Module - II :- Wire Antennas

In this Chapter, we will study the radiation characteristics of wire antennas. These antennas are made of thin, conducting, straight or curved wire segments or hollow tubes and are very easy to construct.
The dipole and the monopole are examples of straight wire antennas; the loop antenna is an example of a curved wire antenna.

One of the assumptions made for this class of antennas is that the radius of the wire is very small compared to the operating wavelength. As a consequence, we can assume that the current has only one component along the wire. The variation of the current along the wire depends on the length and shape of the wire.

The assumed current distribution on the wire enables us to compute the electric and magnetic fields in the far-field region of the antenna using the magnetic vector potential. With the knowledge of the fields, we can compute the antenna characteristics such as Directivity, Radiation resistance, etc.

Short Dipole / Small Dipole $\left[\frac{\lambda}{50} < l \leq \frac{\lambda}{10} \right]$

Consider a short dipole of length 'l' ($l < 0.1\lambda$) and radius 'a' ($a << \lambda$), symmetrically placed about the origin and oriented along Z-axis as shown in figure 1.

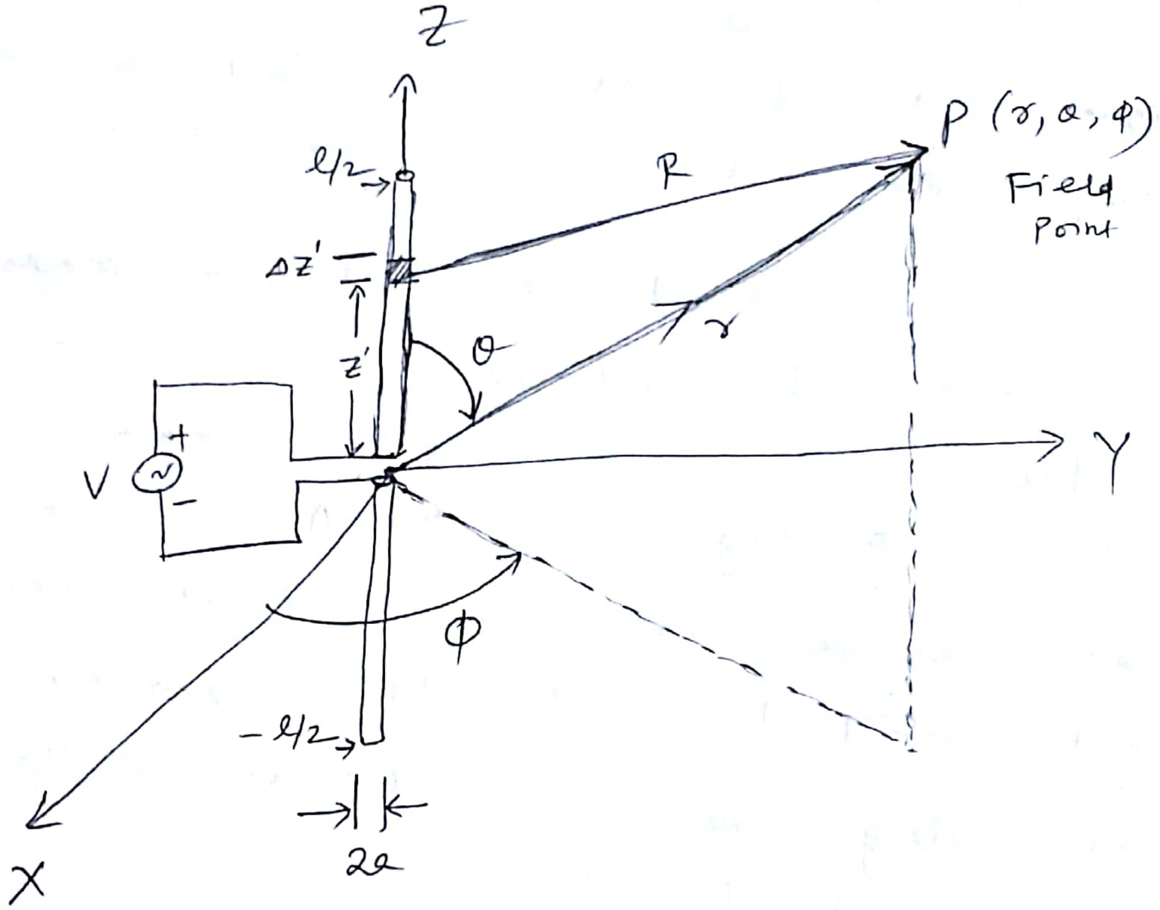


Fig 1: Geometry of a thin wire dipole

Measurements show that the current on a short wire dipole with feed point as shown in fig. 1 has a triangular distribution with a maximum at the center and linearly tapering off to <u>zero</u> at the ends. As shown in figure 2, Mathematically the current on the dipole in the regions $0 \leq z' \leq \frac{l}{2}$ and $-\frac{l}{2} \leq z' \leq 0$ is given by eq^n ①.

