}} = \sqrt{\frac{2\pi f \times 4\pi\times10^{-7}}{2\times5.8\times10^{7}}} = \sqrt{\frac{2\times\pi\times6\times10^{9}\times4\pi\times10^{-7}}{2\times5.8\times10^{7}}}$$

$$\Rightarrow \quad R_s = 0.0202\ \Omega.$$

$\eta = 377\,\Omega$ for free space.

$$\alpha_c = \frac{0.0202}{(3\times10^{-2})^3(2\times10^{-2})(69.46)(125.66)\times377}\left[2\times(2\times10^{-2})\times\pi^2 + (3\times10^{-2})^3(125.66)^2\right]$$

$$\alpha_c = \frac{(0.0202)\left[.3947 + 0.4263\right]}{1.7769}$$

$$\alpha_c = 9.33\times10^{-3}\ \text{Np/m} = 0.081\ \text{dB/m}. \quad \text{(Ans)}$$

$$(\because\ 1\ \text{Np} = 8.686\ \text{dB})$$

3) BPUT-2009

An air filled rectangular waveguide having dimension 4 x 8 operates in the TE_{10} mode. Find out

(i) the cut-off freq

(ii) The Phase velocity at a freq of 4 GHz.

Ans : Given $a = 8$ cm, $b = 4$ cm

$$f_c = \frac{c}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

$$TE_{10}, \quad f_c = \frac{c}{2}\sqrt{\frac{1}{a^2}} = \frac{c}{2a}.$$

(i) $$f_c = \frac{3\times10^8}{2\times 8\times10^{-2}} = \frac{30}{16}\times10^9 = 1.8\ GHz$$

(ii) Phase velocity at freq 4 GHz.

$$u_p = \frac{u'}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}} = \frac{3\times10^8}{\sqrt{1-\left(\frac{1.8}{4}\right)^2}} = 3.36\times10^8\ m/s$$

Phase velocity $= 3.36\times10^8$ m/s

4) RGPV-2008 In an air filled square waveguide with dimensions $a = 1.2$ cms,

$$E_x = -10\sin\left(\frac{2\pi y}{a}\right)\sin(\omega t - 150z)\ \frac{V}{m}.$$

Find (i) Mode of propagation

(ii) the cut-off wavelength

(iii) Calculate the freq of operation

(iv) wave impedance.

~~Ans: Let~~

$$\cancel{E_x = E_A}\cos\left(\frac{m\pi}{a}\right)x\ \sin\left(\frac{n\pi}{b}\right)y\ \cancel{\sin(\omega t-150z)}\quad e^{-\gamma z}$$

~~Comparing with~~ given

$$\cancel{E_x = -10}\ \cos(0)x\ \cancel{\sin}\left(\frac{2\pi}{a}\right)y$$

Ans :- Let

$$E_x = E_A \cos\left(\frac{m\pi}{a}\right)x \; \sin\left(\frac{n\pi}{b}\right)y \; \sin(\omega t - \beta z)$$

given

$$E_x = -10 \; \cos(0)x \; \sin\left(\frac{2\pi}{a}\right)y \; \sin(\omega t - 150z)$$

$(\because \cos(0)x = 1)$

Comparing

$E_A = -10$, $m = 0$, $n = 2$, $\beta = 150$ (For square waveguide $a = b$)

Two possible types

TE_{02} or TM_{02}

If $m=0$, TM_{mn} can n't exist. (All component vanish)

(i) : Mode of propagation is $\boxed{TE_{02}}$.

(ii) Cut off wave length

$$\lambda_c = \frac{2}{\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}} = \frac{2}{\sqrt{0 + \left(\frac{2}{b}\right)^2}} = \frac{2}{\frac{2}{b}}$$

$$\boxed{\lambda_c = b}$$

For square ~~wa~~ wave guide, $b = a = 1.2$ cm.

$$\therefore \boxed{\lambda_c = 1.2 \text{ cm}}$$

(iii)

$$f_c = \frac{c}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} = \frac{c}{2}\sqrt{\left(\frac{2}{b}\right)^2} = \frac{c}{\cancel{2}} \times \frac{\cancel{2}}{b}$$

$$f_c = \frac{c}{b}$$

$$f_c = \frac{3\times10^8}{1.2\times10^{-2}} = 2.5\times10^{10} = 25\ GHz.$$

$$\beta = \beta' \sqrt{1-\left(\frac{f_c}{f}\right)^2}$$

$$\Rightarrow 150 = \omega\sqrt{\mu\epsilon}\sqrt{1-\left(\frac{f_c}{f}\right)^2}$$

$$\Rightarrow 150 = \frac{\omega}{c}\sqrt{1-\left(\frac{f_c}{f}\right)^2}$$

$$\Rightarrow \frac{150c}{\omega} = \sqrt{1-\left(\frac{f_c}{f}\right)^2}$$

$$\Rightarrow \frac{150^2c^2}{\omega^2} = 1-\left(\frac{f_c}{f}\right)^2$$

$$\Rightarrow \frac{150^2c^2}{4\pi^2f^2} = 1-\frac{f_c^2}{f^2}$$

$$\Rightarrow 150^2c^2 = 4\pi^2f^2 - \frac{4\pi^2\cancel{f^2}\cdot f_c^2}{\cancel{f^2}}$$

$$\Rightarrow 150^2c^2 = 4\pi^2f^2 - 4\pi^2f_c^2 = 4\pi^2(f^2-f_c^2)$$

$$\Rightarrow (150)^2\times(3\times10^8)^2 = 4\pi^2\left[f^2-(25\times10^9)^2\right]$$

$$\Rightarrow f^2-(25\times10^9)^2 = \frac{150^2\times9\times10^{16}}{4\pi^2} = 51.29\times10^{18}$$

$$\Rightarrow f^2 = 625\times10^{18} + 51.29\times10^{18} = 676.29\times10^{18}$$

$$\Rightarrow \boxed{f = 26.00\ GHz}$$

(iv) Wave impedance

$$\eta = \frac{\eta'}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}} = \frac{377}{\sqrt{1-\left(\frac{25}{26}\right)^2}} = 1372.55\ \Omega$$

5) An air-filled rectangular waveguide of inside dimension 7 x 3.5 cm operates in the dominant mode TE_{10} mode.

(a) Find the cutoff freq.

(b) Determine phase velocity of the wave in the guide at freq 3.5 GHz.

(c) Determine the guided wavelength at the same freq.

Ans: (a) $f_c = \frac{c'}{2a} = \frac{3\times10^8}{2\times 7\times10^{-2}} = 2.14\ GHz.$

(given $a = 7\ cm$, $b = 3.5\ cm$)

(b) $$u_p = \frac{u'}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}} = \frac{3\times10^8}{\sqrt{1-\left(\frac{2.14}{3.5}\right)^2}} = 3.79\times10^8\ \frac{m}{s}$$

(c) $$\lambda = \frac{\lambda'}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}} = \frac{\frac{c}{f}}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}} = \frac{\frac{3\times10^8}{3.5\times10^9}}{\sqrt{1-\left(\frac{2.14}{3.5}\right)^2}}$$

$$\lambda = \frac{0.857\times10^{-1}}{0.7912} = 1.08\times10^{-1}\ \text{meter}$$

$$\boxed{\lambda = 10.8\ cm}$$

6) Design a rectangular waveguide at cut-off frequency $f_c = 9400$ MHz.

