(20)

'Ch-2'

Frequency Domain Analysis of Signals and Systems

The signals which are used to transmit the information over a communication channel are called information-bearing signals.

In the transmission of an information-bearing signal in a communication channel, the shape of the signal is changed or distorted by channel i.e the received signal is not the exact replica of the i/p signal.

The communication channel is an example of a system; i.e an entity that produces an o/p signal when excited by an i/p signal. A large number of communication channel can be modelled closely by a subclass of systems called linear systems.

Linear System:-

A system that satisfies the superposition principle is said to be a linear system.

Super position principle states that 'The response to a weight sum of i/p s is equal to the weighted sum of o/ps.

e.g: 1) Let the response to $x_1(t) + x_2(t)$ is $y_1(t) + y_2(t)$

2) The response to $a\,x(t)$ will be $a\,y_1(t)$ where 'a' is complex const.

ex: $y(t) = x^2(t)$, $y(t) = \sin[x(t)]$ } Not linear.

SCIENCE AND ENGINEERING NOTES

Fourier Series :-

Fourier Series is used to get frequency spectrum of a time domain signal, when signal is periodic function of time.

→ Fourier transform is applicable to both periodic & non periodic signals.

→ Fourier Series exists only when the function $x(t)$ satisfies the Dirichlet's condition.

Dirichlet's condⁿ :-

1) $x(t)$ is absolutely integrable over its period

ie
$$\int_0^{T_0} |x(t)|\, dt < \infty$$

2. The number of maxima and minima of $x(t)$ in each period is finite.

3. The number of discontinuities of $x(t)$ in each period is finite.

Fourier Series expansion are of 3 types:-

(i) Trigonometric Fourier Series

(ii) Polar Fourier Series

(iii) Complex Fourier Series.

SCIENCE AND ENGINEERING NOTES

Trigonometric Fourier Series:-

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) \right]$$

where

$$a_0 = \frac{1}{T} \int_{t_0}^{t_0+T} x(t)\, dt$$

$$a_n = \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \cos(n\omega_0 t)\, dt$$

$$b_n = \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \sin(n\omega_0 t)\, dt$$

where $T =$ fundamental time period.

If the interval is $[-\infty, \infty]$

$$a_0 = \frac{1}{T} \int_{-\infty}^{\infty} x(t)\, dt$$

$$a_n = \frac{2}{T} \int_{-\infty}^{\infty} x(t) \cos n\omega_0 t\, dt$$

$$b_n = \frac{2}{T} \int_{-\infty}^{\infty} x(t) \sin n\omega_0 t\, dt .$$

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Polar Fourier Series:-

Derivation:-

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) \right]$$

$$\Rightarrow x(t) = a_0 + \sum_{n=1}^{\infty} \sqrt{a_n^2+b_n^2} \left[\frac{a_n}{\sqrt{a_n^2+b_n^2}} \cos(n\omega_0 t) + \frac{b_n}{\sqrt{a_n^2+b_n^2}} \sin(n\omega_0 t) \right]$$

~~$= a_0 +$~~ Substitute $\frac{a_n}{\sqrt{a_n^2+b_n^2}} = \cos\phi_n$ and $\frac{b_n}{\sqrt{a_n^2+b_n^2}} = \sin\phi_n$

then, $\tan\phi_n = \frac{b_n}{a_n}$

$$\Rightarrow \phi_n = \tan^{-1}\left(\frac{b_n}{a_n}\right)$$

$\therefore$ The above eq^n becomes,

$$\Rightarrow x(t) = a_0 + \sum_{n=1}^{\infty} \sqrt{a_n^2+b_n^2} \left[\cos n\omega_0 t \cdot \cos\phi_n + \sin n\omega_0 t \sin\phi_n \right]$$

$$= a_0 + \sum_{n=1}^{\infty} \sqrt{a_n^2+b_n^2} \cos(n\omega_0 t - \phi_n)$$

Now put $a_0 = C_0$, $C_n = \sqrt{a_n^2+b_n^2}$

$$\therefore \boxed{\begin{aligned} & x(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t - \phi_n) \\ & \text{and } \phi_n = \tan^{-1}\left(\frac{b_n}{a_n}\right) \\ & \text{and } C_0 = a_0, \quad C_n = \sqrt{a_n^2+b_n^2} \end{aligned}}$$

C_n = Spectral amplitudes

ϕ_n = Phase information / Phase spectrum.

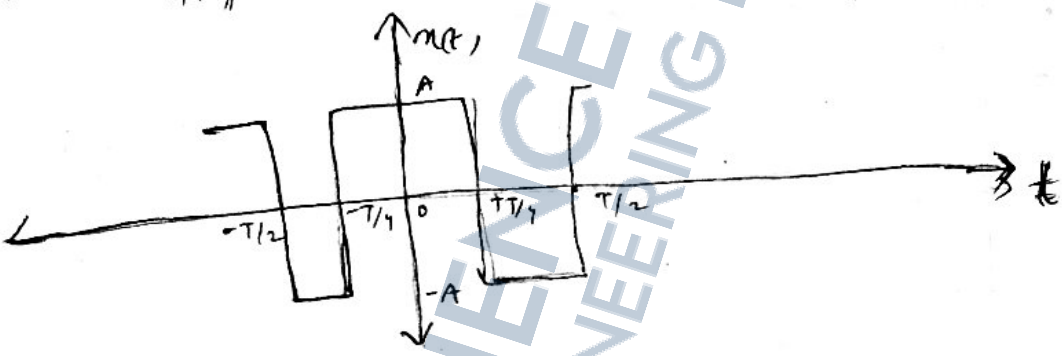
Complex Fourier exponential series:-

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

where $$C_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jn\omega_0 t}\, dt$$

Also $$C_n = \frac{a_n - j b_n}{2}$$ → [Even & odd f^n explanation in page 30]

Ex-1) Find the fourier series representation of the figure shown below. (Given time period = T)



Ans:- Since the function is symmetrical about y-axis, it is a even function.

$$a_0 = \frac{1}{T}\int_{t_0}^{t_0+T} x(t).\, dt$$

$$a_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t) \cos n\omega_0 t\, dt$$

$$b_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t) \sin n\omega_0 t\, dt$$

Here $$b_n = \frac{2}{T}\int_{-T/2}^{T/2} (\text{even } f^n).\, \sin n\omega_0 t\, dt$$ ← odd f^n

$$= \frac{2}{T}\int_{-T/2}^{T/2} (\text{odd } f^n).\, dt$$

$$\boxed{b_n = 0}$$

SCIENCE AND ENGINEERING NOTES

2(b) ⊙ 2(b)

~~$x(t)$~~ $a_0 = \frac{1}{T}\int_{-T/2}^{T/2} (\text{even } f^n)\, dt = \frac{2}{T}\int_0^{T/2} x(t)\, dt$

$\Rightarrow \frac{2}{T}\left[\int_0^{T/4} A\,dt + \int_{T/4}^{T/2} -A\,dt\right] = \boxed{a_0 = 0}$ = Symmetrical about horizontal axis, avg. area is zero.

$\Rightarrow \frac{2A}{T}\left(\frac{T}{4}-\frac{T}{4}\right) = \frac{2A}{T}\cdot 0 = 0$ $= 0$

+ve area $= A \times \frac{T}{2} = \frac{AT}{2}$

-ve area

$= \cancel{2} \times (-A) \times \frac{T}{\cancel{4}\,2}$

$= -\frac{AT}{2}$

$a_0 = \frac{AT}{2} - \frac{AT}{2}$

$a_0 = 0$

$\therefore x(t) = \sum_{n=1}^{\infty} a_n \cos n\omega_0 t$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t) \cos n\omega_0 t\, dt$$

$$x(t) = \begin{cases} -A, & \text{for } -\frac{T}{2} < t < -T/4 \\ A, & \text{for } -\frac{T}{4} < t < T/4 \\ -A, & \text{for } T/4 < t < T/2 \end{cases}$$

$$\therefore a_n = \frac{2}{T}\left[\int_{-T/2}^{-T/4} (-A)\cos n\omega_0 t\, dt + \int_{-T/4}^{T/4} A \cos n\omega_0 t\, dt + \int_{T/4}^{T/2} (-A)\cos n\omega_0 t\, dt\right]$$

$$= -\frac{2A}{T}\left.\frac{\sin n\omega_0 t}{n\omega_0}\right]_{-T/2}^{-T/4} + \frac{2A}{T}\left.\frac{\sin n\omega_0 t}{n\omega_0}\right]_{-T/4}^{T/4}$$

$$-\frac{2A}{T}\cdot\left.\frac{\sin n\omega_0 t}{n\omega_0}\right]_{T/4}^{T/2}$$

$$= \frac{2A}{T.n\omega_0}\left[-\sin n\omega_0 t\Big]_{-T/2}^{-T/4} + \sin n\omega_0 t\Big]_{-T/4}^{T/4} - \sin n\omega_0 t\Big]_{T/4}^{T/2}\right]$$

$$= \frac{2A}{n\omega_0 T}\left[\left(\sin n\omega_0\left(-\frac{T}{2}\right) - \sin n\omega_0\left(\frac{-T}{4}\right)\right) + \left(\sin n\omega_0 \frac{T}{4} - \sin n\omega_0\left(-\frac{T}{4}\right)\right)\right]$$

2(b) (c)

$$- \left(\sin n\omega_0 \frac{T}{2} - \sin n\omega_0 \frac{T}{4} \right) \Big]$$

$$= \frac{2A}{n\omega_0 T}\Big[-\sin n\omega_0\frac{T}{2} + \sin n\omega_0\frac{T}{4} + \sin n\omega_0 \frac{T}{4} + \sin n\omega_0\frac{T}{4} - \sin n\omega_0\frac{T}{2} + \sin n\omega_0 \frac{T}{4}\Big]$$

$$= \frac{2A}{n\omega_0 T}\left[4\sin n\omega_0 \frac{T}{4} - 2\sin n\omega_0\frac{T}{2}\right]$$

$$a_n = \frac{8A}{n\omega_0 T}\sin\left(\frac{n\omega_0 T}{4}\right) - \frac{4A}{n\omega_0 T}\sin\left(\frac{n\omega_0 T}{2}\right)$$

Putting, $\omega_0 = \frac{2\pi}{T}$

$$a_n = \frac{8A}{n\omega_0 T}\sin\left(n\cdot\frac{\cancel{2}\pi}{\cancel{T}}\cdot\frac{\cancel{T}}{\cancel{4}\,2}\right) - \frac{4A}{n\omega_0 T}\sin\left(\frac{n}{\cancel{2}}\cdot\frac{\cancel{2}\pi}{\cancel{T}}\cdot\cancel{T}\right)$$

$$a_n = \frac{8A}{n\omega_0 T}\sin\left(\frac{n\pi}{2}\right) - \xcancel{\frac{4A}{n\omega_0 T}\sin(n\pi)} \rightarrow 0$$

$$a_n = \frac{8A}{n\omega_0 T}\sin\left(\frac{n\pi}{2}\right)$$

$$= \frac{\cancel{8}\,4A\times\cancel{T}}{n\times\cancel{2}\pi\times\cancel{T}}\sin\left(\frac{n\pi}{2}\right)$$

$$a_n = \frac{4A}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

OR

Due to even function,

$$a_n = \frac{4}{T}\int_0^{T/2} x(t)\cos n\omega_0 t\, dt$$

$$= \frac{4}{T}\left[\int_0^{T/4} A\cos n\omega_0 t\,dt + \int_{T/4}^{T/2} -A\cos n\omega_0 t\,dt\right]$$ ... Solving

$$= \frac{4A}{n\pi}\cdot\sin\left(\frac{n\pi}{2}\right)$$

$$x(t) = \sum_{n=1}^{\infty} a_n\cos n\omega_0 t = \sum_{n=1}^{\infty}\frac{4A}{n\pi}\sin\left(\frac{n\pi}{2}\right)\cdot\cos n\omega_0 t$$

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$$x(t) = \frac{4A}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \cdot \sin\left(\frac{n\pi}{2}\right) \cdot \cos n\omega_0 t$$

$$= \frac{4A}{\pi}\left[\cos\omega_0 t + \frac{1}{2}\sin(\pi)\cos 2\omega_0 t + \frac{1}{3}\sin\left(\frac{3\pi}{2}\right)\cdot\cos 3\omega_0 t + \frac{1}{4}\cdot 0 + \frac{1}{5}\sin\left(\frac{5\pi}{2}\right)\cos 5\omega_0 t\right] + \text{---}$$

$$x(t) = \frac{4A}{\pi}\left(\cos\omega_0 t - \frac{1}{3}\cos 3\omega_0 t + \frac{1}{5}\cos 5\omega_0 t \text{---}\right)$$

(Ans)

2) Polar form :-

$$x(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t - \phi_n)$$

$C_0 = a_0$, $C_n = \sqrt{a_n^2 + b_n^2}$, $\phi_n = \tan^{-1}\left(\frac{b_n}{a_n}\right)$

~~$A_0 = 0$~~, Since $x(t)$ is even function

~~$C_0 = 0$~~, $b_n = 0$

$a_0 = 0$ due to symmetrical about horizontal axis, avg area $= 0$.

$C_0 = 0$, $C_n = \sqrt{a_n^2} = a_n$.

$$x(t) = 0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t - 0)$$

$$x(t) = \sum_{n=1}^{\infty} a_n \cos n\omega_0 t$$

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$$a_n = \frac{4A}{n\pi} \sin\frac{n\pi}{2}$$

$$\therefore \boxed{x(t) = \sum_{n=1}^{\infty} \frac{4A}{n\pi} \sin\frac{n\pi}{2} \cdot \cos n\omega_0 t}$$

3) <u>Exponential Fourier Series</u>

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

where

$$C_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t) \cdot e^{-jn\omega_0 t}$$

$$C_n = \frac{1}{T}\left[\int_{-T/2}^{-T/4} (-A) \cdot e^{-jn\omega_0 t}\, dt + \int_{-T/4}^{T/4} A \cdot e^{-jn\omega_0 t}\, dt + \int_{T/4}^{T/2} -A\, e^{-jn\omega_0 t}\, dt\right]$$

$$= \frac{1}{T}\left[-A \times \left[\frac{e^{-jn\omega_0 t}}{-jn\omega_0}\right]_{-T/2}^{-T/4} + A \cdot \left[\frac{e^{-jn\omega_0 t}}{-jn\omega_0}\right]_{-T/4}^{T/4} - A \left[\frac{e^{-jn\omega_0 t}}{-jn\omega_0}\right]_{T/4}^{T/2}\right]$$

= Solving we get

$$C_n = \frac{4A}{2n\pi} \sin\frac{n\pi}{2}$$

2(a) ⊙

OR $C_n = \dfrac{a_n - jb_n}{2}$ ∵ $b_n = 0$.

$C_n = \dfrac{a_n}{2}$ ∵ $a_n = \dfrac{4A}{n\pi}$

$= \dfrac{4A}{2n\pi} \sin\dfrac{n\pi}{2}$

$$x(t) = \sum_{n=-\infty}^{\infty} C_n \, e^{jn\omega_0 t}$$

$$\cancel{= C_0 + 2\sum_{n=-\infty}^{\infty}}$$

$$= C_0 + 2\sum_{n=1}^{\infty} \frac{4A}{2n\pi}\sin\frac{n\pi}{2}\, e^{jn\omega_0 t}$$

$$= 0 + \sum_{n=1}^{\infty} \frac{4A}{n\pi}\sin\frac{n\pi}{2}\, e^{jn\omega_0 t}$$

$$= \sum_{n=1}^{\infty} \frac{4A}{n\pi} \cdot \sin\frac{n\pi}{2} \cdot \left[\cos n\omega_0 t + j\sin n\omega_0 t\right]$$

∴ In exponential form,

$$\boxed{x(t) = \sum_{n=-\infty}^{\infty} \frac{4A}{2n\pi} \sin\frac{n\pi}{2} \cdot e^{jn\omega_0 t}}$$

To verify (∵ $C_0 = a_0/2 = 0/2 = 0$) $= \sum_{n=-\infty}^{-1} \dfrac{4A}{-2n\pi} \cdot \left(-\sin\dfrac{n\pi}{2}\right) \cdot e^{-jn\omega_0 t} + C_0 + \sum_{n=1}^{\infty} \dfrac{4A}{2n\pi} \sin\dfrac{n\pi}{2} \cdot e^{jn\omega_0 t}$

$C_0 = 0$ $= \sum_{n=1}^{\infty} \dfrac{4A}{2n\pi} \sin\dfrac{n\pi}{2} \left[\dfrac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2}\right] \times 2$ (divide 2 & multiply 2)

$$\boxed{x(t) = \sum_{n=1}^{\infty} \frac{4A}{n\pi} \sin\frac{n\pi}{2} \cdot \cos n\omega_0 t}$$

⟨Same as polar⟩

SCIENCE AND ENGINEERING NOTES

Symmetry cond^n :-

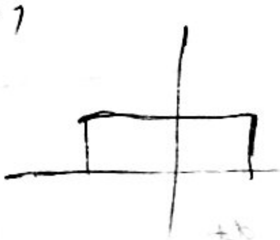
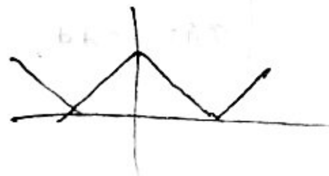
1/ If function $x(t)$ is even

(i) $x(t) = x(-t)$.

ex: t^2, $\cos t$, $t \sin t$

(ii) Symmetric about y-axis.