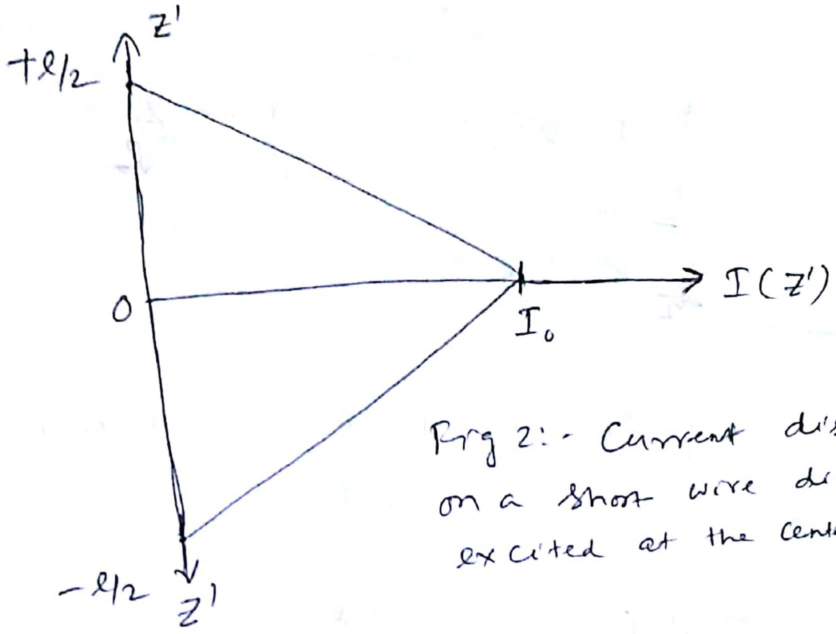
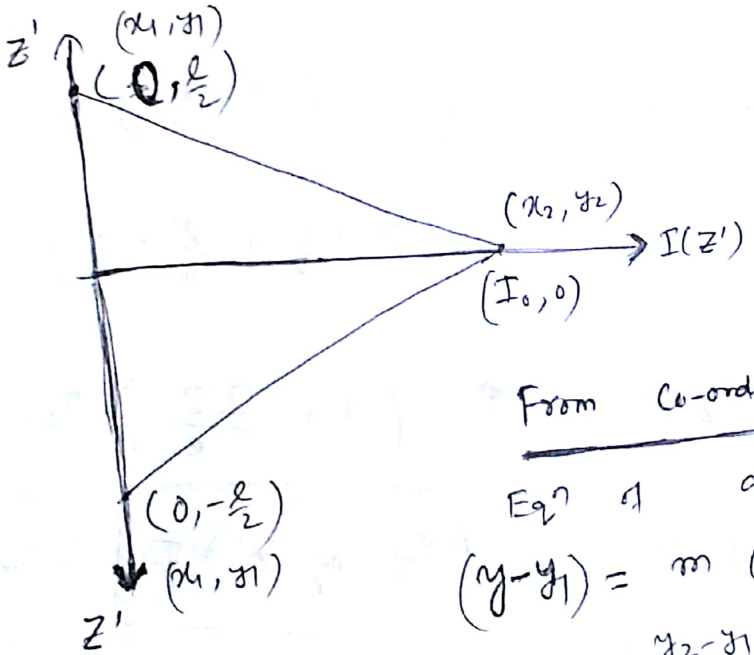


Fig 2:- Current distribution on a short wire dipole excited at the center.

$$I_z(z') = \begin{cases} \left(1 - \frac{2}{l} z'\right) I_0 , & 0 \leq z' \leq \frac{l}{2} \\ \left(1 + \frac{2}{l} z'\right) I_0 , & -\frac{l}{2} \leq z' \leq 0 \end{cases} \quad \text{—— ①}$$