Ans :

$$(f_c)_{10} = \frac{c}{2a}$$

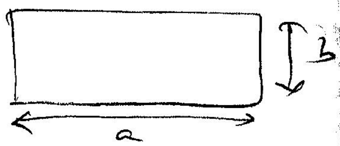
$$\Rightarrow 9400 \times 10^6 = \frac{3 \times 10^8}{2 \times a}$$

$$\Rightarrow 94 \times 2 \times a = 3$$

$$\Rightarrow a = \frac{3}{188} = 0.0159 \text{ meter}$$

$$\Rightarrow a = 1.59 \text{ cm}$$

For ideal case $\boxed{a = 2b}$



$$b = \frac{a}{2} = \frac{1.59}{2} = 0.795 \text{ cm}$$

$$b = 0.795 \text{ cm.}$$

∴ Dimension of Rectangular waveguide (1.59×0.795)

7) A rectangular waveguide with dimension 3 × 2 cm operates in the TM_{11} mode at 10 GHz. Determine the characteristic wave impedance.

Ans : Given $a = 3$ cm, $b = 2$ cm.

$$f_c = \frac{c}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} = \frac{c}{2}\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}$$

$$\Rightarrow f_c = \frac{3 \times 10^8}{2}\sqrt{\frac{1}{(3 \times 10^{-2})^2} + \frac{1}{(2 \times 10^{-2})^2}} = 1.5 \times 10^8 \sqrt{\frac{10^4}{9} + \frac{10^4}{4}}$$

$$\Rightarrow f_c = 1.5 \times 10^8 \times 10^2 \times 0.60 = 9 \text{ GHz.}$$

$$\eta_{TM} = \eta' \sqrt{1-\left(\frac{f_c}{f}\right)^2}$$

$$\eta_{TM} = 377 \sqrt{1-\left(\frac{9}{10}\right)^2}$$

$$\Rightarrow \boxed{\eta_{TM} = 164.33\,\Omega}$$

8) Consider a rectangular waveguide of 8×4 cm. Give critical wavelength of TE_{10} = 16 cm, for 8×4 TM_{11} = 7.16 cm, TM_{21} = 5.6 cm. What mode cm rectangular wave guide propagates at ~~free space~~ guided wavelength of (a) 10 cm (b) 5 cm.

Ans :- $f > f_c$ [For propagation in waveguide].

or $\boxed{\lambda < \lambda_c}$

Case-I $\lambda = 10$ cm

(i) TE_{10}, $\lambda_c = 16$ cm, $\therefore$ $10 < 16$ cm, TE_{10} ✓ Propagates

(ii) TM_{11}, $\lambda_c = 7.16$ cm, $10 \nless 7.16$ cm, TM_{11} ✗

(iii) TM_{21}, $\lambda_c = 5.6$ cm, $10 \nless 5.6$ cm, TM_{21} ✗

Case-II $\lambda = 5$ cm

(i) TE_{10}, $\lambda_c = 16$ cm, $5 < 16$ cm, TE_{10} Propagates.

(ii) TM_{11}, $\lambda_c = 7.16$ cm, $5 < 7.16$ cm, TM_{11} "

(iii) TM_{21}, $\lambda_c = 5.6$ cm, $5 < 5.6$ cm, TM_{21} "

Cylindrical (circular) waveguide :– [Refer Pozar Book]

→ A hollow metal tube of circular cross section also supports TE & TM waveguide modes. Figure 26, shows the cross-section geometry of such a circular waveguide of inner radius 'a'.

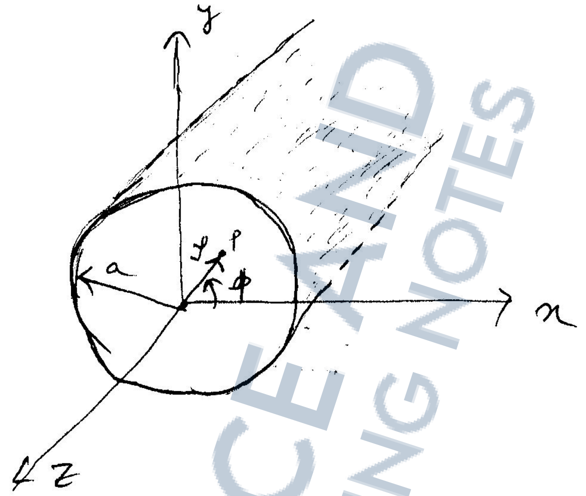


fig 26:- Geometry of a circular waveguide

→ Since a cylindrical geometry is involved, it is appropriate to employ cylindrical coordinates (ρ, ϕ, z).

→ To findout field equations we have applied Maxwell's equations,

i.e
$$\nabla \times E = -j\omega\mu H \qquad \text{——} (1)$$
$$\nabla \times H = j\omega\epsilon E \qquad \text{——} (2)$$

From eq^n (1), we have
$$\nabla \times E = -j\omega\mu H$$

Expanding both the sides of the above equation in terms of cylindrical co-ordinates, we get

$$\Rightarrow \frac{1}{\rho}\begin{vmatrix} \hat{a}_\rho & \rho\hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ E_\rho & \rho E_\phi & E_z \end{vmatrix} = -j\omega\mu\left[H_\rho\hat{a}_\rho + H_\phi\hat{a}_\phi + H_z\hat{a}_z\right]$$

$$\Rightarrow \frac{1}{\rho}\left[\hat{a}_\rho\left(\frac{\partial E_z}{\partial\phi} - \frac{\partial \rho E_\phi}{\partial z}\right) - \rho\hat{a}_\phi\left(\frac{\partial}{\partial\rho}E_z - \frac{\partial}{\partial z}E_\rho\right) + \hat{a}_z\left(\frac{\partial}{\partial\rho}\rho E_\phi - \frac{\partial}{\partial\phi}E_\rho\right)\right]$$

$$= -j\omega\mu H_\rho\hat{a}_\rho - j\omega\mu H_\phi\hat{a}_\phi - j\omega\mu H_z\hat{a}_z$$

Equating the co-efficient of unit vector, both the sides

$$\frac{1}{\rho}\left(\frac{\partial E_z}{\partial\phi} - \frac{\partial(\rho E_\phi)}{\partial z}\right) = -j\omega\mu H_\rho \quad \text{——— 3 (a)}$$

$$\frac{\partial}{\partial z}E_\rho - \frac{\partial E_z}{\partial\rho} = -j\omega\mu H_\phi \quad \text{——— 3(b)}$$

$$\frac{1}{\rho}\left(\frac{\partial}{\partial\rho}(\rho E_\phi) - \frac{\partial E_\rho}{\partial\phi}\right) = -j\omega\mu H_z \quad \text{——— 3(c)}$$