(iii)

 ,  , 

(iv) $$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,dt = \frac{2}{T}\int_0^{T/2} x(t)\,dt$$

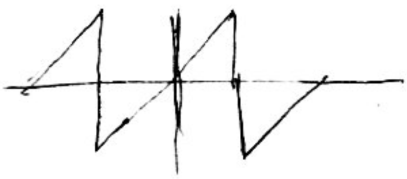
$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos n\omega_0 t\,dt = \frac{4}{T}\int_0^{T/2} x(t)\cos n\omega_0 t\,dt$$

$$b_n = \frac{1}{T}\int_{-T/2}^{T/2} \overbrace{\underset{\text{even}}{x(t)}\,\underset{\text{odd}}{\sin \omega_0 t}}^{\text{odd}}\,dt = 0$$

2/ If f^n $x(t)$ is odd

(i) $x(t) = -x(t)$ (They are symmetrical about origin)

(ii) ex :- $\sin t$, t, t^3, $t\cos t$.

 ,  , 

(iii) $$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,dt = 0.$$

$$a_n = \frac{1}{T}\int_{-T/2}^{T/2} \overbrace{\underset{\text{odd}}{x(t)}\,\underset{\text{even}}{\cos n\omega_0 t}}^{\text{odd}}\,dt = 0$$

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$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} \underbrace{\overset{\text{odd}}{x(t)}\ \overset{\text{odd}}{\sin n\omega_0 t}}_{\text{even}}\, dt = \frac{4}{T}\int_{0}^{T/2} x(t)\sin n\omega_0 t\, dt$$

$x(t)$ even; $a_0 = \frac{2}{T}\int_0^{T/2} x(t)\,dt$, $a_n = \frac{4}{T}\int_0^{T/2} x(t)\cos n\omega_0 t\,dt$

$b_n = 0$

$x(t)$ odd; $a_0 = 0$, $a_n = 0$

$b_n = \frac{4}{T}\int_0^{T/2} x(t)\sin n\omega_0 t\,dt$

3) Sum or product of two or more even function is even.

4) Sum of 2 or more odd function is odd.
product of 2 odd function is even.

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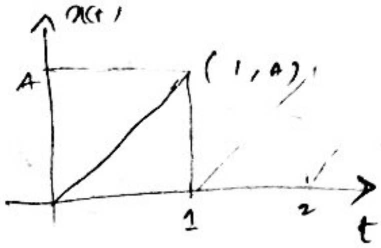
Ex:-2 Expand a function $x(t)$ shown in figure, over the interval $(0,1)$ by using

(i) Trigonometric fourier series.

(ii) Polar fourier series

(iii) Exponential fourier series.



Ans:- Defn of f^n:-

$$y = mx$$

$$x(t) = \left(\frac{A-0}{1-0}\right) \cdot t$$

$$x(t) = At$$

To find

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos n\omega_0 t + b_n \sin n\omega_0 t\right]$$

$$a_0 = \frac{1}{T}\int_0^T x(t)\, dt$$

$$= \frac{1}{T}\int_0^T At\, dt = \frac{A}{T} \cdot \left.\frac{t^2}{2}\right]_0^T$$

$$= \frac{A}{2\cancel{T}} \cdot T^{\cancel{2}} = \frac{AT}{2}.$$

Given time period $T = 1$.

$$\boxed{a_0 = \frac{A}{2}}$$

$$a_n = \frac{2}{T}\int_0^T x(t) \cos n\omega_0 t\, dt$$

$$= \frac{2}{T}\int_0^T At \cos n\omega_0 t\, dt$$

$$= \frac{2A}{\cancel{T}}\int_0^1 t \cos n\omega_0 t\, dt$$

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$$a_n = \frac{2A}{T}\left[t \cdot \frac{\sin n\omega_0 t}{n\omega_0} - \frac{1}{n\omega_0}\cdot\left(-\frac{\cos n\omega_0 t}{n\omega_0}\right)\right]_0^T$$

$$= \frac{2A}{T}\left[t\,\frac{\sin n\omega_0 t}{n\omega_0} + \frac{1}{n^2\omega_0^2}\cos n\omega_0 t\right]_0^T$$

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{1} = 2\pi. \qquad T = 1$$

$$= 2A\left[t\,\frac{\sin 2n\pi t}{2n\pi} + \frac{1}{n^2 4\pi^2}\cos 2\pi n t\right]_0^1$$

$$= 2A\left[\frac{T}{2n\pi}\cdot\sin 2n\pi + \frac{1}{n^2 4\pi^2}\cos 2n\pi - 0 - \frac{1}{n^2 4\pi^2}\cos 0\right]$$

$$= 2A\left[0 + \cancel{\frac{1}{n^2 4\pi^2}} - 0 - \cancel{\frac{1}{n^2 4\pi^2}}\right]$$

$$\boxed{a_n = 0}$$

$$b_n = \frac{2}{T}\int_0^T x(t)\sin n\omega_0 t\,dt$$

$$T = 1, \quad \sin n\omega_0 t = \sin n.\frac{2\pi}{T}t$$

$$\sin n\omega_0 t = \sin 2n\pi t$$

$$b_n = 2\int_0^1 At\cdot\sin 2n\pi t\,dt$$

$$= 2A\int_0^1 t\sin 2n\pi t\,dt$$

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$$b_n = 2A \left[t \cdot \left(-\frac{\cos 2n\pi t}{2n\pi} \right) - (1) \cdot \left(\frac{-1}{2n\pi} \right) \cdot \left(\frac{\sin 2n\pi t}{2n\pi} \right) \right]_0^1$$

$$\left(\because \int U \cdot V = U V_1 - U' V_2 + U'' V_3 - U''' V_4 \cdots \right]$$

$$b_n = 2A \left[\frac{-t}{2n\pi} \cos 2n\pi t + \frac{1}{4 n^2 \pi^2} \sin 2n\pi t \right]_0^1$$

$$= 2A \left[\frac{-1}{2n\pi} \cos 2n\pi + \frac{1}{4n^2\pi^2} \cdot \cancel{\sin 2n\pi}^{\,0} - 0 - 0 \right]$$

$$= \frac{-\cancel{2}A}{\cancel{2}n\pi} \cdot \cos 2n\pi$$

$$= \frac{-A}{n\pi} \cdot 1$$

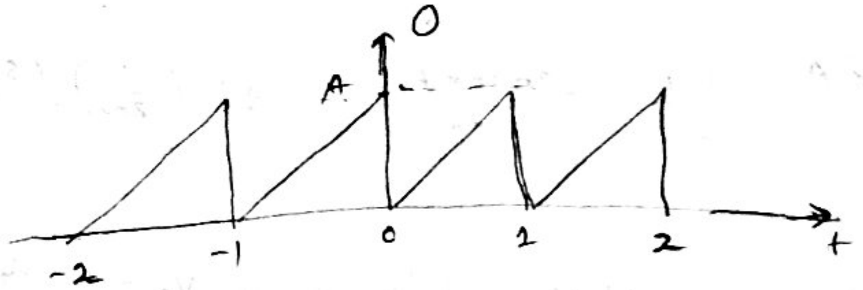
$$\boxed{b_n = -\frac{A}{n\pi}}$$

$$\therefore x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

$$= \frac{A}{2} + \sum_{n=1}^{\infty} \left(-\frac{A}{n\pi} \right) \cdot \sin n\omega_0 t \quad \Big| \because a_n = 0.$$

$$= \frac{A}{2} + \sum_{n=1}^{\infty} \frac{-A}{n\pi} \cdot \sin 2n\pi t$$

$$\therefore \boxed{x(t) = \frac{A}{2} - \frac{A}{\pi} \sum_{n=1}^{\infty} \frac{\sin 2n\pi t}{n}}$$ where $0 < t < 1$.



This function also has the same Fourier series representation, because, it has also period 1 sec.

(ii) <u>Polar form</u> :-

$$x(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t - \phi_n)$$

Where $C_0 = a_0$, $C_n = \sqrt{a_n^2 + b_n^2}$, $\phi_n = \tan^{-1}\left(\frac{b_n}{a_n}\right)$

$a_0 = \frac{A}{2}$, $\Rightarrow C_0 = \frac{A}{2}$,

~~b_n~~ $a_n = 0$, $C_n = \sqrt{b_n^2} = b_n$, $(\because a_n = 0)$

$\phi_n = \tan^{-1}\left(\frac{b_n}{a_n}\right) = \tan^{-1}(\infty) = \frac{\pi}{2}$.

$n\omega_0 t = n \times \frac{2\pi}{T} t = 2n\pi t$ $(\because T = 1)$

$$\therefore x(t) = \frac{A}{2} + \sum_{n=1}^{\infty} b_n \cos\left(2n\pi t - \frac{\pi}{2}\right)$$

$$\boxed{x(t) = \frac{A}{2} + \sum_{n=1}^{\infty} \left(-\frac{A}{n\pi}\right) \cdot \cos\left(2n\pi t - \frac{\pi}{2}\right)}$$

$$= \frac{A}{2} + \sum_{n=1}^{\infty} \left(-\frac{A}{n\pi}\right) \cos\left(\frac{\pi}{2} - 2n\pi t\right)$$

$\because \cos(-\theta) = \cos\theta$

$$\boxed{x(t) = \frac{A}{2} + \sum_{n=1}^{\infty} -\frac{A}{n\pi} \sin 2n\pi t}$$

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3) (iii) **Exponential form:-**

$$x(t) = \sum_{n=-\infty}^{\infty} C_n \, e^{jn\omega_0 t}$$

where $C_n = \dfrac{a_n - jb_n}{2}$

$$C_n = \frac{0 - jb_n}{2} = \frac{-jb_n}{2}$$

$$= -\frac{j}{2} \times \left(-\frac{A}{n\pi}\right)$$

$$= \frac{jA}{2n\pi}$$

$$\boxed{x(t) = \sum_{n=-\infty}^{\infty} \frac{jA}{2n\pi} e^{jn\omega_0 t}}$$

To Verify

$$x(t) = \sum_{n=-\infty}^{-1} \frac{jA}{-2n\pi} e^{-jn\omega_0 t} + C_0 + \sum_{n=1}^{\infty} \frac{jA}{2n\pi} \cdot e^{jn\omega_0 t}$$

$$\cancel{= \sum_{n=1}^{\infty} \frac{jA}{2n\pi} e^{jn\omega_0 t} + C_0 + \sum_{n=1}^{\infty} \frac{jA}{2n\pi} \cdot e^{jn\omega_0 t}}$$

$$= \cancel{\sum_{n=1}^{\infty} \frac{(-1)jA}{2n\pi}} \left(\frac{e^{jn\omega_0 t} - e^{-jn\omega_0 t}}{2j} \right) + C_0$$

$$\cancel{= C_0 + \sum_{n=1}^{\infty} \left(\frac{-A}{n\pi}\right) \sin(n\omega_0 t)}$$

$$= \sum_{n=1}^{\infty} \frac{jA}{2n\pi} \left(e^{jn\omega_0 t} - e^{-jn\omega_0 t} \right) + C_0$$

$$= \sum_{n=1}^{\infty} \frac{jA}{2n\pi} \times 2j \left(\frac{e^{jn\omega_0 t} - e^{-jn\omega_0 t}}{2j} \right) + C_0$$

$$= \sum_{n=1}^{\infty} \frac{-A}{n\pi} \cdot \sin(n\omega_0 t) + C_0$$

$$\boxed{x(t) = \frac{A}{2} + \sum_{n=1}^{\infty} \frac{-A}{n\pi} \sin 2n\pi t}$$

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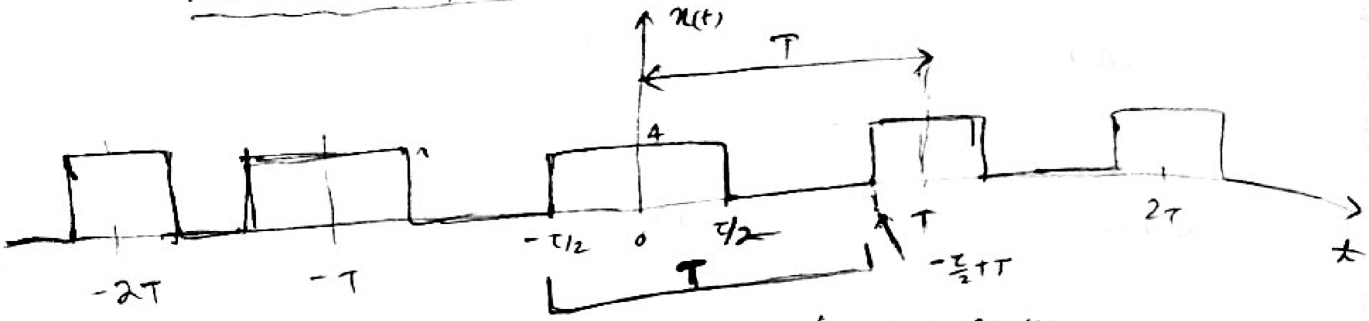
37 ⊙ * Similarly for impulse signal.

3) For the periodic Gate f^n



(i) Find the trigonometric fourier series.

(ii) Find the corresponding exponential fourier series.

Total Period = T

⊓ → 2 × τ/2 = τ

remaining = T − τ

i.e $(-\frac{\tau}{2}+T) + (\frac{\tau}{2})$

$-\frac{\tau}{2}+T-\frac{\tau}{2} = T-\tau$

Ans:

$$x(t) = \begin{cases} A, & -\frac{\tau}{2} < t < \frac{\tau}{2} \\ 0, & \frac{\tau}{2} < t < T-\frac{\tau}{2} \end{cases}$$

(i) $x(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos n\omega_0 t + b_n \sin n\omega_0 t]$

$$a_0 = \frac{1}{T}\int_{-\tau/2}^{\tau/2} x(t)\, dt + \int_{\tau/2}^{T-\frac{\tau}{2}} 0.\, dt = \frac{1}{T}\int_{-\tau/2}^{\tau/2} A\, dt$$

$$= \frac{1}{T} \cdot A \times t\Big]_{-\tau/2}^{\tau/2}$$

$$\boxed{a_0 = \frac{A\tau}{T}}$$

$$a_n = \frac{2}{T}\int_{-\tau/2}^{\tau/2} x(t) \cos n\omega_0 t\, dt + \int_{\tau/2}^{T-\frac{\tau}{2}} 0.\, dt$$

$$= \frac{2}{T}\int_{-\tau/2}^{\tau/2} A \cos n\omega_0 t\, dt$$

$$= \frac{2A}{T} \cdot \frac{\sin n\omega_0 t}{n\omega_0}\Big]_{-\tau/2}^{\tau/2}$$

$$= \frac{2A}{n\omega_0 T}\left[\sin \frac{n\omega_0 \tau}{2} - \sin\left(n\omega_0 \cdot \frac{-\tau}{2}\right)\right]$$

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$$= \frac{2A}{n\omega_0 T} \times 2 \cdot \sin\left(\frac{n\omega_0\tau}{2}\right) \times \left(\frac{\tau}{\tau}\right)$$ | multiplying & dividing τ

$$= \frac{2A\tau}{T} \frac{\sin\left(\frac{n\omega_0\tau}{2}\right)}{\left(\frac{n\omega_0\tau}{2}\right)}.$$

$$a_n = \frac{2A\tau}{T} \frac{\sin\left(\frac{n\omega_0\tau}{2}\right)}{\left(\frac{n\omega_0\tau}{2}\right)}.$$ | $\frac{n\omega_0\tau}{2} = x$.

$$a_n = \frac{2A\tau}{T} \frac{\sin x}{x} = \frac{2A\tau}{T} \cdot Sa(x)$$ | $Sa \rightarrow$ Sampling function.