Proof:-



From Co-ordinate geometry

Eq^n of a straight line

$$(y - y_1) = m\,(x - x_1)$$

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}\,(x - x_1)$$

For $0 \leq z' \leq \frac{l}{2}$

$$z' - \frac{l}{2} = \frac{0 - \frac{l}{2}}{I_0 - 0}\,(I(z') - 0)$$

$$\Rightarrow I(z') = -\left(z' - \frac{l}{2}\right) \times \frac{2 I_0}{l}$$

$$= \left(\frac{l}{2} - z'\right) \times \frac{2 I_0}{l}$$

$$\Rightarrow I(z') = \left(\cancel{\frac{l}{2}} \times \cancel{\frac{2}{l}} - z' \times \frac{2}{l}\right) I_0$$

$$\Rightarrow \boxed{I(z') = \left(1 - \frac{2}{l} z'\right) I_0} \quad \text{--- (i)}$$

For $-\frac{l}{2} \le z' \le 0$

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1)$$

$$z' + \frac{l}{2} = \frac{0 + \frac{l}{2}}{I_0 - 0} (I(z') - 0)$$

$$\Rightarrow z' + \frac{l}{2} = \frac{l}{2 I_0} I(z')$$

$$\Rightarrow I(z') = \frac{2 I_0}{l} \left(z' + \frac{l}{2}\right)$$

$$\Rightarrow I(z') = I_0 \left(z' \times \frac{2}{l} + \cancel{\frac{l}{2}} \times \cancel{\frac{2}{l}}\right)$$

$$\Rightarrow I(z') = \left(1 + \frac{2z'}{l}\right) I_0 \quad \text{---} \cancel{(ii)}$$

$$\Rightarrow \boxed{I(z') = \left(1 + \frac{2}{l} z'\right) I_0} \quad \text{--- (ii)}$$

x

Since the current is Z-directed, the magnetic vector potential, A, has only a Z-component given by

$$A(x, y, z) = a_z \frac{\mu}{4\pi} \int_{-l/2}^{l/2} I_z(z') \frac{e^{-jkR}}{R} dz' \quad \text{--- (2)}$$

where R is the distance from the source point $(x'=0, y'=0, z')$ on the dipole to the field point (x,y,z) and is given by

$$R = \sqrt{x^2 + y^2 + (z-z')^2} \quad \text{—— (3)}$$

Expressing the field point (x,y,z) in spherical co-ordinates using the following transformation eqⁿs,

$$x^2+y^2+z^2 = r^2 \quad \text{—— (4)}$$

$$z = r\cos\theta \quad \text{—— (5)},$$

R can be written as

$$R = \sqrt{r^2 - z^2 + (z-z')^2}$$

$$= \sqrt{r^2 - \cancel{z^2} + \cancel{z^2} + z'^2 - 2zz'}$$

$$= \sqrt{r^2 + z'^2 - 2 . r\cos\theta . z'}$$

$$R = \sqrt{r^2 - 2rz'\cos\theta + z'^2}$$

$$= \sqrt{r^2 \left(1 - \frac{2z'}{r}\cos\theta + \frac{z'^2}{r^2}\right)}$$

$$R = r\left(1 + \left[-\frac{2z'}{r}\cos\theta + \left(\frac{z'}{r}\right)^2\right]\right)^{\frac{1}{2}} \quad \text{—— (6)}$$

For $\underline{r >> z'}$ $\quad \left(\frac{z'}{r}\right)^2 \simeq 0$

$$R = r\left(1 - \frac{2z'}{r}\cos\theta\right)^{\frac{1}{2}}$$

Using the Binomial expansion,

$$(1+x)^n \simeq 1 + nx \quad \text{for } x << 1$$

$$R \simeq r\left(1 - \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}z'}{r}\cos\theta\right)$$

$$R \simeq r\left(1 - \frac{z'}{r}\cos\theta\right)$$

$$R \simeq r - z'\cos\theta \quad \text{——— } (7)$$

In the expression for the magnetic vector potential, A [Eqn (2)], the distance R between the source and the field point appears both in amplitude and phase of integrand. While evaluating the expression in the far field of an antenne, we can use the approximation given in eqn (7), for the phase term

$$e^{-jkR} \simeq e^{-jk(r - z'\cos\theta)} \quad \text{——— } (8)$$

This approximation approximation results in a maximum phase error $\frac{\pi}{8}$ rad (22.5°).

Since both r and R are very large compared to the wavelength, the following approximation is used for amplitude term

$$R \simeq r \quad \text{——— } (9)$$

Which results in a very small error in the amplitude. Equation (7) & (9) are known as the far-field approximation for R.