Similarly from eqn (2), we have

$$\nabla\times H = j\omega\epsilon E$$

Expanding

$$\frac{1}{\rho}\begin{vmatrix} \hat{a}_\rho & \rho\hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ H_\rho & \rho H_\phi & H_z \end{vmatrix} = j\omega\epsilon\left[E_\rho\hat{a}_\rho + E_\phi\hat{a}_\phi + E_z\hat{a}_z\right]$$

$$\Rightarrow \frac{1}{\rho}\left[\hat{a}_\rho \left(\frac{\partial}{\partial \phi} H_z - \frac{\partial}{\partial z}(\rho H_\phi) \right) - \rho \hat{a}_\phi \left(\frac{\partial}{\partial \rho} H_z - \frac{\partial}{\partial z} H_\rho \right) + \hat{a}_z \left(\frac{\partial}{\partial \rho}(\rho H_\phi) - \frac{\partial}{\partial \phi} H_\rho \right) \right]$$

$$= j\omega\epsilon E_\rho \hat{a}_\rho + j\omega\epsilon E_\phi \hat{a}_\phi + j\omega\epsilon E_z \hat{a}_z$$

Now equating the coefficient of unit vector both the sides of the above equation, we have

$$\frac{1}{\rho}\left[\frac{\partial}{\partial \phi} H_z - \frac{\partial}{\partial z}(\rho H_\phi) \right] = j\omega\epsilon E_\rho \quad \text{——— } 4(a)$$

$$\frac{\partial}{\partial z} H_\rho - \frac{\partial}{\partial \rho} H_z = j\omega\epsilon E_\phi \quad \text{——— } 4(b)$$

$$\frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho H_\phi) - \frac{\partial}{\partial \phi} H_\rho \right] = j\omega\epsilon E_z \quad \text{——— } 4(c)$$

Lets assume that the wave is propagating in Z-direction. Then the solution along Z-axis is given by

$$H_z = H_0 e^{-\gamma z}$$

$$\Rightarrow \frac{\partial}{\partial z} H_z = H_0 \cdot e^{-\gamma z} \cdot (-\gamma) = (-\gamma) \cdot H_z$$

$$\Rightarrow \frac{\partial}{\partial z} \cdot \left[H_0 e^{-\gamma z} \right] = (-\gamma) \cdot \left[H_0 \cdot e^{-\gamma z} \right.$$

Comparing the coefficient of $e^{-\gamma z}$, we have

$$\frac{\partial}{\partial z} = -\gamma \quad \text{——— } (5)$$

From equation 3 (a)

$$\frac{1}{\rho}\left[\frac{\partial}{\partial \phi}(E_z) - \frac{\partial}{\partial z}(\rho E_\phi)\right] = -j\omega\mu H_\rho$$

Putting eqn (5) in the above eqn, we have

$$\frac{1}{\rho}\left[\frac{\partial}{\partial \phi}E_z + \gamma(\rho E_\phi)\right] = -j\omega\mu H_\rho$$

$$\Rightarrow H_\rho = \frac{1}{-\rho j\omega\mu}\left[\frac{\partial}{\partial \phi}E_z + \gamma \rho E_\phi\right]$$

$$\Rightarrow H_\rho = -\frac{1}{j\omega\mu}\left[\frac{1}{\rho}\frac{\partial E_z}{\partial \phi} + \gamma E_\phi\right] \quad \text{—— (6)}$$

Similarly from eqn 4(b),

$$\frac{\partial}{\partial z}H_\rho - \frac{\partial}{\partial \rho}H_z = j\omega\epsilon E_\phi$$

Putting eqn (5) in the above eqn, we have

$$(-\gamma).H_\rho - \frac{\partial}{\partial \rho}H_z = j\omega\epsilon E_\phi$$

$$\Rightarrow H_\rho = \frac{-1}{\gamma}\left[j\omega\epsilon E_\phi + \frac{\partial}{\partial \rho}H_z\right] \quad \text{—— (7)}$$

Comparing eqn (6) & (7), we get

$$\frac{1}{j\omega\mu}\left[\frac{1}{\rho}\frac{\partial}{\partial \phi}(E_z) + \gamma E_\phi\right] = \frac{1}{\gamma}\left[j\omega\epsilon E_\phi + \frac{\partial}{\partial \rho}H_z\right]$$

$$\Rightarrow \frac{1}{\rho j\omega\mu}\frac{\partial}{\partial \phi}E_z + \left[\frac{\gamma E_\phi}{j\omega\mu} - \frac{j\omega\epsilon}{\gamma}E_\phi\right] = \frac{1}{\gamma}\frac{\partial}{\partial \rho}H_z$$

$$\Rightarrow \frac{\gamma}{j\omega\mu} E_\phi - \frac{j\omega\epsilon}{\gamma} E_\phi = \frac{1}{\gamma} \frac{\partial}{\partial\rho} H_z - \frac{1}{\rho j\omega\mu} \frac{\partial}{\partial\phi} E_z$$

$$\Rightarrow \left[\frac{\gamma^2 + \omega^2\mu\epsilon}{j\omega\mu\gamma}\right] E_\phi = \frac{1}{\gamma} \frac{\partial}{\partial\rho} H_z - \frac{1}{\rho j\omega\mu} \frac{\partial E_z}{\partial\phi}$$

$$\Rightarrow (\gamma^2 + \omega^2\mu\epsilon) E_\phi = j\omega\mu \frac{\partial}{\partial\rho} H_z - \frac{\gamma}{\rho} \frac{\partial}{\partial\phi} E_z \quad \text{— (8)}$$

We know,

$$h^2 = \gamma^2 + k^2 = \gamma^2 + \omega^2\mu\epsilon \quad \text{— (9)}$$

Let $K_c^2 = \gamma^2 + \omega^2\mu\epsilon$ — (10)

For lossless medium, $\gamma = j\beta$ $(\because \alpha = 0)$

$$\therefore K_c^2 = -\beta^2 + \omega^2\mu\epsilon = -\beta^2 + k^2$$

$$\Rightarrow \beta^2 = k^2 - K_c^2 \quad \text{— (11)}$$

Putting eqn (9), in eqn (8),

$$\Rightarrow K_c^2 E_\phi = j\omega\mu \frac{\partial}{\partial\rho} H_z - \frac{\gamma}{\rho} \frac{\partial}{\partial\phi} E_z$$

Putting eqn (10), in the above eqn

$$\Rightarrow K_c^2 E_\phi = j\omega\mu \frac{\partial}{\partial\rho} H_z - \frac{(j\beta)}{\rho} \frac{\partial}{\partial\phi} E_z$$

$$\Rightarrow E_\phi = \frac{-j\beta}{K_c^2 \rho} \frac{\partial}{\partial\phi} E_z + \frac{j\omega\mu}{K_c^2} \frac{\partial}{\partial\rho} H_z$$

$$\Rightarrow \boxed{E_\phi = \frac{-j}{K_c^2}\left[\frac{\beta}{\rho} \frac{\partial}{\partial\phi} E_z - \omega\mu \frac{\partial}{\partial\rho} H_z\right]} \quad \text{— (12)}$$

Similarly taking other pair of equations, we have

$$E_\rho = \frac{-j}{k_c^2}\left(\beta \frac{\partial E_z}{\partial \rho} + \frac{\omega\mu}{\rho}\frac{\partial H_z}{\partial \phi}\right) \quad —— (13)$$

$$H_\rho = \frac{j}{k_c^2}\left[\frac{\omega\epsilon}{\rho}\frac{\partial E_z}{\partial \phi} - \beta\frac{\partial H_z}{\partial \rho}\right] \quad —— (14)$$

$$H_\phi = \frac{-j}{k_c^2}\left[\omega\epsilon \frac{\partial E_z}{\partial \rho} + \frac{\beta}{\rho}\frac{\partial H_z}{\partial \phi}\right] \quad —— (15)$$

Equation (12), (13), (14), (15) are field equations for cylindrical wave guide.