~~But As~~ Since there is a even f^n, $\boxed{b_n = 0}$

$$\therefore x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t$$

$$\boxed{x(t) = \frac{A\tau}{T} + \frac{2A\tau}{T} \sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\omega_0\tau}{2}\right)}{\left(\frac{n\omega_0\tau}{2}\right)} \cdot \cos n\omega_0 t}$$

(ii) Exponential form:-

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$$C_n = \frac{a_n - jb_n}{2} = \frac{a_n}{2}$$ | $\because b_n = 0$.

$$= \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}A\tau}{T} \cdot \frac{\sin\left(\frac{n\omega_0\tau}{2}\right)}{\left(\frac{n\omega_0\tau}{2}\right)}$$

$$= \frac{A\tau}{T} \frac{\sin\left(\frac{n\omega_0\tau}{2}\right)}{\left(\frac{n\omega_0\tau}{2}\right)}$$

$$\therefore \boxed{x(t) = \sum_{n=-\infty}^{\infty} \frac{A\tau}{T} \cdot \frac{\sin\left(\frac{n\omega_0\tau}{2}\right)}{\left(\frac{n\omega_0\tau}{2}\right)} \cdot e^{jn\omega_0 t}}$$

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OR

$$C_n = \frac{1}{T}\int_{-\tau/2}^{\tau/2} x(t)\, e^{-jn\omega_0 t}\,dt + \int 0 \cdot dt$$

$$= \frac{1}{T}\int_{-\tau/2}^{\tau/2} A \cdot e^{-jn\omega_0 t}\,dt$$

$$= \frac{1}{T} \cdot A \cdot \left.\frac{e^{-jn\omega_0 t}}{-jn\omega_0}\right]_{-\tau/2}^{\tau/2}$$

$$= \frac{A}{-jn\omega_0 T} \times \frac{\left[e^{-jn\omega_0 \frac{\tau}{2}} - e^{+jn\omega_0 \frac{\tau}{2}}\right]}{2j} \times 2j$$

$$= \frac{2A}{n\omega_0 T} \sin\left(n\omega_0 \frac{\tau}{2}\right) \times \frac{\tau}{\tau} \quad \text{(multiplying \& dividing } \tau\text{)}$$

$$= \frac{A\tau}{T} \frac{\sin\left(\frac{n\omega_0\tau}{2}\right)}{\left(\frac{n\omega_0\tau}{2}\right)}$$

$$\therefore \quad x(t) = \sum_{n=-\infty}^{\infty} \frac{A\tau}{T} \frac{\sin\left(\frac{n\omega_0\tau}{2}\right)}{\left(\frac{n\omega_0\tau}{2}\right)} \cdot e^{jn\omega_0 t}$$

$$\boxed{x(t) = \frac{A\tau}{T} \sum_{n=-\infty}^{\infty} \frac{\sin\left(\frac{n\omega_0\tau}{2}\right)}{\left(\frac{n\omega_0\tau}{2}\right)} \cdot e^{jn\omega_0 t}}$$

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Fourier Transform :- (F-T)

Fourier transform is defined as,

$$\boxed{X(\omega) = F[x(t)] = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t}\, dt}$$

$X(\omega)$ is called Fourier transform of $x(t)$.

$X(\omega)$ is frequency domain representation.

$x(t)$ is time domain representation.

Inverse Fourier transform of $X(\omega)$ will generate original signal $x(t)$.

→ For Both periodic & Non periodic signal F.T can be found out.

$$\boxed{x(t) = F^{-1}[X(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) \cdot e^{j\omega t}\, d\omega}$$

→ The fourier transform $X(\omega)$ is complex function of 'w' and can be written as

$$X(\omega) = |X(\omega)| \cdot e^{j\phi(\omega)}$$

Where $|X(\omega)|$ → Amplitude Spectrum.

$\phi(\omega)$ → Phase Spectrum.

→ $x(t)$ & $X(\omega)$ are called Fourier transform pair.

$$x \longleftrightarrow X(\omega)$$

Dirichlet's condⁿ for existance of Fourier transform:-

- The function $x(t)$ must be single valued function with a finite number of maxima & minima with a finite number of discontinuities in any finite interval of time.
- Function must be absolutely integrable, $\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$

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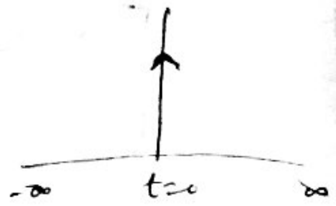
1) Find the Fourier transform of

$x(t) = \delta(t)$



Ans :- $X(\omega) = \int_{-\infty}^{\infty} \delta(t) \cdot e^{-j\omega t} \, dt$

Here

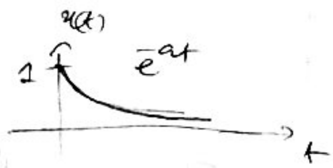
$y(t) = e^{-j\omega t}$

$y(0) = e^{0} = 1.$

$\because \int_{-\infty}^{\infty} y(t) \, \delta(t) \, dt = y(0)$

$$\boxed{X(\omega) = 1}$$

2) F.T of $x(t) = e^{-at} u(t)$



Ans :- $X(\omega) = \int_{0}^{\infty} e^{-at} \cdot 1 \cdot e^{-j\omega t} \, dt$ $\quad \because u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$

$= \int_{0}^{\infty} e^{-(a+j\omega)t} \, dt$

$= \left[\frac{e^{-(a+j\omega)t}}{-(a+j\omega)} \right]_{0}^{\infty}$

$= \frac{1}{-(a+j\omega)} \left[0 - 1 \right]$

$X(\omega) = \frac{1}{a+j\omega}$

$$\boxed{X(\omega) = \frac{a-j\omega}{a^2+\omega^2}}$$

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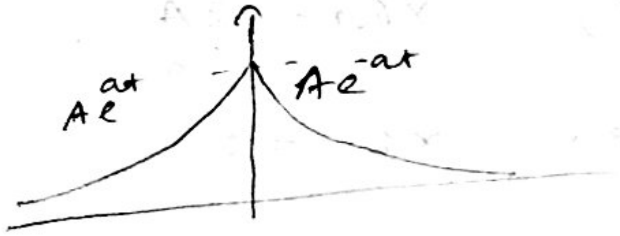
F.T of $A e^{-a|t|}$



Ans :- $x(t) = A e^{-a|t|}$ $\quad |t| = \begin{cases} t, & t>0 \\ -t, & t<0 \end{cases}$

$$= \begin{cases} A e^{-at}, & t>0 \\ A e^{at}, & t<0 \end{cases}$$

$$X(\omega) = \int_{-\infty}^{\infty} A e^{-a|t|} e^{-j\omega t} dt$$

$$= \int_{-\infty}^{0} A e^{at} e^{-j\omega t} dt + \int_{0}^{\infty} A e^{-at} e^{-j\omega t} dt$$

$$= A \cdot \int_{-\infty}^{0} e^{(a-j\omega)t} dt + A \int_{0}^{\infty} e^{-(a+j\omega)} dt$$

$$= A \cdot \left[\frac{e^{(a-j\omega)t}}{a-j\omega}\right]_{-\infty}^{0} + A \left[\frac{e^{-(a+j\omega)}}{-(a+j\omega)}\right]_{0}^{\infty}$$

$$= \frac{A}{a-j\omega}[1-0] + \left(\frac{-A}{a+j\omega}\right)\cdot[0-1]$$

$$= \frac{A}{a-j\omega} + \frac{A}{a+j\omega}$$

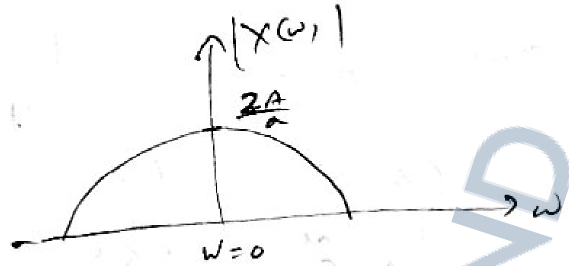
$$= A\left[\frac{a+\cancel{j\omega}+a-\cancel{j\omega}}{(a-j\omega)(a+j\omega)}\right]$$

$$\boxed{X(\omega) = \frac{2Aa}{a^2+\omega^2}}$$

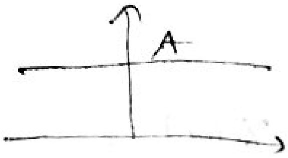
$X(\omega)$ is a real function, $Q(\omega) = 0$.

for $\omega = 0$, $X(\omega) = \frac{2A}{a}$.

$\omega = \infty$, $X(\omega) = 0$.



4. F.T of $x(t) = A$.



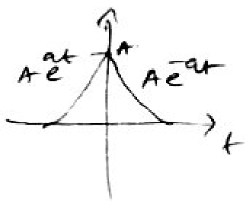
$$X(\omega) = \int_{-\infty}^{\infty} A \cdot e^{-j\omega t}\, dt$$

= does not exist.

Since $X(\omega) \to \infty$.

But its Fourier transform can be find out under certain limit.

$$x(t) = A = \lim_{a \to 0} A \cdot e^{-a|t|}$$



$$X(\omega) = \lim_{a \to 0} \frac{2aA}{a^2 + \omega^2}$$

If $\omega = 0$, $\lim_{a \to 0} \frac{2aA}{a^2} = \lim_{a \to 0} \frac{2A}{a} = \infty$

If $\omega \neq 0$, $\lim_{a \to 0} \frac{2aA}{a^2 + \omega^2} = \frac{2 \cdot 0 \cdot A}{\omega^2} = 0$

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i.e $\lim_{a \to 0} \frac{2aA}{a^2+\omega^2} = \delta(\omega)$ = unit impulse function.

magnitude of $\delta(\omega)$

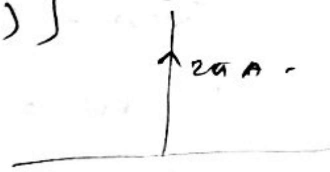
$$= \int_{-\infty}^{\infty} \frac{2aA}{a^2+\omega^2}\, d\omega$$

$$= 2aA \int_{-\infty}^{\infty} \frac{1}{a^2+\omega^2}\, d\omega.$$

$$= 2\cancel{a}A \left[\frac{1}{\cancel{a}} \tan^{-1}\left(\frac{\omega}{a}\right)\right]_{-\infty}^{\infty}$$

$$= 2A \left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right]$$

$$= 2\pi A$$



$\therefore$ ~~magnitude~~

$$\boxed{X(\omega) = 2\pi A\, \delta(\omega)}$$

5) Find the F.T of 1.

Let $A = 1$

$$X(\omega) = 2\pi \cdot 1 \cdot \delta(\omega)$$

$$\boxed{X(\omega) = 2\pi\, \delta(\omega)}$$

6) F.T of $e^{j\omega_0 t}$

Ans: $X(\omega) = \int_{-\infty}^{\infty} e^{j\omega_0 t} \cdot e^{-j\omega t}\, dt$

$$= \int_{-\infty}^{\infty} e^{-j(\omega-\omega_0)t}\, dt$$

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$$\Rightarrow \cancel{X(\omega) = \left[\frac{e^{-j(\omega-\omega_0)t}}{-j(\omega-\omega_0)}\right]_{-\infty}^{\infty}}$$

$$\cancel{= \frac{1}{-j(\omega-\omega_0)}\Big[}$$

$$\Rightarrow X(\omega) = \int_{-\infty}^{\infty} e^{-j(\omega-\omega_0)t}\, dt$$

$$\boxed{\begin{aligned} X(\omega) &= 2\pi \cdot 1 \cdot \delta(\omega-\omega_0) \\ X(\omega) &= 2\pi\, \delta(\omega-\omega_0) \end{aligned}}$$

F.T of A

$$= \int_{-\infty}^{\infty} A \cdot e^{-j\omega t}\, dt$$

$$= 2\pi A\, \delta(\omega)$$

6. Find inverse Fourier transform of $\delta(\omega)$.

Ans :- $F^{-1}[X(\omega)]$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega)\, e^{j\omega t}\, d\omega.$$

$$= \frac{1}{2\pi}\, e^{j\omega t}\Big|_{\omega=0}$$

$$F^{-1}[\delta(\omega)] = \frac{1}{2\pi}$$

$$\int_{-\infty}^{\infty} x(t)\cdot\delta(t)\, dt = x(0)$$

$$\int_{-\infty}^{\infty} X(\omega)\, \delta(\omega)\, d\omega = X(0)$$

$$F[1] = 2\pi\, \delta(\omega)$$

$\therefore$

$$\boxed{\begin{aligned} F[1] &= 2\pi\, \delta(\omega) \\ F^{-1}[2\pi\,\delta(\omega)] &= 1 \\ \Rightarrow F^{-1}[\delta(\omega)] &= \frac{1}{2\pi} \end{aligned}}$$

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7) Find the ~~Impulse~~ Inverse transform of $\delta(\omega-\omega_0)$

Hint :- $\int_{-\infty}^{\infty} x(t) \cdot \delta(t-t_0)\, dt = x(t_0)$.

Ans :- $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega-\omega_0)\, e^{j\omega t}\, d\omega$

$= \frac{1}{2\pi} \times \left. e^{j\omega t}\right]_{\omega=\omega_0}$ $\quad \int_{-\infty}^{\infty} x(\omega) \cdot \delta(\omega-\omega_0)\, d\omega = x(\omega_0)$.

$F^{-1}[\delta(\omega-\omega_0)] = \frac{1}{2\pi} \cdot e^{j\omega_0 t}$

$$\boxed{\begin{aligned} F[e^{j\omega_0 t}] &= 2\pi\, \delta(\omega-\omega_0) \\ F^{-1}[\delta(\omega-\omega_0)] &= \frac{1}{2\pi} e^{j\omega_0 t} \end{aligned}}$$

Note :- $F[e^{j\omega_0 t}] = 2\pi\, \delta(\omega-\omega_0)$

$F[e^{-j\omega_0 t}] = 2\pi\, \delta(\omega+\omega_0)$

8) F.T of $\cos\omega_0 t$, $\sin\omega_0 t$.

$\cos\omega_0 t = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$.

Ans :-

F.T $[\cos\omega_0 t] = \int_{-\infty}^{\infty} \left(\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\right) e^{-j\omega t}\, dt$

$= \frac{1}{2} \left[F.T(e^{j\omega_0 t}) + F.T[e^{-j\omega_0 t}] \right]$

$= \frac{1}{2} \left[2\pi\, \delta(\omega-\omega_0) + 2\pi\, \delta(\omega+\omega_0) \right]$

$$\boxed{F[\cos \omega_0 t] = \pi\,\delta(\omega+\omega_0) + \pi\,\delta(\omega-\omega_0)}$$

$$F[\sin \omega_0 t] = \int_{-\infty}^{\infty} \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} \cdot e^{-j\omega t}\, dt$$

$$= \frac{1}{2j}\left[F.T\left[e^{j\omega_0 t}\right] - F.T\left[e^{-j\omega_0 t}\right] \right]$$

$$= \frac{1}{2j}\left[2\pi\,\delta(\omega-\omega_0) - 2\pi\,\delta(\omega+\omega_0) \right]$$

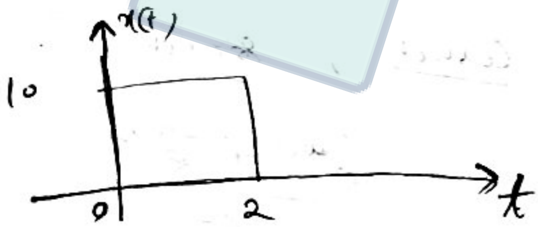
$$= \frac{j}{-2}\left[2\pi\,\delta(\omega-\omega_0) - 2\pi\,\delta(\omega+\omega_0) \right]$$

$$F[\sin \omega_0 t] = \pi\,\delta(\omega+\omega_0) - j\pi\,\delta(\omega-\omega_0)$$

$$\boxed{F[\sin \omega_0 t] = j\pi\,\delta(\omega+\omega_0) - j\pi\,\delta(\omega-\omega_0)}$$

Q) Find the Fourier transform of a rectangular pulse of duration 2 seconds & having magnitude 10 volts.