While evaluating the magnetic vector potential in the far-field region, the term $\frac{e^{-jKR}}{R}$ in the integrand is approximated by

$$\frac{e^{-jKR}}{R} \simeq \frac{e^{-jK(r - z'\cos\theta)}}{r} = \frac{e^{-jKr}}{r} \cdot e^{jKz'\cos\theta} \quad \text{(10)}$$

Geometrically, the far-field approximation implies that the vectors $\underline{R}$ and $\underline{r}$ are parallel to each other and a path difference between the two exists [Fig 3]. $\underline{z'\cos\theta}$

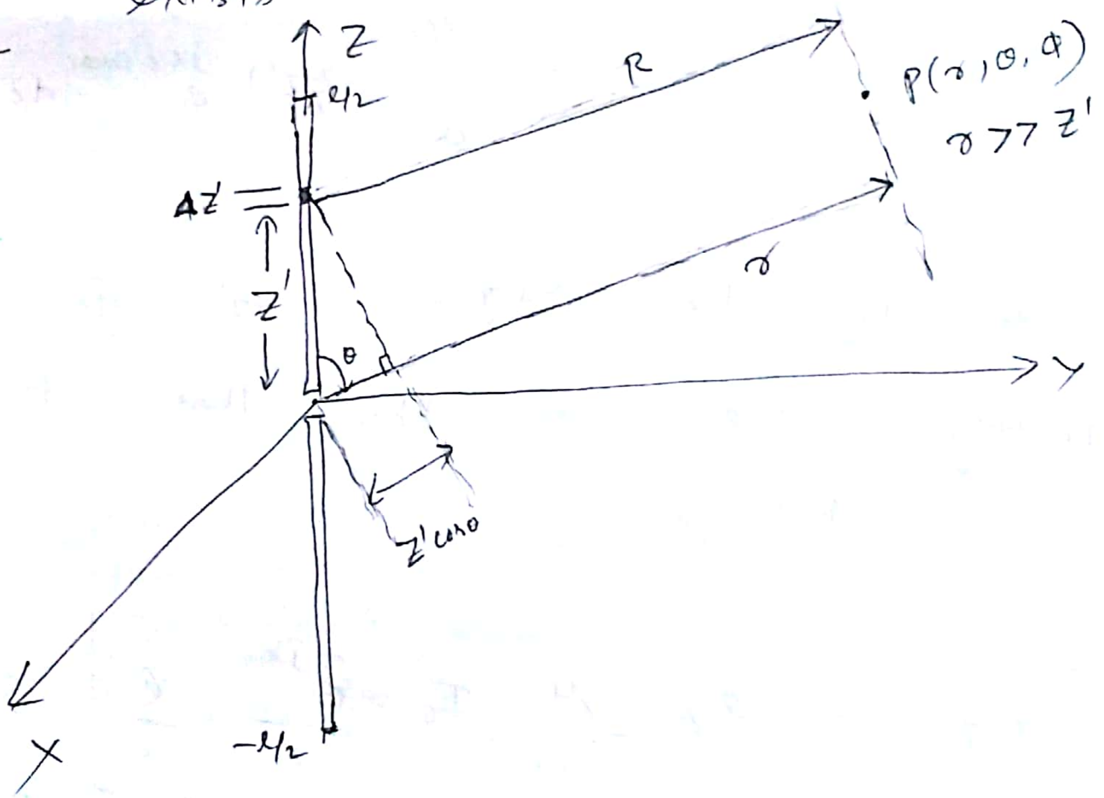


Fig 3:- The far-field approximation.

Now we will evaluate the magnetic vector potential by substituting the current distribution on the dipole given in eqn (1) into eqn (2), ∴ We have

$$A(x,y,z) = a_z \frac{\mu}{4\pi}\left[\int_{-l/2}^{0} \left(1+\frac{2}{l}z'\right) I_0 \frac{e^{-jKR}}{R}\,dz' + \int_{0}^{l/2} \left(1-\frac{2}{l}z'\right) I_0 \frac{e^{-jKR}}{R}\,dz'\right] \quad \text{(11)}$$