TE mode in cylindrical wave guide

For TE mode, $E_z = 0$, $H_z \neq 0$

From the wave equation,

$$\nabla^2 H_z + k^2 H_z = 0 \quad —— (16)$$

Let the wave is travelling along z-direction.

$$\therefore \quad H_z = h_z(\rho, \phi)\, e^{-\gamma z} = h_z(\rho,\phi) \cdot e^{-j\beta z} \quad —— (17)$$

[$\because \gamma = j\beta$ for lossless medm]

Expanding $\nabla^2 H_z$ in cylindrical co-ordinates,

$$\nabla^2 H_z = \frac{1}{\rho}\cdot\frac{\partial}{\partial \rho}\left(\rho \frac{\partial H_z}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 H_z}{\partial \phi^2} + \frac{\partial^2 H_z}{\partial z^2}$$

$$= \frac{1}{\rho}\left[\rho\cdot\frac{\partial^2 H_z}{\partial \rho^2} + \frac{\partial H_z}{\partial \rho}\right] + \frac{1}{\rho^2}\frac{\partial^2 H_z}{\partial \phi^2} + \frac{\partial^2 H_z}{\partial z^2}$$

$$\nabla^2 H_z = \frac{\partial^2}{\partial \rho^2} H_z + \frac{1}{\rho}\frac{\partial}{\partial \rho} H_z + \frac{1}{\rho^2}\frac{\partial^2 H_z}{\partial \phi^2} + \frac{\partial^2 H_z}{\partial z^2} \quad —— (17)$$

Putting eq^n (17), in eq^n (16), we have

$$\frac{\partial^2 H_z}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho}H_z + \frac{1}{\rho^2}\frac{\partial^2 H_z}{\partial \phi^2} + \frac{\partial^2 H_z}{\partial z^2} + K^2 H_z = 0 \quad \text{--- (18)}$$

* Note:- Let $H_z = H_0 \cdot e^{-\gamma z}$

$$H_z = H_0 \cdot e^{-j\beta z} \qquad (\alpha = 0 \text{ for lossless med}^n)$$

$$\frac{\partial}{\partial z} H_z = H_0 \cdot e^{-j\beta z} \cdot (-j\beta)$$

$$\frac{\partial^2 H_z}{\partial z^2} = (-j\beta) H_0 \cdot e^{-j\beta z} (-j\beta) = (-j\beta)^2 H_0 \cdot e^{-j\beta z}$$

$$\frac{\partial^2 H_z}{\partial z^2} = -\beta^2 H_z$$

Comparing the coefficients, $\boxed{\frac{\partial^2}{\partial z^2} = -\beta^2}$ --- (19)

Putting eq^n (19), in eq^n (18), we have

$$\frac{\partial^2 H_z}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho}H_z + \frac{1}{\rho^2}\frac{\partial^2 H_z}{\partial \phi^2} - \beta^2 H_z + K^2 H_z = 0$$

$$\Rightarrow \frac{\partial^2 H_z}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho}H_z + \frac{1}{\rho^2}\frac{\partial^2 H_z}{\partial \phi^2} + (K^2 - \beta^2) H_z = 0$$

$$\Rightarrow \frac{\partial^2 H_z}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho}H_z + \frac{1}{\rho^2}\frac{\partial^2 H_z}{\partial \phi^2} + K_c^2 H_z = 0 \quad \text{--- (20)}$$

(using eq^n (11) $\beta^2 = K^2 - K_c^2$)

The solution of the above equation can be found out using variable separation method.

Let $h_z(\rho, \phi) = R(\rho) \cdot P(\phi)$ be the solution to the eq^n (20), —— (20A)

Then

$$\frac{\partial^2}{\partial \rho^2} R(\rho) \cdot P(\phi) + \frac{1}{\rho} \frac{\partial}{\partial \rho} R(\rho) \cdot P(\phi) + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} R(\rho) P(\phi)$$

$$+ K_c^2 \left(R(\rho) \cdot P(\phi) \right) = 0$$

$$\Rightarrow P(\phi) \cdot \frac{\partial^2}{\partial \rho^2} R(\rho) + \frac{P(\phi)}{\rho} \frac{\partial}{\partial \rho} R(\rho) + \frac{R(\rho)}{\rho^2} \frac{\partial^2 P(\phi)}{\partial \phi^2} + K_c^2 R(\rho) P(\phi) = 0$$

Dividing both the sides by $R(\rho) \cdot P(\phi)$

$$\Rightarrow \frac{1}{R} \frac{\partial^2 R}{\partial \rho^2} + \frac{1}{\rho R} \frac{\partial R}{\partial \rho} + \frac{1}{P \rho^2} \frac{\partial^2 P}{\partial \phi^2} + K_c^2 = 0$$

Multiplying ρ^2 both the sides,

$$\frac{\rho^2}{R(\rho)} \frac{\partial^2 R(\rho)}{\partial \rho^2} + \frac{\rho}{R(\rho)} \frac{\partial R(\rho)}{\partial \rho} + \rho^2 K_c^2 = -\frac{1}{P(\phi)} \frac{\partial^2 P(\phi)}{\partial \phi^2} \quad \text{—— (21)}$$

The left side of the above equation depends only on (ρ) not on (ϕ) and the right side depends only on (ϕ). Therefore each side of the above equation must be equal to a constant, lets say K_ϕ^2. Then

$$-\frac{1}{P(\phi)} \frac{\partial^2 P(\phi)}{\partial \phi^2} = K_\phi^2 \quad \text{—— (22)}$$

$$\Rightarrow \frac{\partial^2 P}{\partial \phi^2} + K_\phi^2 P = 0 \quad \text{—— (23)}$$

The solution to the eq^n (23),

$$P(\phi) = A' \sin(K_\phi \phi) + B' \cos(K_\phi \phi) \quad \text{--- (24)}$$

~~Similarly~~, ~~left side of~~ $\therefore$ eq^n (21) using eq^n (22), ~~is a constant~~ becomes

$$\frac{\rho^2}{R}\frac{\partial^2 R}{\partial \rho^2} + \frac{\rho}{R}\frac{\partial R}{\partial \rho} + \rho^2 K_c^2 = K_\phi^2$$

$$\Rightarrow \rho^2 \frac{\partial^2 R}{\partial \rho^2} + \rho \frac{\partial R}{\partial \rho} + (\rho^2 K_c^2 - K_\phi^2) R = 0$$

~~Which is recognized as Bessel's differential equation~~.