Ans →



$$F[x(t)] = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t}\, dt$$

$$= \int_0^2 10\, e^{-j\omega t}\, dt$$

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$$\Rightarrow X(\omega) = 10 \cdot \left.\frac{e^{-j\omega t}}{-j\omega}\right]_0^2$$

$$= \frac{10}{-j\omega}\left[e^{-j\omega \cdot 2} - e^{0}\right]$$

$$= \frac{10}{-j\omega}\left[e^{-2j\omega} - 1\right]$$

$$= \frac{10}{-j\omega}\left[\frac{e^{-j\omega}}{e^{+j\omega}} - 1\right]$$

$$= \frac{10}{-j\omega}\left[\frac{e^{-j\omega} - e^{+j\omega}}{e^{j\omega}}\right]$$

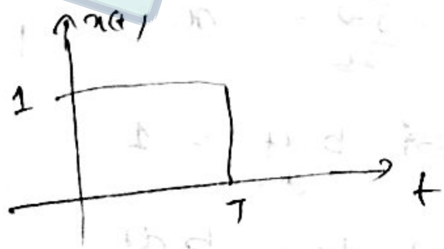
$$= \frac{10 \times 2}{2j\omega}\left[\frac{e^{j\omega} - e^{-j\omega}}{e^{j\omega}}\right]$$

$$= \frac{20}{\omega\, e^{j\omega}}\left[\frac{e^{j\omega} - e^{-j\omega}}{2j}\right]$$

$$= \frac{20}{e^{j\omega}}\left(\frac{\sin\omega}{\omega}\right)$$

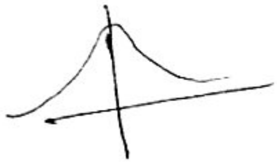
$$\Rightarrow X(\omega) = 20\, e^{-j\omega}\, Sa(\omega)$$

10) <u>Gate function</u>



$$x(t) = \begin{cases} 1, & 0 < t < T \\ 0, & \text{elsewhere.} \end{cases}$$

11. Find the Fourier transform of Gaussian Pulse.



Ans:

$$x(t) = e^{-b^2t^2}$$

$$F[x(t)] = X(\omega) = \int_{-\infty}^{\infty} e^{-b^2t^2} e^{-j\omega t}\, dt.$$

$$= \int_{-\infty}^{\infty} e^{-(b^2t^2 + j\omega t)}\, dt$$

Put

$$b^2t^2 + j\omega t = (bt)^2 + 2.\,bt.\,\frac{j\omega t}{2bt}$$

$$= bt + 2(bt).\left(\frac{j\omega}{2b}\right) + \left(\frac{j\omega}{2b}\right)^2 - \left(\frac{j\omega}{2b}\right)^2$$

$$= \left(bt + \frac{j\omega}{2b}\right)^2 + \frac{\omega^2}{4b^2}$$

$$X(\omega) = \int_{-\infty}^{\infty} e^{-\left(bt + \frac{j\omega}{2b}\right)^2} . e^{-\frac{\omega^2}{4b^2}}\, dt$$

$$= e^{-\frac{\omega^2}{4b^2}} \int_{-\infty}^{\infty} e^{-\left(bt + \frac{j\omega}{2b}\right)^2}\, dt.$$

put $bt + \frac{j\omega}{2b} = x$

$b\frac{dt}{dx} = 1$

$\Rightarrow dx = b\,dt$

$\Rightarrow dt = \frac{dx}{b}.$

$t \to -\infty$, $x \to -\infty$

$t \to \infty$, $x \to \infty$

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$$X(\omega) = e^{-\frac{\omega^2}{4b^2}} \int_{-\infty}^{\infty} e^{-x^2} \frac{dx}{b}$$

$$= \frac{e^{-\frac{\omega^2}{4b^2}}}{b} \cdot 2 \times \int_0^{\infty} e^{-x^2} dx.$$

$$= \frac{e^{-\frac{\omega^2}{4b^2}}}{b} \times \cancel{2} \times \frac{\sqrt{\pi}}{\cancel{2}}$$

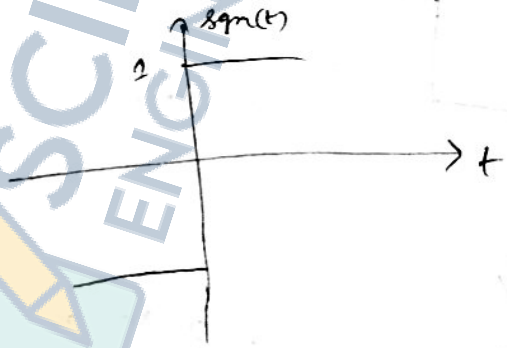
$$\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

$$X(\omega) = \frac{\sqrt{\pi}}{b} \cdot e^{-\frac{\omega^2}{4b^2}}$$

$$\boxed{F\left[e^{-b^2 t^2}\right] = \frac{\sqrt{\pi}}{b} \cdot e^{-\left(\frac{\omega}{2b}\right)^2}}$$

12. Find the Fourier transform of $sgn(t)$.

Ans:



$$sgn(t) = x(t) = \begin{cases} -1, & -\infty < t < 0 \\ 0, & t = 0 \\ 1, & t > 0 \end{cases}$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt$$

$$= \int_{-\infty}^{0} (-1) \cdot e^{-j\omega t} dt + \int_0^{\infty} 1 \cdot e^{-j\omega t} dt$$

* Alternative method

$sgn(t)$

$$= \lim_{a \to 0} e^{-at} u(t) - e^{at} u(-t)$$

$\Rightarrow$

$F[sgn(t)]$

$$= \lim_{a \to 0} \frac{1}{a + j\omega} - \frac{1}{a - j\omega}$$

$$= \lim_{a \to 0} \frac{\cancel{a} - j\omega - \cancel{a} - j\omega}{a^2 + \omega^2}$$

$$= \lim_{a \to 0} \frac{-2j\omega}{a^2 + \omega^2}$$

$$= \frac{-2j\omega}{\omega \times \omega} \times \frac{j}{j}$$

$$= \frac{2}{j\omega}$$

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$$X(\omega) = (-1)\,\frac{e^{-j\omega t}}{-j\omega}\Big]_{-\infty}^{0} + \frac{e^{-j\omega t}}{-j\omega}\Big]_{0}^{\infty}$$

$$= \frac{1}{j\omega}\left[1 + \infty\right] + \left(\frac{-1}{j\omega}\right)\left[0 - 1\right]$$

$$= ??$$

$$= \frac{1}{j\omega} + \frac{1}{j\omega}$$

$$X(\omega) = \frac{2}{j\omega}\ \infty \quad \times$$

$$\boxed{F\left[\operatorname{sig}(t)\right] = \frac{2}{j\omega}} \quad \times$$

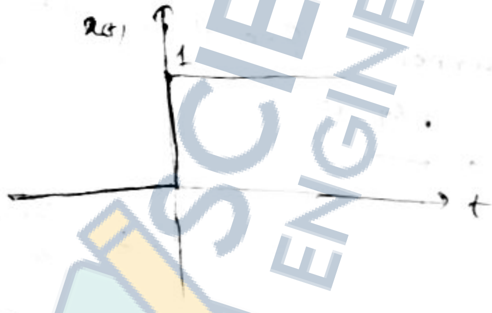
(In this method we can't find Fourier Transform

* Alternative method previous page)

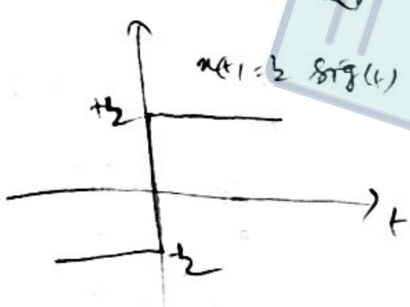
14

13.

$x(t) = u(t)$ find $X(\omega) = ?$



$\Downarrow$



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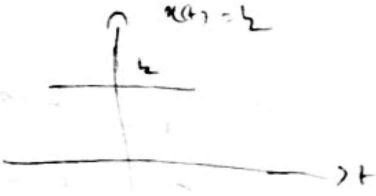

$$u(t) = \frac{1}{2}\operatorname{sig}(t) + \frac{1}{2}$$

$$F\left[u(t)\right] = \frac{1}{2}\,F\left[\operatorname{sig}(t)\right] + F\left[\frac{1}{2}\right]$$

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$$\Rightarrow X(\omega) = \cancel{\frac{1}{2}} \times \left(\frac{2}{j\omega}\right) + \cancel{\frac{1}{2}}\left(\cancel{2}\pi\,\delta(\omega)\right)$$

$$\Rightarrow X(\omega) = \pi\,\delta(\omega) + \frac{1}{j\omega}$$

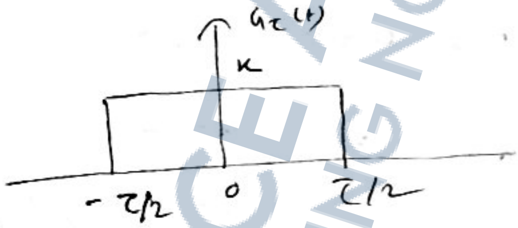
$$\therefore \boxed{F[u(t)] = \pi\,\delta(\omega) + \frac{1}{j\omega}}$$

... method

...n't find ... transform

(... ternative method ... vious page)

14. Find the F.T. of Gate f^n, given below



Ans :- $x(t) = \begin{cases} K, & -\frac{\tau}{2} < t < \frac{\tau}{2} \\ 0, & \text{elsewhere.} \end{cases}$

$$X(\omega) = \int_{-\tau/2}^{\tau/2} K \cdot e^{-j\omega t}\, dt$$

$$= K \left[\frac{e^{-j\omega t}}{-j\omega}\right]_{-\tau/2}^{+\tau/2}$$

$$= \frac{K}{j\omega}\left[e^{-j\omega t}\right]_{\tau/2}^{-\tau/2}$$ [Changes the limit by absorbing -ve sign]

$$= \frac{K}{j\omega}\;\frac{\left[e^{j\omega\frac{\tau}{2}} - e^{-j\omega\frac{\tau}{2}}\right]}{2} \times 2$$ | Dividing & multiplying 2

$$= \frac{2K}{\omega} \cdot \sin\left(\frac{\omega\tau}{2}\right)$$ $\left[\because \sin\left(\frac{\omega\tau}{2}\right) = \frac{e^{j\omega\frac{\tau}{2}} - e^{-j\omega\frac{\tau}{2}}}{2j}\right]$

(Multiplying & dividing τ) $\rightarrow$ $$= \frac{K\tau}{\frac{\omega\tau}{2}} \cdot \sin\left(\frac{\omega\tau}{2}\right) = K\tau\,\frac{\sin\left(\frac{\omega\tau}{2}\right)}{\left(\frac{\omega\tau}{2}\right)}$$

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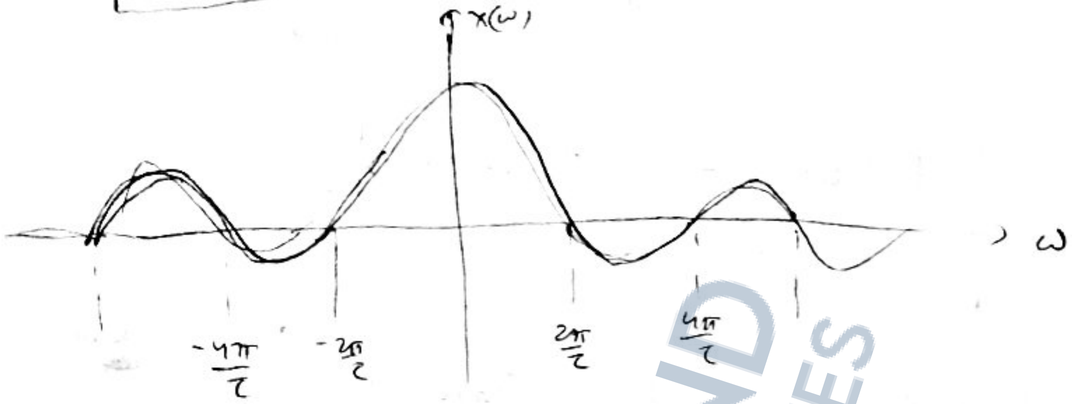
$$\boxed{X(\omega) = K\tau \; Sa\left(\frac{\omega\tau}{2}\right)}$$



$\underline{\sin\theta \to 0}$ when $\theta = \pi, 2\pi, 3\pi$ - -

$\sin\left(\frac{\omega\tau}{2}\right) = 0$ when $\frac{\omega\tau}{2} = \pi, 2\pi$ - -

$\Rightarrow \omega = \frac{2\pi}{\tau}, \frac{4\pi}{\tau}$ - -

Properties of Fourier Transform :-

1) **Linearity property :-**

A linear combination of 2 functions $x_1(t)$ & $x_2(t)$ is also linear combination in frequency domain.

If $x_1(t) \longleftrightarrow X_1(\omega)$,

$x_2(t) \longleftrightarrow X_2(\omega)$

Then $F\left[a_1 x_1(t) + a_2 x_2(t)\right] = a_1 X_1(\omega) + a_2 X_2(\omega)$

Proof :- $F\left[a_1 x_1(t) + a_2 x_2(t)\right]$

$$= \int_{-\infty}^{\infty} \left[a_1 x_1(t) + a_2 x_2(t)\right] e^{-j\omega t} \, dt$$

$$= a_1 \int_{-\infty}^{\infty} x_1(t) \, e^{-j\omega t} \, dt + a_2 \int_{-\infty}^{\infty} x_2(t) \, e^{-j\omega t} \, dt$$

$$= a_1 X_1(\omega) + a_2 X_2(\omega)$$

$$\Rightarrow \boxed{F[a_1 x_1(t) + a_2 x_2(t)] = a_1 X_1(\omega) + a_2 X_2(\omega)}$$ Proved.

2) <u>Time delay / Time shifting Property.</u>

If a signal $x(t)$ is delayed (shifted) in time by $\underline{t_0}$ sec, then the spectrum is modified by a ~~[illegible]~~ phase shift of $\underline{-\omega t_0}$.

or A shift in time domain by an amount t_0 is equivalent to multiplication $e^{-j\omega t_0}$ in frequency domain.

i.e $$F[x(t-t_0)] = X(\omega) \cdot e^{-j\omega t_0}.$$

<u>Proof:</u> $$F[x(t-t_0)] = \int_{-\infty}^{\infty} x(t-t_0) \cdot e^{-j\omega t} \, dt.$$

Let $t - t_0 = \theta \Rightarrow t = \theta + t_0$

$dt = d\theta$.

$t \to -\infty, \quad \theta \to -\infty$

$t \to \infty, \quad \theta \to \infty$

$$= \int_{-\infty}^{\infty} x(\theta) \cdot e^{-j\omega(\theta + t_0)} \, d\theta$$

$$= \int_{-\infty}^{\infty} x(\theta) \cdot e^{-j\omega\theta} \cdot e^{-j\omega t_0} \, d\theta$$

$$= e^{-j\omega t_0} \cdot \int_{-\infty}^{\infty} x(\theta) \cdot e^{-j\omega\theta} \, d\theta$$

$$\boxed{F[x(t-t_0)] = e^{-j\omega t_0} \cdot X(\omega)}$$

$$\Rightarrow x(t-t_0) \longleftrightarrow e^{-j\omega t_0} X(\omega)$$

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3) Frequency shifting Property :-

It states that multiplication of function $x(t)$ by $\underline{e^{j\omega_0 t}}$ is equivalent to shifting the Fourier transform $X(\omega)$ in the +ve direction by an amount $\underline{\omega_0}$

i.e $\quad x(t) \leftrightarrow X(\omega)$

Then $\quad e^{j\omega_0 t} x(t) \leftrightarrow X(\omega-\omega_0)$

Proof :-

$$F\left[e^{j\omega_0 t} x(t)\right] = \int_{-\infty}^{\infty} e^{j\omega_0 t} x(t) \cdot e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} x(t) \cdot e^{-j(\omega-\omega_0)t} dt$$

$$F\left[e^{j\omega_0 t} x(t)\right] = X(\omega-\omega_0)$$

$$\therefore \quad e^{j\omega_0 t} x(t) \longleftrightarrow X(\omega-\omega_0)$$

4) Time differentiation property :-

Time differentiation property states that the differentiation of function $x(t)$ in the time domain is equivalent to multiplication of its Fourier transform by a factor $\underline{j\omega}$.

i.e $\quad x(t) \leftrightarrow X(\omega)$

$$\frac{d}{dt} x(t) \leftrightarrow j\omega X(\omega).$$

Proof :-

$$F^{-1}[X(\omega)] = x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \cdot e^{j\omega t} d\omega$$

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$$\frac{d}{dt} x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) . \underline{\frac{d}{dt}}(e^{j\omega t}) . d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{X(\omega) . (j\omega)} e^{j\omega t} d\omega$$

$$\frac{d}{dt} x(t) = F^{-1}[j\omega X(\omega)] \quad \Bigg| \quad \because F^{-1}[X(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) . e^{j\omega t} d\omega$$

Taking Fourier transform both the sides,

$$F\left[\frac{d}{dt} x(t)\right] = j\omega X(\omega)$$

$$\frac{d}{dt} x(t) \longleftrightarrow j\omega X(\omega)$$

<u>Note</u>: $$\frac{d^n}{dt^n} x(t) \longleftrightarrow (j\omega)^n . X(\omega)$$

(ii) <u>Differentiation in frequency domain</u>

<u>if</u> $x(t) \longleftrightarrow X(f)$

$$(-j2\pi t) x(t) \longleftrightarrow \frac{d}{df} X(f)$$

<u>Proof</u>: $$X(f) = \int_{-\infty}^{\infty} x(t) . e^{-j2\pi f t} dt$$

$$\frac{d}{df} X(f) = \int_{-\infty}^{\infty} \{x(t) . (-j2\pi t)\} e^{-j2\pi f t} dt$$

$$\frac{d}{df} X(f) = F[x(t)(-j2\pi t)]$$

$$(-j2\pi t) x(t) \longleftrightarrow \frac{d}{df} X(f)$$

3/ **Duality or Symmetry property :-**

$$x(t) \leftrightarrow X(\omega)$$

$$X(t) \leftrightarrow 2\pi x(-\omega)$$

Proof :-

$$F^{-1}[X(\omega)] = x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) \cdot e^{j\omega t}\, d\omega$$

$$x(-t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) \cdot e^{-j\omega t}\, d\omega$$

$$\Rightarrow 2\pi\, x(-t) = \int_{-\infty}^{\infty} X(\omega)\, e^{-j\omega t}\, d\omega$$