Introducing the far-field approximation as given in eqn (10), eqn (11) becomes

$$A(x,y,z) = a_z \frac{\mu}{4\pi}\left[\int_{-l/2}^{0}\left(1+\frac{2}{l}z'\right) I_o \cdot \frac{e^{-jkr}}{r} \cdot e^{jkz'\cos\theta}\, dz' + \int_{0}^{l/2}\left(1-\frac{2}{l}\right)z' I_o \cdot \frac{e^{-jkr}}{r} \cdot e^{jkz'\cos\theta}\, dz'\right]$$

$$A(x,y,z) = \hat{a}_z \cdot \frac{\mu}{4\pi} \cdot \frac{e^{-jkr}}{r} \cdot I_o \left[\int_{-l/2}^{0}\left(1+\frac{2}{l}z'\right) e^{jkz'\cos\theta}\, dz' + \int_{0}^{l/2}\left(1-\frac{2}{l}z'\right) e^{jkz'\cos\theta}\, dz'\right] \quad \text{--- (12)}$$

By evaluating the integral in eqn (12), and simplifying we can show that for

$$\frac{kl}{4} << 1$$

$$\boxed{A(x,y,z) \simeq a_z \frac{\mu}{4\pi} I_o \cdot \frac{e^{-jkr}}{r} \cdot \frac{l}{2}} \quad \text{--- (13)}$$

{ For Proof :- Integration of the terms in the square brackets Refer to Book

[Page No 100 : A.R. Harish & M. Sachidananda) Antenna & Wave Propagation }

Solution: Let us denote the term in the square brackets by

$$I = \left[\int_{-l/2}^{0}\left(1+\frac{2}{l}z'\right)e^{jkz'\cos\theta}dz' + \int_{0}^{l/2}\left(1-\frac{2}{l}z'\right)e^{jkz'\cos\theta}dz'\right]$$

Substituting $z' = -z''$ in the first integral, and interchanging the limits

$$I = \int_{0}^{l/2}\left(1-\frac{2}{l}z''\right)e^{-jkz''\cos\theta}dz'' + \int_{0}^{l/2}\left(1-\frac{2}{l}z'\right)e^{jkz'\cos\theta}dz'$$

Since both the integrals have the limits 0 to $l/2$, we can write

$$I = \int_{0}^{l/2}\left(1-\frac{2}{l}z'\right)(e^{jkz'\cos\theta}+e^{-jkz'\cos\theta})dz'$$

Now we can simplify the integrand by using the identity $e^{jx}+e^{-jx} = 2\cos x$, to get

$$I = \int_{0}^{l/2}\left(1-\frac{2}{l}z'\right)2\cos(kz'\cos\theta)dz'$$

Performing the integration

$$I = 2\left[\frac{\sin(kz'\cos\theta)}{k\cos\theta}\right]_0^{l/2} - \frac{4}{l}\left[z'\frac{\sin(kz'\cos\theta)}{k\cos\theta}+\frac{\cos(kz'\cos\theta)}{(k\cos\theta)^2}\right]_0^{l/2}$$

Substituting the limits and simplifying, we get

$$I = \frac{4}{l(k\cos\theta)^2}\left[1-\cos\left(\frac{kl}{2}\cos\theta\right)\right]$$

Using the identity $\cos(2\theta) = 1 - 2\sin^2\theta$, the above expression can be written as

$$I = \frac{4}{l(k\cos\theta)^2}2\sin^2\left(\frac{kl}{4}\cos\theta\right)$$

For $kl/4 \ll 1$

$$\sin^2\left(\frac{kl}{4}\cos\theta\right) \simeq \left(\frac{kl}{4}\cos\theta\right)^2$$

and hence the integral reduces to

$$I \simeq \frac{l}{2}$$

Following the procedure as described for Hertzian dipole, we first express the components of the magnetic vector potential in spherical co-ordinates as

$$A_r = A_z \cos\theta = \frac{\mu}{4\pi} I_0 \cdot \frac{e^{-jkr}}{r} \frac{l}{2} \cos\theta \quad \text{--- (14)}$$

$$A_\theta = -A_z \sin\theta = -\frac{\mu}{4\pi} I_0 \cdot \frac{e^{-jkr}}{r} \cdot \frac{l}{2} \sin\theta \quad \text{--- (15)}$$

$$A_\phi = 0 \quad \text{--- (16)}.$$

$$H = \frac{1}{\mu} (\nabla \times A)$$

$$\nabla \times A = \frac{1}{r^2 \sin\theta} \begin{vmatrix} a_r & r a_\theta & r\sin\theta\, a_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & r A_\theta & r\sin\theta\, A_\phi \end{vmatrix}$$

$$= \frac{1}{r^2 \sin\theta} \begin{vmatrix} a_r & r a_\theta & r\sin\theta\, a_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & r A_\theta & 0 \end{vmatrix}$$