Let $K_\phi = n$ (Integer)

$\because$ since h_z must be periodic in ϕ
$h_z(\rho, \phi) = h_z(\rho, \phi \pm 2m\pi)$,
K_ϕ must be an integer, n

$$\Rightarrow \rho^2 \frac{d^2 R}{\partial \rho^2} + \rho \frac{\partial R}{\partial \rho} + (\rho^2 K_c^2 - n^2) R$$

which is recognized as Bessel's differential eq^n,

The solution is

$$R(\rho) = C_n J_n(K_c \rho) + D\, Y_n(K_c \rho),$$

BPUT-2018

where $J_n(x)$ & $Y_n(x)$ are the Bessel functions of first and second kinds, respectively. Since $Y_n(K_c \rho)$ becomes infinite at $\rho = 0$, this term is physically unacceptable for the circular waveguide problem, so that $D = 0$. The sol^n for $R(\rho)$ can be written as,

$$R(\rho) = C_n J_n(K_c \rho) \quad \text{--- (25)}$$

Using eqn (24) & (25) in eqn (20A), we have

$$h_z(\rho,\varphi) = \{A' \sin(n\varphi) + B'\cos(n\varphi)\}\{C_n J_n(K_c\rho)\}$$

$(\because K_\varphi = n)$

From eqn (17),

$$H_z = h_z(\rho,\varphi)\cdot e^{-j\beta z}$$

$$\Rightarrow H_z = C_n J_n(K_c\rho)\cdot\{A' \sin(n\varphi) + B'\cos(n\varphi)\}\cdot e^{-j\beta z}$$

$$\Rightarrow H_z = \{A \sin n\varphi + B \cos n\varphi\} J_n(K_c\rho)\cdot e^{-j\beta z} \quad \text{——— (18)}$$

where $A = A'C_n$
$B = B'C_n$

From eqn (12),

$$E_\varphi = \frac{-j}{K_c^2}\left[\frac{\beta}{\rho}\frac{\partial}{\partial\varphi}E_z - \omega\mu\frac{\partial}{\partial\rho}H_z\right]$$

For TE mode, $E_z = 0$

$$\Rightarrow E_\varphi = \frac{+j\,\omega\mu}{K_c^2}\frac{\partial}{\partial\rho}H_z \quad \text{——— (19)}$$

Putting the boundary condition, $[E_\varphi = 0,\ \text{at } \rho = a]$

$$\frac{j\omega\mu}{K_c^2}\frac{\partial}{\partial\rho}\left[A\sin n\varphi + B\cos n\varphi\right] J_n(K_c\rho)\cdot e^{-j\beta z}\Big|_{\rho=a} = 0$$

$$\Rightarrow \frac{j\omega\mu}{K_c^2}\left[A \sin n\varphi + B\cos n\varphi\right] J_n'(K_c\rho)\cdot e^{-j\beta z} = 0.$$

$$\therefore J_n'(K_c\rho)\Big|_{\rho=a} = 0$$

where $J_n'(K_c\rho)$ refers to derivative of J_n with respect to its argument

$\Rightarrow J_n'(k_c a) = 0$

The roots of $J_n'(x)$ are defined as P'_{nm}, so that $J_n'(P'_{nm}) = 0$, where P'_{nm} is the $\underline{m^{th}}$ root of J_n'.

$$\therefore k_c a = P'_{nm}$$

$$\Rightarrow k_c = \frac{P'_{nm}}{a} \qquad (20)$$

Cutoff freq :-

At cutoff freq, No propagation takes place, $\beta = 0$,
→ For loss less medium, $\alpha = 0$.

$$\therefore \gamma = 0$$

From eqn (9),

$$k_c^2 = \omega^2 \mu \epsilon \qquad [\because \gamma = 0]$$

$$\Rightarrow \left(\frac{P'_{nm}}{a}\right)^2 = \omega_c^2 \mu \epsilon \qquad [\text{using eqn } (20)]$$

$$\Rightarrow \frac{P'_{nm}}{a} = \omega_c \sqrt{\mu \epsilon}$$

$$\Rightarrow \omega_c = \frac{P'_{nm}}{a\sqrt{\mu \epsilon}}$$

Imp

$$\Rightarrow \boxed{f_c = \frac{P'_{nm}}{2\pi a \sqrt{\mu \epsilon}}} \qquad (21)$$

, f_c = cutoff freq

$\longleftarrow p'_{nm} \longrightarrow$

n ↓ \ m →	1	2	3
0	3.832	7.016	10.174
1	1.841	5.331	8.536
2	3.054	6.706	9.970

p'_{nm} minimum is 1.841 ie p'_{11}.

$(f_c)_{11}$ is minimum $\left[\because f_c \propto p'_{nm} \right]$

$$f_c = \frac{p'_{11}}{2\pi a\sqrt{\mu\epsilon}} = \frac{1.841}{2\pi a\sqrt{\mu\epsilon}}$$

Imp

$$\boxed{f_c = \frac{1.841}{2\pi a\sqrt{\mu\epsilon}}} \quad \text{--- (22)}$$

$\therefore$ The dominant mode is $\boxed{TE_{11}}$

<u>The field eqn are</u>

$$H_z = \left\{ A \sin n\phi + B \cos n\phi \right\} J_n(k_c \rho)\ \cancel{J_n\left(\frac{p'_{nm}\rho}{a}\right)}\ e^{-j\beta z} \quad \text{--- (23)}$$

$$E_\rho = \frac{-j}{K_c^2}\left(\frac{\omega\mu}{\rho} \frac{\partial H_z}{\partial \phi} \right) \qquad \left[\begin{array}{l} \text{From eqn 13} \\ \text{and } E_z = 0 \end{array} \right]$$

$$= \frac{-j}{K_c^2}\left[\frac{\omega\mu}{\rho} \frac{\partial}{\partial\phi} \left\{ [A \sin n\phi + B \cos n\phi]\ J_n(K_c\rho)\ e^{-j\beta z} \right\} \right]$$

$$= \frac{-j}{K_c^2} \frac{\omega\mu}{\rho} \cdot [nA \cos n\phi - nB \sin n\phi] . J_n(K_c\rho)\ e^{-j\beta z}$$

$$E_\rho = \frac{-j\omega\mu n}{K_c^2 \rho} \left[A \cos n\phi - B \sin n\phi \right] J_n(K_c\rho) . e^{-j\beta z} \quad \text{--- (24)}$$

$$E_\phi = \frac{-j}{K_c^2}\left[-\omega\mu \frac{\partial}{\partial \rho} H_z\right] \qquad \left[\text{From eq}^n \ (12) \text{ and } E_z = 0\right]$$

$$= \frac{+j\omega\mu}{K_c^2} \frac{\partial}{\partial \rho}\left[A \sin n\phi + B \cos n\phi\right] J_n(K_c \rho)\, e^{-j\beta z}$$

$$E_\phi = \frac{j\omega\mu}{K_c^2}\left[A \sin n\phi + B \cos n\phi\right] . J_n'(K_c \rho)\, e^{-j\beta z} \qquad (25)$$

$$H_\rho = \frac{j}{K_c^2}\left[-\beta . \frac{\partial}{\partial \rho} H_z\right] \qquad \left[\text{From eq}^n \ (14) \text{ and } E_z = 0\right]$$