Since ω is a dummy variable, interchanging t & ω we get,

$$\Rightarrow 2\pi\, x(-\omega) = \int_{-\infty}^{\infty} X(t) \cdot e^{-j\omega t}\, dt = F[X(t)]$$

$$\Rightarrow \boxed{F[X(t)] = 2\pi\, x(-\omega)}$$

c) **Scale change property :-**

$$x(at) \leftrightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

Proof :-

$$F[x(at)] = \int_{-\infty}^{\infty} x(at)\, e^{-j\omega t}\, dt$$

Let $at = \theta$, $t = \theta/a$

$$a\frac{dt}{d\theta} = 1 \Rightarrow dt = \frac{d\theta}{a}$$

$t \to -\infty$, $\theta \to -\infty$

$t \to \infty$, $\theta \to \infty$

$$F[x(at)] = \frac{1}{a}\int_{-\infty}^{\infty} x(\theta) \cdot e^{-j\omega\frac{\theta}{a}}\, d\theta$$

$= \frac{1}{}$ ~~Case-1~~ ~~when $a>0$~~

$$F[x(at)] = \frac{1}{a}\int_{-\infty}^{\infty} x(\theta)\cdot e^{-j\left(\frac{\omega}{a}\right)\theta}\, d\theta$$

$$= \frac{1}{a}\, X\left(\frac{\omega}{a}\right)$$

When $a>0$

$$F[x(at)] = \frac{1}{a}\, X\left(\frac{\omega}{a}\right)$$

When $a<0$

$$F[x(at)] = -\frac{1}{a}\, X\left(\frac{\omega}{a}\right)$$

In general

$$F[x(at)] = \frac{1}{|a|}\, X\left(\frac{\omega}{a}\right).$$

7) ~~Multiplication~~ ~~property~~/Modulation property:-

$$x(t)\cdot \cos\omega_c t \longleftrightarrow \frac{1}{2}\left[X(\omega-\omega_c) + X(\omega+\omega_c)\right]$$

Proof:-

$$x(t)\cos\omega_c t = x(t)\cdot\left[\frac{e^{j\omega_c t} + e^{-j\omega_c t}}{2}\right]$$

$$F[x(t)\cos\omega_c t] = \int_{-\infty}^{\infty} \frac{x(t)}{2}\left[e^{j\omega_c t} + e^{-j\omega_c t}\right] e^{-j\omega t}\, dt$$

$$= \frac{1}{2}\left[\int_{-\infty}^{\infty} x(t)\cdot e^{-j(\omega-\omega_c)t}\, dt + \int_{-\infty}^{\infty} x(t)\, e^{-j(\omega+\omega_c)t}\, dt\right]$$

$$= \frac{1}{2}\left[X(\omega-\omega_c) + X(\omega+\omega_c)\right]$$

$$\therefore\; x(t)\cos\omega_c t \longleftrightarrow \frac{1}{2}\left[X(\omega-\omega_c) + X(\omega+\omega_c)\right]$$

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8/ Integration property:-

$$F\left[\int_{-\infty}^{t} x(\tau)\,d\tau\right] = \frac{1}{j\omega} X(\omega) \quad \text{, Provided } X(0)=0$$

Proof :-

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\cdot e^{j\omega t}\,d\omega$$

Then

$$x(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega \tau}\,d\omega$$

$$\Rightarrow \int_{-\infty}^{t} x(\tau)\,d\tau = \int_{-\infty}^{t}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\cdot e^{j\omega\tau}\,d\omega\right]d\tau$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{t} \underbrace{X(\omega)}_{\text{const.}}\cdot \underbrace{e^{j\omega\tau}\,d\tau}_{\tau \text{ is variable}}\right)d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\left.\frac{e^{j\omega\tau}}{j\omega}\right]_{-\infty}^{t}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\,\frac{\left[e^{j\omega t}\right]}{j\omega}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{X(\omega)}{j\omega}\right) e^{j\omega t}\,d\omega$$

$$= F^{-1}\left[\frac{X(\omega)}{j\omega}\right]$$

Taking Fourier transform both the sides,

$$\Rightarrow F\left[\int_{-\infty}^{t} x(\tau)\,d\tau\right] = \frac{1}{j\omega} X(\omega)$$

$$\boxed{\int_{-\infty}^{t} x(\tau)\,d\tau \longleftrightarrow \frac{1}{j\omega} X(\omega)}$$

(6) Q) Convolution them:-

(1) Time convolution them:-

It states that convolution in time domain is equivalent to multiplication of their spectra in frequency domain.

If $x_1(t) \longleftrightarrow X_1(\omega)$

$x_2(t) \longleftrightarrow X_2(\omega)$

$$x_1(t) * x_2(t) \longleftrightarrow X_1(\omega) \cdot X_2(\omega)$$

Proof:- $F[x_1(t) * x_2(t)]$

$$= \int_{-\infty}^{\infty} (x_1(t) * x_2(t)) \cdot e^{-j\omega t} \, dt$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x_1(\tau) \cdot x_2(t-\tau) \, d\tau\right] \cdot e^{-j\omega t} \, dt.$$

Changing the order of Integration:-

$$= \int_{-\infty}^{\infty} x_1(\tau) \left[\int_{-\infty}^{\infty} x_2(t-\tau) \cdot e^{-j\omega t} \, dt\right] d\tau$$

By using time shifting property,

$$F[x(t-t_0)] = e^{-j\omega t_0} X(\omega).$$

$$= \int_{-\infty}^{\infty} x_1(\tau) \cdot e^{-j\omega\tau} X_2(\omega) \, d\tau.$$

$$= X_2(\omega) \cdot \int_{-\infty}^{\infty} x_1(\tau) \cdot e^{-j\omega\tau} \, d\tau = X_2(\omega) \cdot X_1(\omega)$$

$\therefore \; x_1(t) * x_2(t) \longleftrightarrow X_1(\omega) \cdot X_2(\omega)$

Freq convolution thm:-

The multiplication of 2 functions in time domain is equivalent to convolution of their spectra in frequency domain. 10)

i.e

$$x_1(t) \leftrightarrow X_1(\omega)$$

$$x_2(t) \leftrightarrow X_2(\omega)$$

then $$F[x_1(t) \cdot x_2(t)] = \frac{1}{2\pi}\left[X_1(\omega) * X_2(\omega)\right]$$

Proof :- $F[x_1(t) \cdot x_2(t)]$

$$= \int_{-\infty}^{\infty} x_1(t) \cdot x_2(t) \cdot e^{-j\omega t}\, dt$$

Let $$x_2(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_2(\omega') \cdot e^{j\omega' t}\, d\omega'$$

$$= \int_{-\infty}^{\infty} x_1(t) \cdot \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X_2(\omega') \cdot e^{j\omega' t}\, d\omega'\right] e^{-j\omega t}\, dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_2(\omega') \left[\int_{-\infty}^{\infty} x_1(t) \cdot e^{-j(\omega-\omega')t}\, dt\right] d\omega'$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_2(\omega') \cdot X_1(\omega-\omega')\, d\omega'$$

$$= \frac{1}{2\pi}\left[X_2(\omega) * X_1(\omega)\right] = \frac{1}{2\pi}\left[X_1(\omega) * X_2(\omega)\right]$$

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$$x_1(t) \cdot x_2(t) \longleftrightarrow \frac{1}{2\pi}\left[X_1(\omega) * X_2(\omega)\right]$$

time

their

10) Area under curve:-

(i) The area under function $x(t)$ is equal to the value of its Fourier transform $X(\omega)$ at ~~ω~~ $\omega = 0$

i.e if $x(t) \longleftrightarrow X(\omega)$

$$\int_{-\infty}^{\infty} x(t)\,dt = \frac{1}{2\pi} X(0).$$

(ii) Similarly, area under Fourier transform $X(\omega)$ of a function $x(t)$ is equal to the value of function $x(t)$ at $t=0$,

i.e $\int_{-\infty}^{\infty} X(\omega)\,d\omega = x(0).$

~~[illegible]~~ Fourier transform of periodic f^n.

→ Fourier series is used to analyze periodic waveform.

Let $x(t)$ is a periodic function with period T. Then exponential Fourier series of $x(t)$ is

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}, \quad \text{where } \omega_0 = \frac{2\pi}{T}.$$

dt

~~$F[x(t)] = \sum_{n=-\infty}^{\infty} C_n F[$~~

$$F[x(t)] = F\left[\sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}\right]$$

$$= \sum_{n=-\infty}^{\infty} C_n F\left[e^{jn\omega_0 t}\right]$$

$$= \sum_{n=-\infty}^{\infty} C_n F\left[1 \cdot e^{jn\omega_0 t}\right]$$

$X_2(\omega)]$

$$F[1] = 2\pi\,\delta(\omega)$$

$$F[1.e^{jn\omega_0 t}] = 2\pi\,\delta(\omega - n\omega_0)$$

$$\therefore F[x(t)] = \sum_{n=-\infty}^{\infty} C_n .\ 2\pi\ \delta(\omega - n\omega_0)$$

$$\boxed{X(\omega) = 2\pi \sum_{n=-\infty}^{\infty} C_n\ \delta(\omega - n\omega_0)}$$

So fourier transform of a periodic function consists of a train of equally spaced impulses.

* 11) <u>Perseval's property :-</u>

If fourier transform of the signals $x(t)$ and $y(t)$ are denoted by $X(f)$ and $Y(f)$, respectively then

$$\int_{-\infty}^{\infty} x(t)\ y^*(t)\ dt = \int_{-\infty}^{\infty} X(f) . Y^*(f)\ df$$

* 12) <u>Rayleigh's property :-</u>

$$\int_{-\infty}^{\infty} |x(t)|^2\, dt = \int_{-\infty}^{\infty} |X(f)|^2\, df$$

* 13) <u>Auto Correlation property :-</u>

The time auto correlation function of the signal $x(t)$ is denoted by $R_x(\tau)$ and is definded by

$$R_x(\tau) = \int_{-\infty}^{\infty} x(t)\ x^*(t-\tau)\ dt$$

Then auto correlation property states that

$$F[R_x(\tau)] = |X(f)|^2$$

* 14) <u>Time Reversal</u>

$F[x(-t)] = X(-\omega)$ i.e $x(-t) \leftrightarrow X(-\omega)$

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Properties :-

Property	$x(t)$		$X(\omega)$
1) Linearity	$a_1 x_1(t) + a_2 x_2(t)$	$\rightarrow$	$a_1 X_1(\omega) + a_2 X_2(\omega)$
2) Time shifting	$x(t-t_0)$	$\rightarrow$	$X(\omega) \cdot e^{-j\omega t_0}$
3) Time reversal	$x(-t)$	$\rightarrow$	$X(-\omega)$
4) Duality	$X(t)$	$\rightarrow$	$2\pi\, x(-\omega)$
	$X(t)$	$\rightarrow$	$x(-f)$
5) Freq shifting	$e^{j\omega_0 t} x(t)$	$\rightarrow$	$X(\omega-\omega_0)$
6) Modulation	$x(t)\cos\omega_c t$	$\rightarrow$	$\frac{1}{2}\left[X(\omega-\omega_c) + X(\omega+\omega_c)\right]$
7) Scaling	$x(at)$	$\rightarrow$	$\frac{1}{\lvert a\rvert} X\left(\frac{\omega}{a}\right)$
8) Differentiation	$\frac{d}{dt} x(t)$	$\rightarrow$	$j\omega\, X(\omega)$
	$-j2\pi t\, x(t)$	$\rightarrow$	$\frac{d}{df} X(f)$ i.e $X'(f)$
9) Integration	$\left[\int_{-\infty}^{t} x(\tau)\, d\tau\right]$	$\rightarrow$	$\frac{1}{j\omega} X(\omega)$ (provided $X(0)=0$) $= \frac{1}{j\omega} X(\omega) + \pi X(0)\delta(\omega)$
10) Convolution	$x_1(t) \circledast x_2(t)$	$\rightarrow$	$X_1(\omega) \cdot X_2(\omega)$
11) Multiplication	$x_1(t) \cdot x_2(t)$	$\rightarrow$	$\frac{1}{2\pi}\left[X_1(\omega) * X_2(\omega)\right]$

12) Moment property (proofs) :- If $F[x(t)] = X(f)$, then $\int_{-\infty}^{\infty} t^n x(t)\, dt$,

n^{th} moment of $x(t)$ can be obtained from the relation,

$$\int_{-\infty}^{\infty} t^n x(t)\, dt = \left(\frac{j}{2\pi}\right)^n \frac{d^n}{df^n} X(f)\Big|_{f=0}$$

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Fourier transform of different functions :-

	$x(t)$		$X(\omega)$
1)	$\delta(t)$	$\rightarrow$	1
2)	$\delta(t-t_0)$	$\rightarrow$	$e^{-j\omega t_0}$
3)	1	$\rightarrow$	$2\pi\,\delta(\omega)$
4)	$u(t)$	$\rightarrow$	$\pi\,\delta(\omega) + \frac{1}{j\omega}$
5)	$sgn(t)$	$\rightarrow$	$\frac{2}{j\omega}$
6)	$e^{j\omega_0 t}$	$\rightarrow$	$2\pi\,\delta(\omega-\omega_0)$
7)	$\cos\omega_0 t$	$\rightarrow$	$\pi\,[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)]$
8)	$\sin\omega_0 t$	$\rightarrow$	$j\pi\,[\delta(\omega+\omega_0) - \delta(\omega-\omega_0)]$
9)	$e^{-at}u(t),\ a>0$	$\rightarrow$	$\frac{1}{a+j\omega}$
10)	$t\,e^{-at}u(t),\ a>0$	$\rightarrow$	$\frac{1}{(a+j\omega)^2}$
* 11)	$e^{-a\lvert t\rvert}$	$\rightarrow$	$\frac{2a}{a^2+\omega^2}$
12)	$e^{-a^2t^2}$	$\rightarrow$	$\frac{\sqrt{\pi}}{a}\,e^{-\left(\frac{\omega}{2a}\right)^2}$

* 13) Rectangular function

$$\Pi(t) = \begin{cases} 1, & \lvert t\rvert < \frac{1}{2} \\ \frac{1}{2}, & t = \pm\frac{1}{2} \\ 0, & \text{otherwise} \end{cases} \rightarrow sinc(f)$$

14) (Triangular function)

$$\Lambda(t) = \begin{cases} t+1, & -1 \le t < 0 \\ -t+1, & 0 \le t \le 1 \\ 0, & \text{otherwise} \end{cases} \rightarrow sinc^2(f)$$

15) $\frac{1}{t} \rightarrow -j\pi\,sgn(f)$

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(66)

15) —

16)

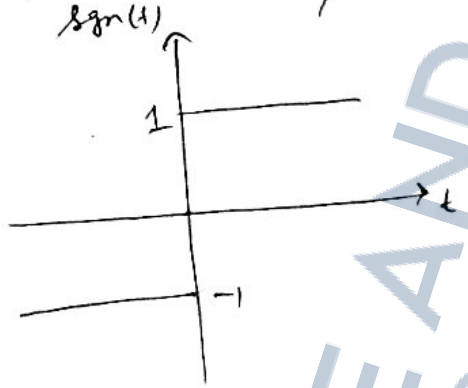
$\odot$

$x(t)$	$X(\omega)$
5) ~~$\sum_{n=-\infty}^{\infty} \delta(t - nT_0)$~~	~~$\frac{1}{T_0} \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T_0}\right)$~~
5) $\sum_{n=-\infty}^{\infty} \delta(t - nT_0)$	$\frac{1}{T_0} \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T_0}\right)$

More examples on Fourier Transform.

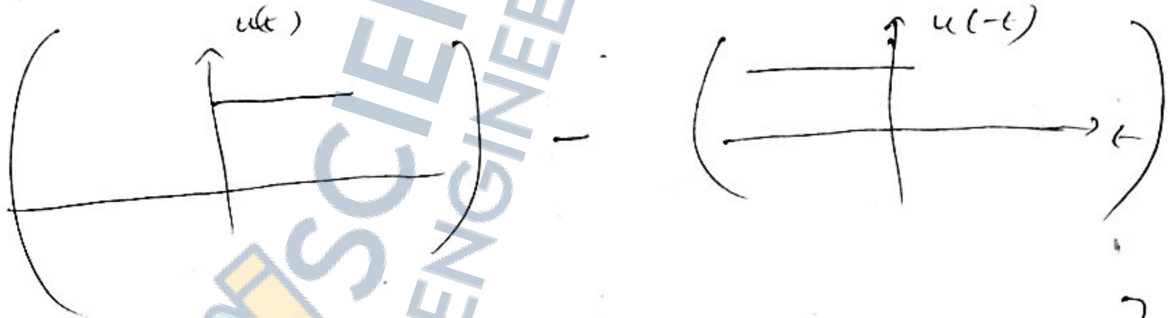
1) $F\left[sgn(t) \right] = ?$ [Signum function]

Ans: $sgn(t) = \begin{cases} -1, & -\infty < t < 0 \\ 0, & t = 0 \\ 1, & t > 0 \end{cases}$



$$sgn(t) = \lim_{a \to 0} \left[e^{-at} u(t) - e^{at} u(-t) \right]$$

i.e



$$X(\omega) = \lim_{a \to 0} \left[F.T\left(e^{-at} u(t) \right] - F.T\left[e^{at} u(-t) \right] \right]$$

$$* = \lim_{a \to 0} \left[\int_0^\infty e^{-at} \cdot e^{-j\omega t} \cdot dt - \int_{-\infty}^0 e^{at} \cdot e^{-j\omega t} \cdot 1\, dt \right]$$

$$= \lim_{a \to 0} \left(\int_0^\infty e^{-(a+j\omega)t} dt - \int_{-\infty}^0 e^{(a-j\omega)t} dt \right]$$

$$= \lim_{a \to 0} \left(\left. \frac{e^{-(a+j\omega)t}}{-(a+j\omega)} \right|_0^\infty - \left. \frac{e^{(a-j\omega)t}}{a-j\omega} \right|_{-\infty}^0 \right)$$

$$= \lim_{a \to 0} \left[\frac{-1}{a+j\omega} \left[0 - 1 \right] - \frac{1}{a-j\omega} \left(1 - 0 \right) \right]$$

SCIENCE AND ENGINEERING NOTES

$$X(\omega) = \lim_{a \to 0} \left[\frac{1}{a+j\omega} - \frac{1}{a-j\omega} \right]$$

(*You can write this step directly)

$$= \lim_{a \to 0} \left[\frac{\cancel{a} - j\omega - \cancel{a} - j\omega}{(a+j\omega)(a-j\omega)} \right]$$

$$= \lim_{a \to 0} \frac{-2j\omega}{a^2+\omega^2}$$

$$= \frac{-2j\cancel{\omega}}{\omega^{\cancel{2}}}$$

$$= -\frac{2j}{\omega}$$

$$= -\frac{2j \times j}{j\omega}$$

$$\boxed{X(\omega) = \frac{2}{j\omega}}$$

$$F[\,\mathrm{sgn}(t)\,] = \frac{2}{j\omega}$$

2) F.T of $\Pi(t)$ (Rectangular f^n)

$$\Pi(t) = \begin{cases} 1, & |t| < \frac{1}{2} \\ \frac{1}{2}, & t = \pm\frac{1}{2} \\ 0, & \text{otherwise.} \end{cases}$$

Note: In some books. F.T is defined as

$$F[x(t)] = \int_{-\infty}^{\infty} x(t).\, e^{-j2\pi f t}\, dt$$

$\frac{1}{a-j\omega} \Big]$

ter directly)

$\frac{e-j\omega}{(\ldots)(a-j\omega)} \Big]$

(Rectangular f^n)

rse.

. F.T is defined as

$\ldots e^{-j2\pi ft} dt$

I.F.T

$$x(t) = F^{-1}[X(f)] = \int_{-\infty}^{\infty} X(f) \cdot e^{+j2\pi ft}\, df$$

→ Instead of $\omega \rightarrow 2\pi f$

In IFT, No $\left(\frac{1}{2\pi}\right)$ term will come. because in ω, 2π term

Ans: $$F[\pi(t)] = \int_{-\infty}^{\infty} \pi(t)\, e^{-j2\pi ft}\, dt$$

$$= \int_{-1/2}^{1/2} 1 \cdot e^{-j2\pi ft}\, dt$$

$$= \left[\frac{e^{-j2\pi ft}}{-j2\pi f}\right]_{-1/2}^{1/2}$$

$$= \frac{-1}{j2\pi f}\left[e^{-j2\pi f\cdot\frac{1}{2}} - e^{-j2\pi f\cdot\left(-\frac{1}{2}\right)}\right]$$

$$= \frac{1}{\pi f}\left[\frac{e^{j\pi f} - e^{-j\pi f}}{2j}\right]$$

$$= \frac{1}{\pi f} \times \sin(\pi f)$$

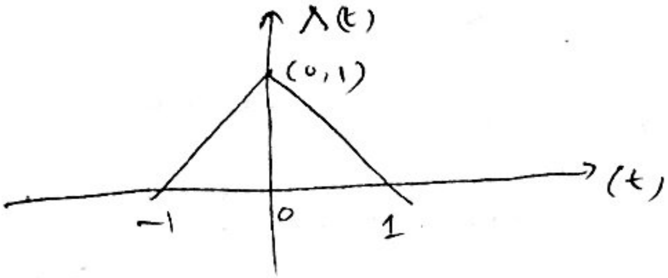
$$= \mathrm{sinc}(f)$$

$$\boxed{\therefore F[\Pi(t)] = \mathrm{sinc}(f)}$$

SCIENCE AND ENGINEERING NOTES

3) Find F.T of $\Lambda(t)$ [Triangular function]

H. Work. Assignment

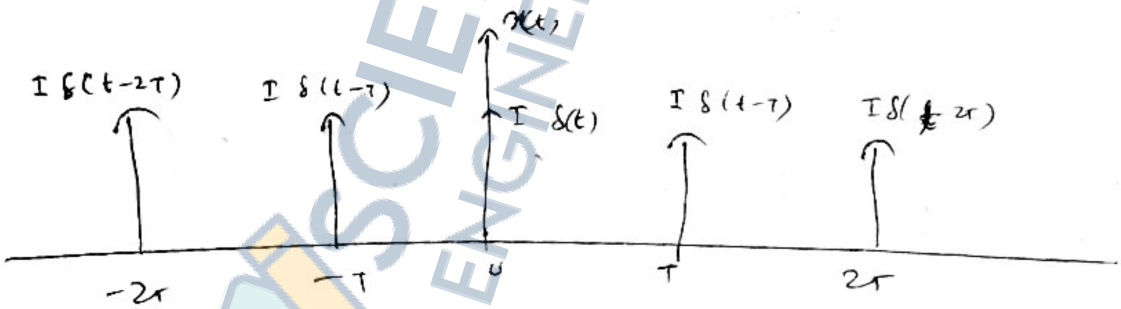


$$\Lambda(t) = \begin{cases} t+1, & -1 < t < 0 \\ -t+1, & 0 < t < 1 \\ 0, & \text{Otherwise.} \end{cases}$$

Ans.

$$F[\Lambda(t)] = \left(\frac{\sin \pi f}{\pi f}\right)^2 = \operatorname{sinc}^2(f)$$

4) Fourier Series of ~~Ratte~~ train of Pulses



I - Amplitude of Impulse.

Ans.
$$a_0 = \frac{1}{T} \cdot \int_{-T/2}^{T/2} I\, \delta(t)\, dt$$

$$= \frac{I}{T} \int_{-T/2}^{T/2} \delta(t)\, dt \qquad \left(\because \int_{-T/2}^{T/2} \delta(t) = 1\right)$$

$$\boxed{a_0 = \frac{I}{T}}$$

$$a_n = \frac{2}{T} \int_{-T/2}^{T/2} I\, \delta(t) \cdot \cos(n\omega_0 t)\, dt$$

$$= \frac{2I}{T} \int_{-T/2}^{T/2} \cos(n\omega_0 t)\, \delta(t)\, dt$$

SCIENCE AND ENGINEERING NOTES

$$a_n = \frac{2I}{T} \cdot \cos n\omega_0 t \Big|_{t=0} \qquad \Big| \quad \because \int_{-T/2}^{T/2} x(t)\,\delta(t) = x(0)$$

$$\boxed{a_n = \frac{2I}{T}}$$

Similarly, $b_n = \frac{2I}{T} \cdot \sin(n\omega_0 t)\Big|_{t=0}$

$$\boxed{b_n = 0}$$

$$C_n = \frac{a_n - jb_n}{2} = \frac{a_n}{2} = \frac{2I}{T} \times \frac{1}{2} = \frac{I}{T}.$$

$$x(t) = \sum_{n=-\infty}^{\infty} C_n \cdot e^{jn\omega_0 t} \qquad \text{(exponential form)}$$

$$x(t) = \sum_{n=-\infty}^{\infty} \frac{I}{T} \cdot e^{jn\omega_0 t}$$

$$\boxed{x(t) = \frac{I}{T} \sum_{n=-\infty}^{\infty} e^{jn\omega_0 t}}$$

57. F.T of $\left(\frac{1}{t}\right)$

Using Duality Property

$$F[X(t)] = x(-f) \qquad \text{—— (1)}$$

Let
$$x(t) = \operatorname{sgn}(t) \qquad \text{—— (2)}$$

$$X(f) = \frac{2}{j2\pi f} \qquad \text{—— (3)}$$

$$\Rightarrow X(t) = \frac{2}{j2\pi t} \qquad \text{—— (4)}$$

From eqn (2), $x(-f) = \operatorname{sgn}(-f)$ —— (5)

From eq (1)

$$\therefore \quad F[x(t)] = x(-f)$$

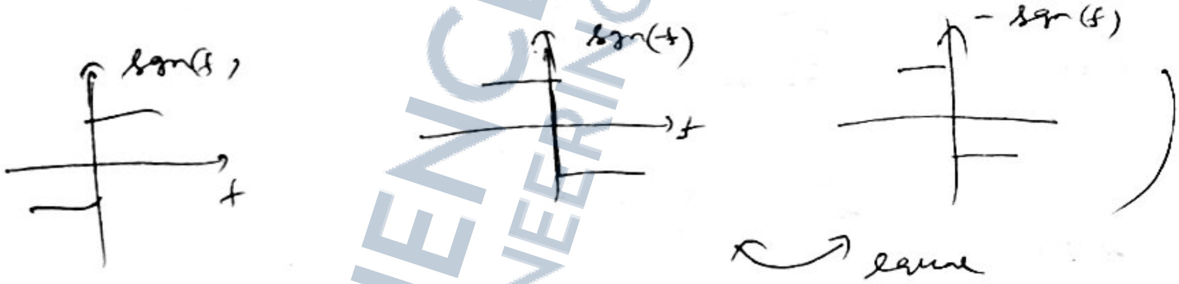
Using eqn (4) & (5)

$$F\left[\frac{2}{j2\pi t}\right] = \text{sgn}(-f)$$

$$\Rightarrow \quad F\left[\frac{1}{t}\right] = j\pi \,\text{sgn}(-f)$$

$$= -j\pi \,\text{sgn}(f)$$

$$\left(\because \quad \text{sgn}(-f) = -\text{sgn}(f) \right.$$



$$\Rightarrow \quad \boxed{F\left[\frac{1}{t}\right] = -j\pi \,\text{sgn}(f)}$$

(Ans)

8) Prop: 4 Time reversal

$$F[x(-t)] = X(-\omega)$$

Ans:

$$F[x(-t)] = \int_{-\infty}^{\infty} x(-t) \cdot e^{-j\omega t} \, dt$$

Put $\quad -t = \alpha, \quad dt = -d\alpha$.

$$t \to -\infty, \quad \alpha \to \infty$$

$$t \to \infty, \quad \alpha \to -\infty$$

SCIENCE AND ENGINEERING NOTES

$$F[x(-t)] = \int_{\infty}^{-\infty} x(\theta) \cdot e^{+j\omega\theta} (-d\theta)$$

$$= \int_{-\infty}^{\infty} x(\theta) \cdot e^{j\omega\theta} \, d\theta$$

Change the limit by removing '-' from $d\theta$

$$= \int_{-\infty}^{\infty} x(\theta) \cdot e^{-j(-\omega)\theta} \, d\theta$$

$$= X(-\omega)$$

$$\therefore \boxed{F[x(-t)] = X(-\omega)}$$

(67)

Power & Energy:-

Spectral energy:- It is distribution of power or energy of a signal per unit bandwidth as a function of frequency.

Energy signal:-

Signal which has finite energy $(0 < E < \infty)$ and zero power is called energy signals.

Power signal :-

Signal which has finite avg. power $(0 < P < \infty)$ and infinite energy $(E = \infty)$ is called power signal.

Normalized energy:-

The normalized energy of a signal $x(t)$ is defined as the energy dissipated by a voltage signal applied across 1Ω resistor (or by current signal flowing through 1Ω resistor)

Mathematically

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

$0 < E < \infty$, E is finite.

Perseval's them for energy signal (Rayleigh energy them)

$$E = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega = \int_{-\infty}^{\infty} |X(f)|^2 \, df$$

SCIENCE AND ENGINEERING NOTES

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Proof :

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

$$= \int_{-\infty}^{\infty} x(t) \cdot x^*(t) \, dt$$

$$= \int_{-\infty}^{\infty} x^*(t) \cdot x(t) \, dt$$

$$= \int_{-\infty}^{\infty} x^*(t) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \cdot e^{j\omega t} \, d\omega \right] dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \left[\int_{-\infty}^{\infty} x^*(t) \, e^{j\omega t} \, dt \right] d\omega$$

$$F[x(t)] = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} \, dt$$

$$F[x^*(t)] = \int_{-\infty}^{\infty} x^*(t) \cdot e^{j\omega t} \, dt \, .$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \cdot X^*(\omega) \, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega$$

$\omega = 2\pi f$

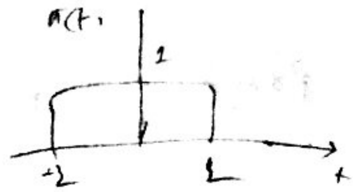
$d\omega = 2\pi \, df$

$$\boxed{\begin{aligned} E &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega \\ E &= 1 \int_{-\infty}^{\infty} |X(f)|^2 \, df \end{aligned}}$$

SCIENCE AND ENGINEERING NOTES

(1)

1) Find F.T of $\Pi(t)$



Ans:

$$\Pi(t) = \begin{cases} 1, & |t| < \frac{1}{2} \\ \frac{1}{2}, & t = \pm\frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$$

$$F[\Pi(t)] = \int_{-\infty}^{\infty} \Pi(t)\, e^{-j2\pi f t}\, dt$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} 1 \cdot e^{-j2\pi f t}\, dt$$

$$= \left.\frac{e^{-j2\pi f t}}{-j2\pi f}\right]_{-\frac{1}{2}}^{\frac{1}{2}}$$

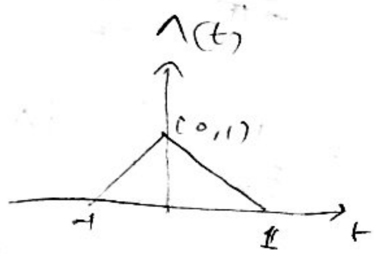
$$= \frac{-1}{j2\pi f}\left[e^{\frac{-j2\pi f}{2}} - e^{\frac{j2\pi f}{2}}\right]$$

$$= \frac{1}{j2\pi f}\left[e^{j\pi f} - e^{-j\pi f}\right]$$

$$= \frac{\sin \pi f}{\pi f}$$

$$F[\Pi(t)] = \text{sinc}(f)$$

2/ F.T of $\Lambda(t)$



$$\Lambda(t) = \begin{cases} t+1, & -1 < t < 0 \\ -t+1, & 0 < t < 1 \\ 0, & \text{otherwise.} \end{cases}$$

$$F[\Lambda(t)] = \int_{-\infty}^{\infty} \Lambda(t)\, e^{-j2\pi f t}\, dt$$

$$= \int_{-1}^{0} (t+1)\, e^{-j2\pi f t}\, dt + \int_{0}^{1} (-t+1)\, e^{-j2\pi f t}\, dt$$

SCIENCE AND ENGINEERING NOTES

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①

Consider

$$\int t \cdot e^{-j2\pi f t}\, dt$$

$$= t \cdot \frac{e^{-j2\pi f t}}{-j2\pi f} - \left(\frac{1}{-j2\pi f}\right) \cdot \frac{e^{-j2\pi f t}}{(-j2\pi f)}$$

$$= \frac{-t\, e^{-j2\pi f t}}{j2\pi f} - \frac{e^{-j2\pi f t}}{(j2\pi f)^2}$$

$$= \frac{e^{-j2\pi f t}}{(2\pi f)^2} - \frac{t\, e^{-j2\pi f t}}{j2\pi f}$$

$$= \int_{-1}^{0} t\, e^{-j2\pi f t}\, dt + \int_{-1}^{0} e^{-j2\pi f t}\, dt - \int_{0}^{1} t\, e^{-j2\pi f t}\, dt + \int_{0}^{1} e^{-j2\pi f t}\, dt$$

$$= \int_{-1}^{0} t\, e^{-j2\pi f t}\, dt - \int_{0}^{1} t\, e^{-j2\pi f t}\, dt + \int_{-1}^{1} e^{-j2\pi f t}\, dt$$

$$= \left[\frac{e^{-j2\pi f t}}{(2\pi f)^2} - \frac{t\, e^{-j2\pi f t}}{j2\pi f}\right]_{-1}^{0} - \left[\frac{e^{-j2\pi f t}}{(2\pi f)^2} - \frac{t\, e^{-j2\pi f t}}{j2\pi f}\right]_{0}^{1}$$

$$+ \left[\frac{e^{-j2\pi f t}}{-j2\pi f}\right]_{-1}^{1}$$

$$= \left[\frac{1}{(2\pi f)^2} - 0 - \frac{e^{+j2\pi f}}{(2\pi f)^2} - \frac{e^{+j2\pi f}}{j2\pi f}\right]$$

$$- \left[\frac{e^{j2\pi f}}{(2\pi f)^2} - \frac{e^{-j2\pi f}}{j2\pi f} - \frac{1}{(2\pi f)^2} + 0\right] + \left[\frac{e^{-j2\pi f} - e^{+j2\pi f}}{-j2\pi f}\right]$$

$$= 2 \cdot \frac{1}{(2\pi f)^2} - \frac{e^{j2\pi f}}{(2\pi f)^2} - \frac{e^{-j2\pi f}}{(2\pi f)^2} - \cancel{\frac{e^{j2\pi f}}{j2\pi f}} + \cancel{\frac{e^{-j2\pi f}}{j2\pi f}} - \cancel{\frac{e^{-j2\pi f}}{j2\pi f}} + \cancel{\frac{e^{j2\pi f}}{j2\pi f}}$$

$$= -\left[\left(\frac{e^{j\pi f}}{2\pi f}\right)^2 + \left(\frac{e^{-j\pi f}}{2\pi f}\right)^2 - 2\cdot\left(\frac{e^{j\pi f}}{2\pi f}\right)\left(\frac{e^{-j\pi f}}{2\pi f}\right)\right] = \frac{1}{j2}\left(\frac{e^{j\pi f} - e^{-j\pi f}}{2\pi f}\right)^2$$

SCIENCE AND ENGINEERING NOTES

(7)

$$F[\Lambda(t)] = \left(\frac{e^{j\pi f} - e^{-j\pi f}}{2j\pi f}\right)^2$$

$$= \left(\frac{\sin \pi f}{\pi f}\right)^2$$

$= \cancel{\operatorname{sinc}(f)}$

$$F[\Lambda(t)] = \operatorname{sinc}^2(f)$$

3)

ACF (Auto Correlation Function)

1) ACF of energy signal.

$$R(\tau) = \int_{-\infty}^{\infty} x(t)\, x^*(t-\tau)\, dt \quad \text{—— ①}$$

where $\tau \rightarrow$ Defined as time-delay parameter.