$$= \frac{-j\beta}{K_c^2}\left(\frac{\partial}{\partial \rho}\left[A \sin n\phi + B \cos n\phi\right] J_n(K_c \rho) . e^{-j\beta z}\right)$$

$$H_\rho = \frac{-j\beta}{K_c^2}\left[A \sin n\phi + B \cos n\phi\right] J_n'(K_c \rho) . e^{-j\beta z} \qquad (26)$$

$$H_\phi = \frac{-j}{K_c^2}\left[\frac{\beta}{\rho} \frac{\partial}{\partial \phi} H_z\right] \qquad \left[\text{From eq}^n \ (15) \text{ and } E_z = 0\right]$$

$$H_\phi = \frac{-j}{K_c^2}\left[\frac{\beta}{\rho} \frac{\partial}{\partial \phi}\right]\left[A \sin n\phi + B \cos n\phi\right] . J_n(K_c \rho) . e^{-j\beta z}$$

$$= \frac{-j}{K_c^2}\left[\frac{\beta}{\rho}\left[n A \cos n\phi - B n \sin n\phi\right] . J_n(K_c \rho) . e^{-j\beta z}\right]$$

$$H_\phi = \frac{-j\beta n}{K_c^2 \rho}\left[A \cos n\phi - B \sin n\phi\right] J_n(K_c \rho)\, e^{-j\beta z} \qquad (27)$$

In the above eq^{n}s there are 2 remaining arbitrary amplitude constants A and B. The co-ordinate system can be rotated about Z-axis to obtain either $A=0$ or $B=0$

Now consider the dominant mode $\boxed{TE_{11}}$ with an excitation such that $\boxed{B=0}$. The fields can be written as, [$TE_{11} \rightarrow n=1$, and $B=0$]

(a) $H_Z = A \sin\phi \, J_1(K_c \rho) \cdot e^{-j\beta z}$ [From eq^n 23]

(b) $E_\rho = \frac{-j\omega\mu}{K_c^2 \rho} [A \cos\phi] J_1(K_c\rho) e^{-j\beta z}$ [From eq^n 24]

(c) $E_\phi = \frac{j\omega\mu}{K_c^2} [A \sin\phi] J_1'(K_c\rho) e^{-j\beta z}$ [From eq^n 25]

28 (d) $H_\rho = \frac{-j\beta}{K_c^2} [A \sin\phi] J_1'(K_c\rho) e^{-j\beta z}$ [From eq^n 26]

(e) $H_\phi = \frac{-j\beta}{K_c^2 \rho} A \cos\phi \, J_1(K_c\rho) e^{-j\beta z}$ [From eq^n 27]

(f) $E_Z = 0$ (For TE mode)

where $\beta = \sqrt{K^2 - K_c^2}$ [From eq^n 11]

$$\boxed{\beta_{nm} = \sqrt{K^2 - \left(\frac{P'_{nm}}{a}\right)^2}} \quad \text{—— (29)}$$

Again, $h^2 = \gamma^2 + K^2 = K_c^2$ [$h^2 = K_c^2$]

$\therefore K_c^2 = \gamma^2 + K^2 = \gamma^2 - \omega^2\mu\epsilon = -\beta^2 - \omega^2\mu\epsilon$.

$(\because \gamma = j\beta)$

Note: Remember any one eq^n e.g. $h^2 = \gamma^2 + K^2$

Then you know, $h = K_c$, $\gamma = j\beta$, $K = \omega\sqrt{\mu\epsilon}$

you can derive the eq^n as per your requirement.

Cut off freq & wavelength

From eqn (21), we have

$$f_c = \frac{p'_{nm}}{2\pi a\sqrt{\mu\epsilon}}$$

$$\Rightarrow \lambda_c = \frac{c}{f_c} = \frac{\frac{1}{\sqrt{\mu\epsilon}}}{\frac{p'_{nm}}{2\pi a\sqrt{\mu\epsilon}}} = \frac{1}{\cancel{\sqrt{\mu\epsilon}}} \times \frac{2\pi a\cancel{\sqrt{\mu\epsilon}}}{p'_{nm}}$$

Imp

$$\Rightarrow \boxed{\lambda_c = \frac{2\pi a}{p'_{nm}}} \quad \text{—(30)}$$

Imp

For dominant mode $\boxed{\lambda_c = \frac{2\pi a}{1.841}}$ —(31)

Attenuation Constant

$$\alpha_d = \frac{k^2 \tan\delta}{2\beta} \; Np/m \quad \text{—(32)}$$

[As described for rectangular waveguide]

$$\alpha_c = \frac{R_s}{a k \eta \beta}\left(k_c^2 + \frac{k^2}{{p'_{11}}^2 - 1}\right) \; Np/m. \quad \text{—(33)}$$

$$, \; R_s = \sqrt{\frac{\omega\mu}{2\sigma}}$$

Total attenuation $\alpha = \alpha_c + \alpha_d$.

Wave Impedance

$$Z_{TE} = \frac{E_\rho}{H_\phi} = \frac{-E_\phi}{H_\rho} = \frac{\omega\mu}{\beta}$$

[Dividing eqn $\frac{24}{27}$ and $\frac{25}{26}$

$$Z_{TE} = \frac{\frac{k}{\sqrt{\mu\epsilon}}\cdot\mu}{\beta} = \frac{k\sqrt{\frac{\mu}{\epsilon}}}{\beta} = \frac{k\eta'}{\beta}$$

[$\because$ $k = \omega\sqrt{\mu\epsilon} \Rightarrow \omega = \frac{k}{\sqrt{\mu\epsilon}}$

$\eta' = \sqrt{\frac{\mu}{\epsilon}}$

$$\boxed{Z_{TE} = \frac{k\eta'}{\beta}} \quad \text{--- (34)}$$

→ Propagation const.

$$\beta = \sqrt{k^2 - k_c^2} \qquad \text{[From eqn 11]}$$

$$= \sqrt{k^2\left[1 - \left(\frac{k_c}{k}\right)^2\right]}$$

$$= k\sqrt{1 - \left\{\frac{\left(\frac{p'_{nm}}{a}\right)}{\omega\sqrt{\mu\epsilon}}\right\}^2} \qquad \left(\because k_c = \frac{p'_{nm}}{a},\ k = \omega\sqrt{\mu\epsilon}\right)$$

$$= k\sqrt{1 - \left(\frac{p'_{nm}}{a\, 2\pi f\sqrt{\mu\epsilon}}\right)^2}$$

$$= k\sqrt{1 - \left(\frac{\frac{p'_{nm}}{2\pi a\sqrt{\mu\epsilon}}}{f}\right)^2}$$

$$\beta = k\sqrt{1 - \left(\frac{f_c}{f}\right)^2} \qquad \left(\because f_c = \frac{p'_{nm}}{2\pi a\sqrt{\mu\epsilon}}\right)$$

$$\boxed{\beta = \beta'\sqrt{1 - \left(\frac{f_c}{f}\right)^2}} \quad \text{--- (35)} \qquad (\because k = \beta')$$

$$\therefore Z_{TE} = \frac{k\eta'}{\beta} = \frac{\cancel{k}\cdot\eta'}{\cancel{\beta'}\sqrt{1 - \left(\frac{f_c}{f}\right)^2}}$$

$$\boxed{Z_{TE} = \frac{\eta'}{\sqrt{1 - \left(\frac{f_c}{f}\right)^2}}} \quad \text{--- (36)} \qquad \text{[Same as Rectangular waveguide]}$$

Similarly, $$u_p = \frac{u'}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}} \quad -(37)$$

$$\lambda = \frac{\lambda'}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}} \quad -(38)$$

TM mode in cylindrical waveguide

→ For TM mode, $H_z = 0$, $E_z \neq 0$

→ So we have to solve ~~the~~ for E_z from the wave eqn $\nabla^2 E_z + k^2 E_z = 0$, in cylindrical Co-ordinate System.