In the above eqn, the complex valued signal $x(t)$ is delayed in +ve-direction.

If the signal $x(t)$ is shifted by the same period 'τ' in the -ve direction, then the auto correlation function in eqn ① becomes,

$$R(\tau) = \int_{-\infty}^{\infty} x(t+\tau)\, x^*(t)\, dt .$$

2) ACF of power signal :-

Let $x(t)$ be a periodic signal with period T_0. The auto-correlation function of periodic signals over one period is given as,

$$R(\tau) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) . x^*(t-\tau)\, dt$$

If τ is -ve direction :-

$$R(\tau) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t+\tau)\, x^*(t)\, dt .$$

For any period T, $R(\tau)$ is given as

$$\boxed{R(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t). x^*(t-\tau)\, dt}$$

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1) Verify Perseval's them for energy signal

$x(t) = e^{-at} u(t)$, $a > 0$

Ans :- Total energy of signal,

$$E = \int_{-\infty}^{\infty} x^2(t)\, dt$$

$$= \int_{-\infty}^{\infty} \{ e^{-at} u(t) \}^2 \, dt .$$

but u(t) only exist, $t > 0$.

$$= \int_{0}^{\infty} e^{-2at} \, dt$$

$$= \left. \frac{e^{-2at}}{-2a} \right]_0^{\infty} = \frac{-1}{2a} [0 - 1] = \frac{1}{2a} \quad \text{--- (1)}$$

using perseval's them

$$E = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega .$$

Now :-

$$X(\omega) = \int_{-\infty}^{\infty} e^{-at} u(t) \, e^{-j\omega t} \, dt$$

u(t) exist only for, $t > 0$

$$= \int_{0}^{\infty} e^{-at} e^{-j\omega t} \, dt$$

$$= \int_{0}^{\infty} e^{-(a+j\omega)t} \, dt$$

$$= \left. \frac{e^{-(a+j\omega)t}}{-(a+j\omega)} \right]_0^{\infty} = \frac{1}{-(a+j\omega)} \cdot [0 - 1]$$

$$= \frac{1}{a+j\omega}$$

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$X(\omega) =$

$|X(\omega)|$

$|X(\omega)|$

$|X(\omega)|$

using perseva

$E =$

E

From e

them

SCIENCE AND ENGINEERING NOTES

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$$X(\omega) = \frac{1}{a+j\omega}$$

$$\{X(\omega)\} = \frac{a-j\omega}{a^2+\omega^2} = \frac{a}{a^2+\omega^2} - j\,\frac{\omega}{a^2+\omega^2}$$

$$|X(\omega)| = \sqrt{\frac{a^2}{(a^2+\omega^2)^2} + \frac{\omega^2}{(a^2+\omega^2)^2}} = \sqrt{\frac{\cancel{(a^2+\omega^2)}}{(a^2+\omega^2)^{\cancel{2}}}}$$

$$|X(\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}$$

Using perseval's theorem

$$E = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{a^2+\omega^2}}\right)^2 d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{a^2+\omega^2}\,d\omega$$

$$= \frac{1}{2\pi}\cdot\frac{1}{a}\cdot\tan^{-1}\left(\frac{\omega}{a}\right)\Big]_{-\infty}^{\infty}$$

$$= \frac{1}{2\pi a}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right]$$

$$= \frac{1}{2\cancel{\pi}a} \times \cancel{\pi}$$

$$E = \frac{1}{2a} \quad \text{—— (2)}$$

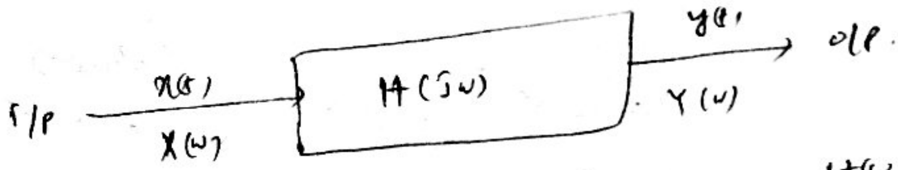
From eqn (1) & (2) Perseval's theorem is verified.

SCIENCE AND ENGINEERING NOTES

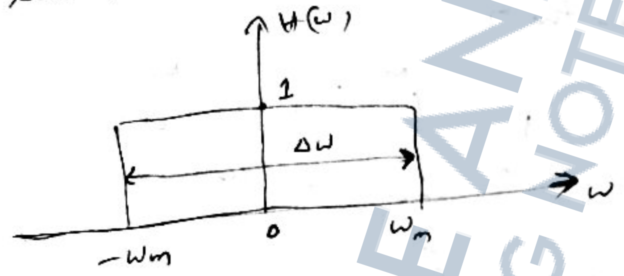
(*) ⊙

Energy Spectral density:- (ESD)

Let us consider a signal $x(t)$ which is applied to an ideal low pass filter.



The graph of transfer function $H(\omega)$ of the ideal low pass filter is shown below.



The o/p of the system is expressed as,

$$Y(\omega) = X(\omega) \cdot H(\omega)$$

where $X(\omega)$ = F.T of $x(t)$

$Y(\omega)$ = F.T of $y(t)$

Using perseval's thm, E_o of o/p signal $y(t)$

$$E_o = \frac{1}{2\pi} \int_{-\infty}^{\infty} |Y(\omega)|^2 \, d\omega$$

$$\Rightarrow E_o = \frac{1}{2\pi} \int_{-\infty}^{\infty} | H(\omega) \cdot X(\omega) |^2 \, d\omega$$

$H(\omega) = 0$, for all frequencies except for the narrow band $-\omega_m$ to ω_m for which it is unity.

$$E_o = \frac{1}{2\pi} \int_{-\omega_m}^{\omega_m} |1 \cdot X(\omega)|^2 \, d\omega$$

$$= \frac{1}{2\pi} \int^{\omega_m} |X(\omega)|^2 \, d\omega$$

SCIENCE AND ENGINEERING NOTES

(76) We may assume $X(\omega)$ is a const. with freq for narrow band $-\omega_m$ to ω_m.

$$\therefore \quad E_0 = \frac{1}{2\pi}|X(\omega)|^2 \int_{-\omega_m}^{\omega_m} d\omega$$

$$= \frac{1}{2\pi}|X(\omega)|^2 \cdot 2\omega_m$$

Putting $2\omega_m = \Delta\omega$,

$$E_0 = \frac{1}{2\pi}|X(\omega)|^2 \cdot \Delta\omega$$

$$E_0 = |X(\omega)|^2 \cdot \Delta f$$

$$\Rightarrow \boxed{\frac{E_0}{\Delta f} = |X(\omega)|^2}.$$

Hence, $|X(\omega)|^2$ represent energy per unit bandwidth and is known as Energy Spectral Density or Energy Density Spectrum.

→ It is denoted by $\boxed{\Psi(\omega) = |X(\omega)|^2}$

→ By parseval's theorem, $E = \frac{1}{2\pi}\int_{-\infty}^{\infty}|X(\omega)|^2 d\omega$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\Psi(\omega)\, d\omega$$

For Real Signal $|X(\omega)|^2 = |X(-\omega)|^2$.

$$E = \frac{2}{2\pi}\int_{0}^{\infty}|X(\omega)|^2 d\omega$$

For Real signal, $E = \frac{1}{\pi}\int_{0}^{\infty}\Psi(\omega)\, d\omega$

Reln betn i/p & o/p energy spectral density:-

$$Y(\omega) = H(\omega) \cdot X(\omega)$$

$$|Y(\omega)|^2 = |H(\omega) \cdot X(\omega)|^2 = |H(\omega)|^2 \cdot |X(\omega)|^2 \quad \text{---} (1)$$

Let $\psi_y(\omega)$ be the ESD of o/p $y(t)$ &

$\psi_x(\omega)$ be the ESD of i/p $x(t)$.

$$\psi_y(\omega) = |Y(\omega)|^2 \quad \text{---} (2)$$

$$\psi_x(\omega) = |X(\omega)|^2 \quad \text{---} (3)$$

Putting eqn (2) & (3) in eqn (1),

$$\cancel{|Y(\omega)|^2} = \quad \boxed{\psi_y(\omega) = |H(\omega)|^2 \cdot \psi_x(\omega)}$$

Properties of ESD :-

1) The total area under ESD function is equal to the total energy of that signal.

i.e $$E = \int_{-\infty}^{\infty} \psi(f)\, df.$$

2) If the $x(t)$ is i/p to a linear time variant (LTI) system with transfer function $H(\omega)$, then i/p and o/p ESD function are related as

$$\psi_o(\omega) = |H(\omega)|^2 \psi_i(\omega)$$

where $\psi_o(\omega)$ = o/p ESD function

$\psi_i(\omega)$ = i/p ESD function.

$|H(\omega)|^2$ = energy gain at frequency ω

3) The ACF $R(\tau)$ and ESD $\psi(\omega)$ form

(Auto Correlation Function)

1 a Fourier transform pair.

$$\boxed{R(\tau) \longleftrightarrow \psi(\omega)}$$

Power Signal

Power signal is one which has finite average power and infinite energy.

i.e
$$\boxed{\begin{matrix} 0 < P < \infty \\ \text{and} \quad E = \infty \end{matrix}}$$

P = Avg. Power

E = Energy.

→ Almost all practical periodic signals are power signals since their avg. power is finite and non-zero. For continuous time signal, the avg power P of a signal is defined as

$$\boxed{P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t)\, dt}$$

Perseval's thm for power signal :-

Perseval's thm for power signals states that the power of a signal may be defined in terms of its Fourier series co-efficients.

Proof:- Let there be a function $x(t)$

we know $|x(t)|^2 = x(t)\, x^*(t)$

Here $x^*(t)$ is the complex conjugate of the function $x(t)$

The power of a signal $x(t)$ over a cycle is expressed as $P = \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt$

SCIENCE AND ENGINEERING NOTES

$$P = \frac{1}{T} \int_{-T/2}^{T/2} x(t) \; x^*(t) \; dt$$

Replacing $x^*(t)$ by its complex Fourier exponential series, we get

$$P = \frac{1}{T} \int_{-T/2}^{T/2} x^*(t) \left[\sum_{n=-\infty}^{\infty} C_n \, e^{jn\omega_0 t} \right] dt$$

where $\omega_0 = \frac{2\pi}{T}$

Interchanging the order of Integration and summation, we get

$$P = \frac{1}{T} \sum_{n=-\infty}^{\infty} C_n \int_{-T/2}^{T/2} x^*(t) \, e^{jn\omega_0 t} \, dt$$

Since $C_n = \frac{1}{T} \int_{-T/2}^{T/2} x(t) \, e^{-jn\omega_0 t} \, dt$

So $C_n^* = \frac{1}{T} \int_{-T/2}^{T/2} x^*(t) \, e^{jn\omega_0 t} \, dt$

$$P = \frac{1}{T} \sum_{n=-\infty}^{\infty} C_n \cdot \left(T \, C_n^* \right)$$

$$= \sum_{n=-\infty}^{\infty} C_n \cdot C_n^*$$

$$\boxed{P = \sum_{n=-\infty}^{\infty} |C_n|^2}$$

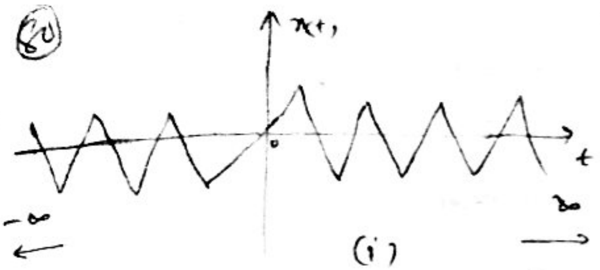
The above equation is known as Perseval's th^m for power signals.

The Power Spectral Density (PSD) :-

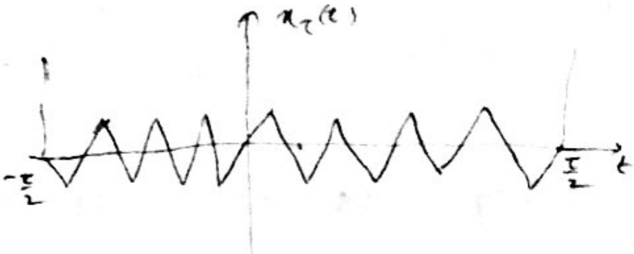
The expression for PSD may be derived by assuming the power signal as a limiting case of energy signal.

SCIENCE AND ENGINEERING NOTES

(80)



(i)



(ii)

Let us consider a power signal x(t) which is extended to infinity.
Lets terminate this signal such that it is zero outside the interval, $\pm\frac{\tau}{2}$.
Let this terminated signal denoted by $\underline{x_\tau(t)}$.

$$x_\tau(t) = \begin{cases} x(t), & |t| < \frac{\tau}{2} \\ 0, & \text{elsewhere} \end{cases}$$

Now since the terminated signal $x_\tau(t)$ is of finite duration τ, ~~therof~~ therefore it is a energy signal

$$\therefore E_\tau = \int_{-\infty}^{\infty} |x_\tau(t)|^2 dt = \int_{-\infty}^{\infty} |X_\tau(\omega)|^2 df \quad \text{(Perseval's thm)} \quad \text{——— (1)}$$

We have,
$$\int_{-\infty}^{\infty} |x_\tau(t)|^2 dt = \int_{-\tau/2}^{\tau/2} |x(t)|^2 dt \quad \text{——— (2)}$$

(because $x_\tau(t)$ exist only in $[-\frac{\tau}{2}, \frac{\tau}{2}]$ outside it is zero.)

Therefore

$$\frac{1}{\tau}\int_{-\tau/2}^{\tau/2} |x(t)|^2 dt = \frac{1}{\tau}\cdot\int_{-\infty}^{\infty} |X_\tau(\omega)|^2 df \quad \text{(from eqn (1))}$$

$$\Rightarrow \lim_{\tau\to\infty} \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} |x(t)|^2 dt = \lim_{\tau\to\infty} \frac{1}{\tau} \int_{-\infty}^{\infty} |X_\tau(\omega)|^2 df$$

$$\Rightarrow P = \int_{-\infty}^{\infty} \lim_{\tau\to\infty} \frac{1}{\tau} |X_\tau(\omega)|^2 df \quad \text{——— (3)}$$

In the limit $\tau\to\infty$, the ratio $\frac{|X_\tau(\omega)|^2}{\tau}$ may be approach to a finite value. Let the finite value be $\underline{S(\omega)}$

SCIENCE AND ENGINEERING NOTES

So that $$\boxed{S(\omega) = \lim_{\tau \to \infty} \frac{|X_\tau(\omega)|^2}{\tau}} \quad \text{---(4)}$$

using eqn (4) in eqn (3), we have

$$P = \int_{-\infty}^{\infty} S(\omega)\, df = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\, d\omega .$$

According to above equation, the total power of the signal is obtained by multiplying the bandwidth $\Delta\omega$ or $d\omega$ & integrating over the entire bandwidth. Hence $\boxed{S(\omega)}$, may be treated as avg. power per unit BW (Band width) & so it is called PSD or power density spectrum.

Since $|X_\tau(\omega)|^2 = |X_\tau(-\omega)|^2$,

Therefore, the power contribution by +ve & -ve frequencies are identical.

$$\therefore P = \int_{-\infty}^{\infty} S(\omega)\, df = 2\int_{0}^{\infty} S(\omega)\, df = \frac{1}{\pi}\int_{0}^{\infty} S(\omega)\, d\omega$$

Relⁿ betⁿ i/p & o/p PSD.

$$S_y(\omega) = |H(\omega)|^2 \; S_x(\omega)$$

where $S_y(\omega)$ = PSD of response $y(t)$ (o/p).

$S_x(\omega)$ = PSD of input $x(t)$

$H(\omega)$ = transfer f^n of the system.

Properties of PSD:-

1) The area under PSD f^n is equal to the avg. power of that signal

$$\therefore P = \int_{-\infty}^{\infty} S(\omega)\, df$$

SCIENCE AND ENGINEERING NOTES

2) If $x(t)$ is i/p to LTI (Linear time invariant) system with transfer function $H(j\omega)$ then i/p & o/p PSD are related as

$$S_o(\omega) = |H(\omega)|^2 S_i(\omega)$$

where $S_o(\omega) =$ o/p PSD.

$S_i(\omega) =$ i/p PSD

$|H(\omega)|^2 =$ power gain of LTI system.

3) The Autocorrelation function $R(\tau)$ & PSD $S(\omega)$ form Fourier transform pair.

$$R(\tau) \leftrightarrow S(\omega)$$

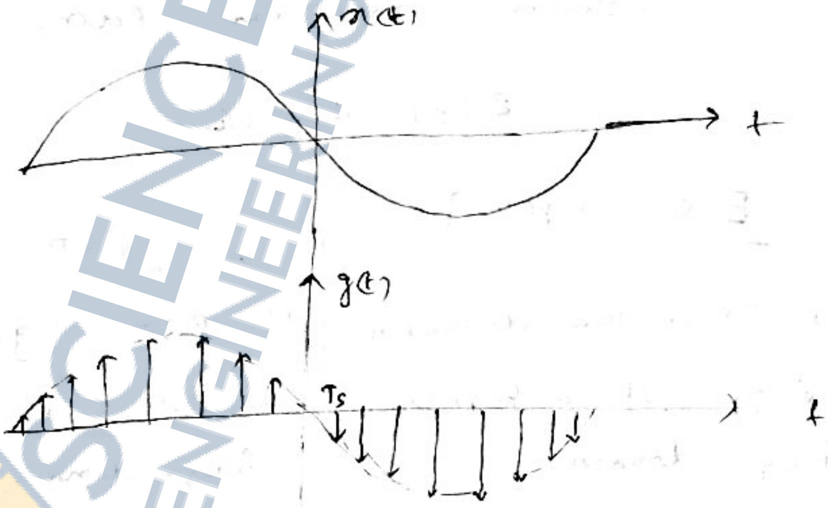
ESD $[\psi(\omega)]$	PSD $[S(\omega)]$
1) ESD gives the distribution of energy of a signal in freq domain.	1) PSD gives the distribution of power of a signal in freq-domain.
2) ESD is given as $\psi(\omega) = \|X(\omega)\|^2$	2) PSD is given as $S(\omega) = \lim_{\tau\to\infty} \frac{\|X_\tau(\omega)\|^2}{\tau}$
3) Total energy $E = \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi(\omega)\, d\omega = \int_{-\infty}^{\infty} \psi(f)\, df$	3) Total power $P = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\, d\omega = \int_{-\infty}^{\infty} S(f)\, df.$
4) Autocorrelation function (ACF) & ESD form a Fourier transform pair $R(\tau) \leftrightarrow \psi(\omega)$	4) ACF & PSD form a Fourier transform pair. $R(\tau) \leftrightarrow S(\omega)$

SCIENCE AND ENGINEERING NOTES

Sampling of bandlimited signals:-
(Low freq signals)

Sampling ~~[illegible]~~ them:-

A Continuous-time signal may be completely represented in its samples and recovered back if the sampling frequency is $f_s \geq 2f_m$, where f_s is the sampling frequency & f_m is the max^m frequency present in the signal.



i.e (i) A band limited signal of finite energy, which has no frequency component higher than f_m Hz, is completely described by its samples values at uniform intervals less than equal to $\frac{1}{2f_m}$ seconds apart.

Since $\boxed{f_s \geq 2f_m}$

$$\Rightarrow \boxed{T_s \leq \frac{1}{2f_m}}$$

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8) Nyquist rate & Nyquist interval:-

When the sampling rate becomes exactly equal to $2f_m$ samples per second, then it is called Nyquist rate. Nyquist rate is also called <u>min^m sampling rate.</u>

$$\boxed{f_s = 2f_m}$$

<u>Similarly</u>, max^m sampling interval is called the ~~max^m~~ Nyquist interval.

$$\text{Nyquist interval } (T_s) = \frac{1}{2f_m}$$

Ex 1) Find <u>Nyquist rate</u>

$$x(t) = 3\cos 50\pi t + 10\sin 300\pi t - \cos 100\pi t.$$

Ans:

$$f_{m1} = \frac{50\pi}{2\pi} = 25\text{ HZ} \qquad \left(\because \omega = 50\pi,\ 2\pi f = 50\pi \Rightarrow f = \frac{50\pi}{2\pi} = 25 \right)$$

$$f_{m2} = \frac{300\pi}{2\pi} = 150\text{ HZ}$$

$$f_{m3} = \frac{100\pi}{2\pi} = 50\text{ HZ}.$$

$$f_s = 2\,f_m|_{max}$$

$$= 2 \times 150\text{ HZ}.$$

$$\boxed{f_s = 300\text{ HZ}.}$$

2) $x(t) = \frac{1}{2\pi}\cos(4000\pi t)\cdot\cos(1000\pi t)$, find <u>$f_s$</u>, <u>$T_s$</u>

Ans:

$$x(t) = \frac{1}{2\pi} \times \frac{1}{2} \times 2\cos(4000\pi t)\cdot\cos(1000\pi t)$$

$$= \frac{1}{4\pi}\cdot\left[\cos 3000\pi t + \cos 5000\pi t\right]$$

$$\omega = 5000\pi, \quad f = \frac{5000\pi}{2\pi} = 2500.$$

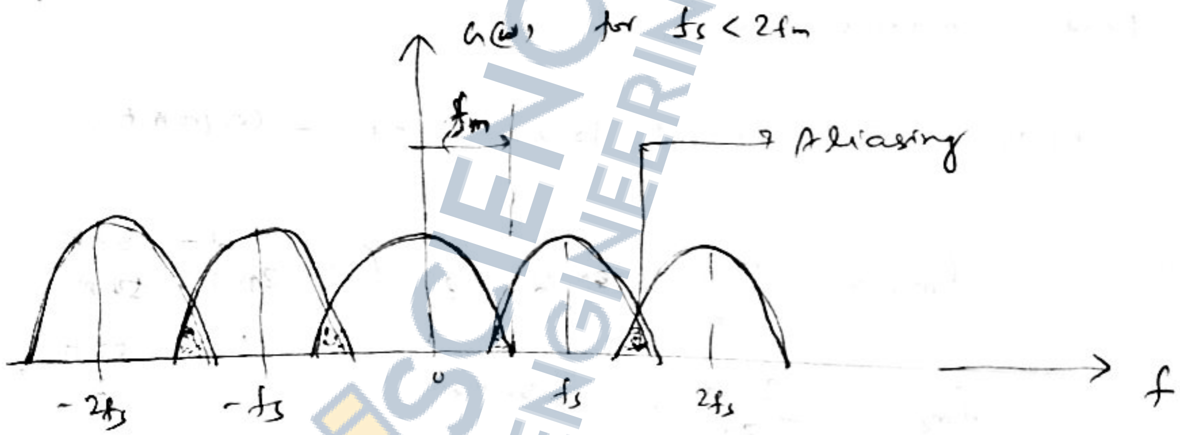
$$f_s = 2f_m = 2 \times 2500 = 5000\text{ HZ} = 5\text{ KHZ}.$$

SCIENCE AND ENGINEERING NOTES

Nyquist interval $(T_s) = \frac{1}{f_s} = \frac{1}{5000} = 0.2$ msec.

Aliasing effect:-

When a continuous-time band-limited signal is sampled at a rate lower than Nyquist rate i.e $f_s < 2f_m$, the successive cycle of the spectrum $G(\omega)$ of sampled signal gets overlap with each other. This type of distortion that results from under sampling is known as aliasing error or aliasing distortion



Spectrum of sampled signals for the case $f_s < 2f_m$

To avoid aliasing:-

(i) Prealias filter must be used to limit band of frequency of signal to f_m HZ.

(ii) Sampling frequency 'f_s' must be selected such that $f_s > 2f_m$

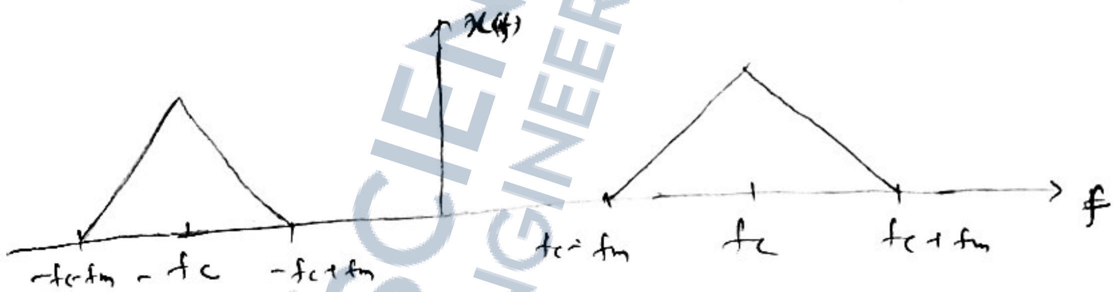
86 Sampling of bandpass signal :-

(which has a Lower & upper cutoff freq)
→ one particular band of freq.

The bandpass signal $x(t)$ whose max^m bandwidth is $2f_m$ can be completely ~~recovered~~ represented into & recovered from its samples if it is sampled at the min^m rate of twice the bandwidth.

f_m → Max^m frequency component present in the signal.

Hence, if the BW is $2f_m$, min^m sampling rate for bandpass signal must be $4f_m$ samples per second.



Min^m sampling rate = Twice of BW

$$= 2 \times 2f_m$$

$$f_s = 4f_m \text{ Samples / second}$$

✓ ex :- The spectral range of f^n extends from 10.0 to 10.2 MHZ. Find the min^m sampling rate & max^m sampling time.

Ans :- $f_s = 2 \times (f_H - f_L) = 2 \times (10.2 - 10)$ MHZ

$$= 2 \times 0.2 \times 10^6 = 0.4 \text{ MHZ}$$

$$T_s = \frac{1}{f_s} = \frac{1}{0.4 \times 10^6} = 2.5 \ \mu s.$$

SCIENCE AND ENGINEERING NOTES

Note :- 1) ACF of energy signal

$$R_{x_E}(\tau) = \int_{-\infty}^{\infty} x(t)\, x^*(t-\tau)\, dt$$

If $\tau = 0$, $R_{x_E}(0) = \int_{-\infty}^{\infty} x(t) \cdot x^*(t)\, dt$

$$= \int_{-\infty}^{\infty} |x(t)|^2\, dt$$

= Energy of the signal

$$\therefore \boxed{E = R_{x_E}(0)}$$

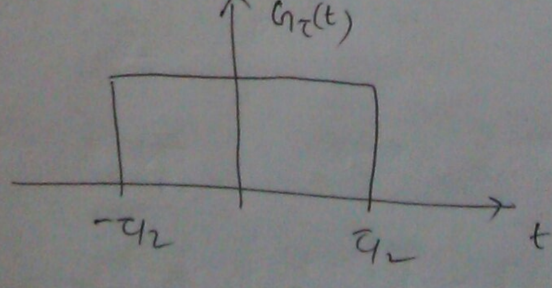
2) ACF of power signal

$$R_{x_P}(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t) \cdot x^*(t-\tau)\, dt$$

$\tau = 0$, $R_{x_P}(0) = \lim_{T \to \infty} \int_{-T/2}^{T/2} |x(t)|^2\, dt$ = power of a signal.

$$\therefore \boxed{P = R_{x_P}(0)}$$

3) Find ESD of Gate function



Ans :-

$$ESD = |X(\omega)|^2$$

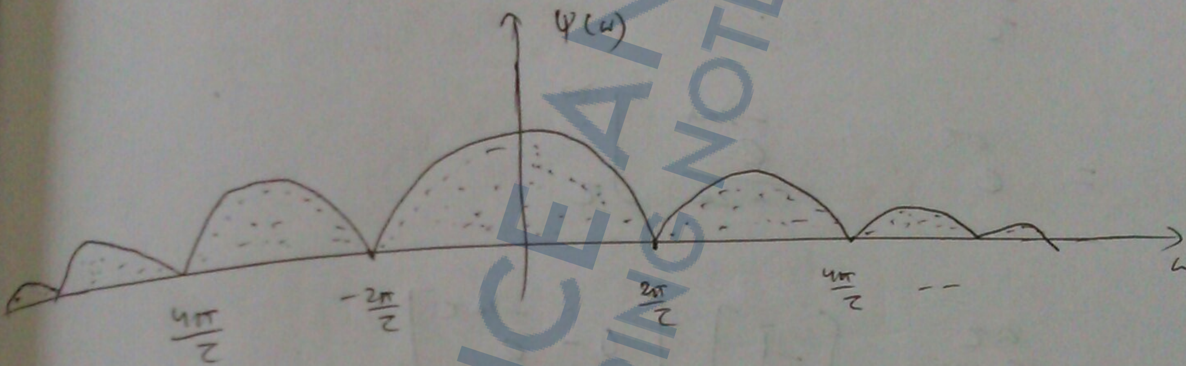
SCIENCE AND ENGINEERING NOTES

$X(\omega)$ of gate f^n $= A\tau \; Sa\left(\frac{\omega\tau}{2}\right)$

[Derived earlier].

$$ESD = |X(\omega)|^2 = \left|A\tau \; Sa\left(\frac{\omega\tau}{2}\right)\right|^2$$
$(\Psi(\omega))$

$$\boxed{ESD = A^2\tau^2 \; Sa^2\left(\frac{\omega\tau}{2}\right)}$$



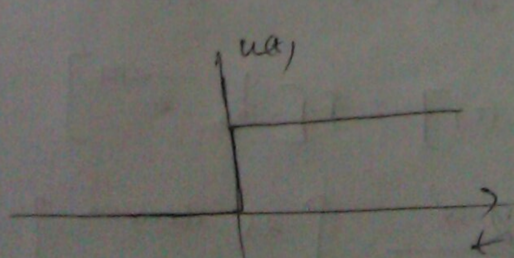
4) Find the time auto correlation function of the signal $g(t) = e^{-at}\,u(t)$, and from this obtain ESD of $g(t)$

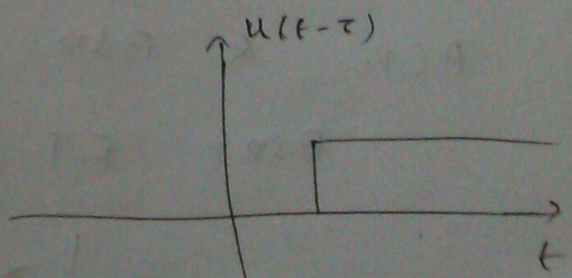
Ans:-

$$R(\tau) = \int_{-\infty}^{\infty} g(t) \cdot g^*(t-\tau)\, dt$$

$$g(t) = e^{-at}\,u(t)$$

$$g(t-\tau) = e^{-a(t-\tau)}\,u(t-\tau)$$





SCIENCE AND ENGINEERING NOTES

Common area betⁿ $u(t)$ & $u(t-\tau)$

R's τ to ∞.

$$R(\tau) = \int_{\tau}^{\infty} e^{-at} \cdot e^{-a(t-\tau)}\, dt$$

$$= \int_{\tau}^{\infty} e^{-2at} \cdot e^{a\tau}\, dt$$

$$= e^{a\tau} \cdot \left.\frac{e^{-2at}}{-2a}\right]_{\tau}^{\infty}$$

$$= e^{a\tau} \left[-\frac{1}{2a}\right]\left[0 - e^{-2a\tau}\right]$$

$$= \frac{e^{a\tau} \times e^{-2a\tau}}{2a}$$

$$R(\tau) = \frac{e^{-a\tau}}{2a}$$

Since $R(\tau) = R(-\tau)$

$$R(\tau) = \frac{1}{2a} e^{-a|\tau|}$$

ACR & ESD are F.T pair.

$$ESD = F.T\left[R(\tau)\right] = F\left[\frac{1}{2a} \cdot e^{-a|\tau|}\right]$$

$$= \frac{1}{\cancel{2a}} \times \frac{\cancel{2}a}{a^2+\omega^2}$$ [as derived earlier]

ESD

or.

$$\boxed{ESD = \frac{1}{a^2+\omega^2}}$$

or

$$ESD = |X(\omega)|^2$$

$$x(t) = e^{-at}u(t)$$

$$X(\omega) = \frac{1}{a+j\omega}$$

$$|X(\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}$$

$$|X(\omega)|^2 = \frac{1}{a^2+\omega^2}$$

$$\therefore \boxed{ESD = \frac{1}{a^2+\omega^2}}$$

SCIENCE AND ENGINEERING NOTES