→ If the wave travels in z-direction, then

$$E_z = e_z(\rho, \phi) \cdot e^{-j\beta z} \quad -(39)$$ [Similar to TE mode]

and $$\frac{\partial^2 E_z}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial E_z}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 E_z}{\partial \phi^2} + \frac{\partial^2 E_z}{\partial z^2} + k^2 E_z = 0 \quad -(40)$$

→ Taking the similar approach to that of TE mode [Replace H_z by E_z]

The general solution of the above eqn

$$E_z(\rho, \phi, z) = (A \sin n\phi + B \cos n\phi) \cdot J_n(k_c \rho) \cdot e^{-j\beta z} \quad -(41)$$

→ The difference between the TE solution and present solution is that the boundary conditions.

$$E_z = 0, \quad \text{at } \rho = a$$

Then we have

$$J_n(K_c \rho) = 0$$

$$\boxed{K_c = \frac{P_{nm}}{a}} \quad -(42)$$

[Remember in TE

$K_c = \frac{P'_{nm}}{a}$]

where P_{nm}, m^{th} root of $J_n(x)$.

$$\text{Cut off freq } \boxed{(f_c) = \frac{P_{nm}}{2\pi a\sqrt{\mu\epsilon}}} \quad -(43)$$

n ↓ \ m →	1	2	3
0	2.405	5.520	8.654
1	3.832	7.016	10.174
2	5.135	8.417	11.620

P_{01} is lowest = 2.405.

∴ Dominant mode ~~for~~ $\boxed{TM_{01}}$

$$\boxed{f_c = \frac{2.405}{2\pi a\sqrt{\mu\epsilon}}} \quad -(44)$$

$$\beta = \beta' \sqrt{1 - \left(\frac{f_c}{f}\right)^2} \quad -(45)$$

$$u_p = \frac{u'}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}} \quad -(46)$$

Same as that of TE

$$\lambda_c = \frac{2\pi a}{P_{nm}} = \frac{2\pi a}{2.405} \quad -(47)$$

Since $p'_{11} = 1.841$ and $p_{01} = 2.405$

the overall dominant mode in circular waveguide is

$$\boxed{TE_{11}}$$

$$E_z = (A \sin n\phi + B \cos n\phi)\, J_n(k_c \rho)\, e^{-j\beta z} \quad -48(a)$$

The other field components can be found out using eqn 12, 13, 14, 15, with $\underline{H_z = 0}$ [TM mode]

$$E_\rho = \frac{-j\beta}{k_c^2} (A \sin n\phi + B \cos n\phi)\, J_n'(k_c \rho)\, e^{-j\beta z} \quad (b)$$

$$E_\phi = \frac{-j\beta n}{k_c^2 \rho} [A \cos n\phi - B \sin n\phi]\, J_n(k_c \rho)\, e^{-j\beta z} \quad (c)$$

$$H_\rho = \frac{j\omega \epsilon n}{k_c^2 \rho} [A \cos n\phi - B \sin n\phi]\, J_n(k_c \rho)\, e^{-j\beta z} \quad (d)$$

$$H_\phi = \frac{-j\omega \epsilon}{k_c^2} [A \sin n\phi + B \cos n\phi]\, J_n'(k_c \rho)\, e^{-j\beta z} \quad (e)$$

48

Wave impedance

$$Z_{TM} = \frac{E_\rho}{H_\phi} = \frac{\beta}{\omega \epsilon} = -\frac{E_\phi}{H_\rho}$$

$$Z_{TM} = \frac{\beta}{\frac{k}{\sqrt{\mu\epsilon}} \cdot \epsilon} = \frac{\beta}{k \cdot \sqrt{\frac{\epsilon}{\mu}}} = \frac{\beta}{k \cdot \frac{1}{\eta'}}$$

$(\because k = \omega\sqrt{\mu\epsilon} \Rightarrow \omega = \frac{k}{\sqrt{\mu\epsilon}}$

$(\because \eta' = \sqrt{\frac{\mu}{\epsilon}}$

$$\Rightarrow \boxed{Z_{TM} = \frac{\eta' \beta}{k}} \quad -(49)$$

$$Z_{TM} = \frac{\eta' \times \cancel{\beta'}\sqrt{1-\left(\frac{f_c}{f}\right)^2}}{\cancel{K}}$$

$$\Rightarrow Z_{TM} = \eta' \sqrt{1-\left(\frac{f_c}{f}\right)^2} \quad \text{—— (50)}$$

Similar to Rectangular waveguide.

$$Z_{TE} \times Z_{TM} = (\eta')^2 \quad \text{—— (51)}$$

Note :-

1) For Rectangular wave guide, $\boxed{TE_{10} \text{ \& } TM_{11}}$ are dominant mode.
2) For Circular " " , $\boxed{TE_{11} \text{ \& } TM_{01}}$ are dominant mode

ex :- 1) Find the cutoff frequencies of the first two propagating modes of a Teflon-filled circular waveguide with a = 0.5 cm. If the interior of the guide is gold plated, calculate the overall loss in dB for a 30 cm length operating at 14 GHz.

pozar

[For Teflon, $\epsilon_r = 2.08$, $\tan\delta = 0.0004$

$\sigma_{Gold} = 4.1 \times 10^7 \frac{S}{m}$]

Ans : First 2 propagating modes of a circular waveguides are TE_{11} & TM_{01}.

$$\cancel{TE_{11} : f_c = \frac{2\pi a}{1.841} = \frac{2\pi \times 0.5 \times 10^{-2}}{1.841}}$$

$$TE_{11} : f_c = \frac{p'_{nm}}{2\pi a\sqrt{\mu\epsilon}} = \frac{1.841}{2\pi a\sqrt{\mu_0 \epsilon_0 \epsilon_r}} \quad \left(\because \mu = \mu_0, \ \epsilon_r = \frac{\epsilon}{\epsilon_0} \right)$$

$$\Rightarrow f_c = \frac{1.841 \times c}{2\pi a \sqrt{\epsilon_r}} \qquad \left(\because \frac{1}{\sqrt{\mu_0 \epsilon_0}} = c\right)$$

$$= \frac{1.841 \times 3\times10^8}{2\pi \times 0.5\times10^{-2} \times \sqrt{2.08}}$$

$$\boxed{f_c = 12.189 \text{ GHz}}$$

TM_{01} :
$$f_c = \frac{2.405 \times c}{2\pi a \sqrt{\epsilon_r}}$$

$$= \frac{2.405 \times 3\times10^8}{2\pi \times 0.5\times10^{-2} \sqrt{2.08}}$$

$$f_c = 15.92 \text{ GHz}$$

$$\rightarrow K = \omega\sqrt{\mu\epsilon} = 2\pi f \sqrt{\mu_0 \epsilon_0 \epsilon_r} = \frac{2\pi f}{c}\sqrt{\epsilon_r} \qquad \left(\because \sqrt{\mu_0\epsilon_0} = \frac{1}{c}\right)$$

$$K = \frac{2\pi \times 14\times10^9 \sqrt{2.08}}{3\times10^8} = 422.9 \text{ m}^{-1}$$

$$\rightarrow \beta = \beta' \sqrt{1 - \left(\frac{f_c}{f}\right)^2} = K\sqrt{1-\left(\frac{f_c}{f}\right)^2} = 422.9 \sqrt{1 - \left(\frac{12.19}{14}\right)^2}$$

$$\Rightarrow \boxed{\beta = 208 \text{ m}^{-1}}$$

$$\alpha_d = \frac{k^2 \tan\delta}{2\beta} = \frac{(422.9)^2 \times 0.0004}{2\times 208} = 0.172 \text{ Np/m}$$

$$= 1.49 \text{ dB/m}.$$

$$R_s = \sqrt{\frac{\omega\mu}{2\sigma}} = \sqrt{\frac{\omega\mu_0}{2\sigma}} = \sqrt{\frac{2\pi \times 14 \times 10^9 \times 4\pi \times 10^{-7}}{2 \times 4.1 \times 10^7}}$$

$$R_s = .0367\ \Omega$$

$$\alpha_c = \frac{R_s}{a k \eta \beta}\left(k_c^2 + \frac{k^2}{p_{11}'^2 - 1}\right) = \frac{R_s}{a k \sqrt{\frac{\mu}{\epsilon}}\,\beta}\left(k_c^2 + \frac{k^2}{p_{11}'^2 - 1}\right)$$

$$\alpha_c = \frac{0.0367 \times \sqrt{2.08}}{0.5 \times 10^{-2} \times 422.9 \times 377 \times 208} \times \left[\left(\frac{1.841}{0.5 \times 10^{-2}}\right)^2 + \frac{(422.9)^2}{1.841^2 - 1}\right]$$

$$\left[\because \sqrt{\frac{\mu}{\epsilon}} = \sqrt{\frac{\mu_0}{\epsilon_0 \epsilon_r}} = \sqrt{\frac{\mu_0}{\epsilon_0}} \cdot \frac{1}{\sqrt{\epsilon_r}} = 377 \times \frac{1}{\sqrt{2.08}} \qquad k_c = \frac{p_{nm}'}{a} = \frac{1.841}{0.5 \times 10^{-2}}\right]$$

$$\alpha_c = 0.0672\ \text{Np/m} = 0.583\ \text{dB/m}.$$

Total attenuation $= \alpha_c + \alpha_d = 0.583 + 1.49$

$$= 2.073\ \text{dB/m}.$$

Then loss in 30 ~~meter~~ cm length of guide is

$$\text{attenuation (dB)} = 30 \times 10^{-2}\ \text{m} \times 2.073\ \frac{\text{dB}}{\text{m}}$$

$$= 0.62\ \text{dB}.$$

2) ABUT-2010

For a 50 cm length of Cylindrical wave guide operating at 13 GHz with $a = 0.5$ cm $\epsilon_r = 2.25$. Find the Propagating mode of TE If the guide is silver plated and dielectric loss tangent is 0.001. Also Calculate

(i) Wave number for Propagating mode

(ii) Propagation Constant

(iii) Surface Resistance, (Given $\sigma = 6.17 \times 10^7 \frac{S}{m}$).

(iv) Attenuation (α_d) in dB due to ~~dielectric~~ dielectric loss.

Given $P'_{01} = 3.832$, $P'_{11} = 1.841$, $P'_{21} = 3.054$

Ans:- For TE mode

$$(f_c) = \frac{P'_{nm}}{2\pi a\sqrt{\mu\epsilon}} = \frac{P'_{nm}}{2\pi a\sqrt{\mu_0\epsilon_0\epsilon_r}} = \frac{P'_{nm} \cdot c}{2\pi a\sqrt{\epsilon_r}}$$

$$(f_c)_{01} = \frac{P'_{01} \times 3\times10^8}{2\pi \times 0.5 \times 10^{-2} \times \sqrt{2.25}} = \frac{3.832 \times 3 \times 10^8}{2\pi \times 0.5 \times 10^{-2} \times \sqrt{2.25}} = 24.39 \text{ GHz}$$

$f < (f_c)_{01}$, TE 01 Not possible.

13 GHz < 24.39 GHz.

$$(f_c)_{21} = \frac{P'_{21} \times 3\times10^8}{2\pi \times 0.5 \times 10^{-2} \times \sqrt{2.25}} = \frac{3.054 \times 3\times10^8}{2\pi \times 0.5 \times 10^{-2} \times \sqrt{2.25}} = 19.44 \text{ GHz}$$

$f < (f_c)_{21}$, TE_{21} is Not possible.

$$(f_c)_{11} = \frac{1.841 \times 3\times10^8}{2\pi \times 0.5\times10^{-2}\sqrt{2.25}} = 11.71 \text{ GHz}.$$

$$f > (f_c)_{11}$$

$13 \text{ GHz} > (f_c)_{11}$. Since operating freq greater than cutoff freq. (TE_{11}) mode is propagating.

(i) Wave number $= k = \omega\sqrt{\mu\epsilon} = \omega\sqrt{\mu_0\epsilon_0\epsilon_r}$

$$\Rightarrow k = 2\pi f\sqrt{\mu_0\epsilon_0}\,\sqrt{2.25}$$

$$= \frac{2\pi \times 13\times10^9}{3\times10^8}\sqrt{2.25} \qquad \left(\because \sqrt{\mu_0\epsilon_0} = \frac{1}{c}\right)$$

$$\boxed{k = 408.40} \ m^{-1}$$

(ii) $\beta = \beta'\sqrt{1-\left(\frac{f_c}{f}\right)^2} = k\sqrt{1-\left(\frac{f_c}{f}\right)^2}$

$$= 408.40\sqrt{1-\left(\frac{11.71}{13}\right)^2}$$

$$\beta = 177.36 \ m^{-1} \text{ or rad}/m^{-1}.$$

or $\beta = \sqrt{k^2 - k_c^2} = \sqrt{(408.40)^2 - \left(\frac{1.841}{0.5\times10^{-2}}\right)^2}$

$$= 176.68 \ m^{-1}$$

(iii) $R_s = \sqrt{\frac{\omega\mu}{2\sigma}} = \sqrt{\frac{\omega\mu_0}{2\sigma}} = \sqrt{\frac{2\pi\times13\times10^9\times4\pi\times10^{-7}}{2\times6.17\times10^7}} = 0.029\,\Omega$

$\therefore R_s = 0.029\,\Omega$

(iv) $$\alpha_a = \frac{K^2 \tan\delta}{2\beta}$$

$$\alpha_a = \frac{(408-40)^2 \times 0.001}{2 \times 176.68} = 0.47\ Np/m.$$

(Ans).

—o—