$$= \frac{1}{r^2 \sin\theta} \left[a_r \left(0 - \frac{\partial}{\partial \phi} r A_\theta\right) - r a_\theta \left(0 - \frac{\partial}{\partial \phi} A_r\right) + r\sin\theta\, a_\phi \left(\frac{\partial}{\partial r} \cdot r A_\theta - \frac{\partial}{\partial \theta} A_r\right)\right]$$

$$= \frac{1}{r^2 \sin\theta} \left[a_r \cdot 0 - r a_\theta \cdot 0 - r\sin\theta\, a_\phi \left(\frac{\partial}{\partial r}(r A_\theta) - \frac{\partial}{\partial \theta} A_r\right)\right]$$

[$\because$ A_θ & A_r independent of ϕ, from eqn 15 & 14]

$$\therefore \nabla \times A = \frac{1}{r\,\cancel{r^2 \sin\theta}}\left[\cancel{r \sin\theta}\, a_\phi \left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial}{\partial \theta} A_r\right)\right]$$

$$\nabla \times A = \frac{1}{r}\left[\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial}{\partial\theta} A_r\right)\right] a_\phi \quad \text{—(17)}$$

$$\therefore H = \frac{1}{\mu}(\nabla \times A) = \frac{1}{\mu}\,\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial}{\partial\theta} A_r\right] a_\phi$$

$$\therefore H = a_\phi \frac{1}{r\mu}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial}{\partial\theta} A_r\right] \quad \text{—(18)}$$

Substituting eqn (14) & (15) in eqn (18), we have

$$H = a_\phi \frac{1}{r\mu}\left[\frac{\partial}{\partial r}\left(\cancel{r} \times -\frac{\mu}{4\pi} I_0 \cdot \frac{e^{-jkr}}{\cancel{r}} \frac{l}{2}\sin\theta\right) - \frac{\partial}{\partial\theta}\left(\frac{\mu}{4\pi} I_0 \frac{e^{-jkr}}{r}\frac{l}{2}\cos\theta\right)\right]$$

$$= a_\phi \times \frac{1}{r\cancel{\mu}} \times -\frac{\cancel{\mu}}{4\pi} I_0 \times \frac{l}{2}\left[\left(\frac{\partial}{\partial r} e^{-jkr}\right)\cdot \sin\theta + \frac{\partial}{\partial\theta}(\cos\theta)\cdot\left(\frac{e^{-jkr}}{r}\right)\right]$$

$$= a_\phi \times \frac{1}{r} \times \frac{-I_0}{4\pi} \times \frac{l}{2}\left[e^{-jkr}\cdot(-jk)\cdot\sin\theta - \sin\theta\,\frac{e^{-jkr}}{r}\right]$$

$$= a_\phi \frac{1}{r} \times \frac{I_0}{4\pi} \times \frac{l}{2} \times \sin\theta \times e^{-jkr}\left[jk + \frac{1}{r}\right]$$

$$\therefore H = \frac{jKI_0 l}{8\pi} \frac{e^{-jkr}}{r} \left[1 + \frac{1}{jkr}\right] \sin\theta \, a_\phi$$

Since $\vec{H}$ contains only $\hat{a}_\phi$ component,

$$\therefore H_r = 0 \quad \text{---} (19)$$

$$H_\theta = 0 \quad \text{---} (20)$$

$$H_\phi = \frac{jKI_0 l}{8\pi} \frac{e^{-jkr}}{r} \left[1 + \frac{1}{jkr}\right] \sin\theta \quad \text{---} (21)$$

$$= \frac{jKI_0 l}{8\pi} \cdot e^{-jkr} \left[\frac{1}{r} + \frac{1}{jkr^2}\right] \sin\theta$$

In the far field region of the antenna, we can neglect the term containing $\frac{1}{r^2}$, hence ~~dipole~~ magnetic field the ϕ component of the reduce to

$$H_\phi = \frac{jKI_0 l}{8\pi} \frac{e^{-jkr}}{r} \sin\theta \quad \text{---} (22)$$

The electric field can be computed by substituting the value of magnetic field into the Maxwell's curl equation for source-free region

$$E = \frac{1}{j\omega\epsilon} (\nabla \times H) \quad \text{---} (23)$$

Maxwell's eqn

$$\because \nabla \times H = J + \frac{\partial D}{\partial t}$$

$$= 0 + \frac{\partial D}{\partial t}$$

$$= \frac{\partial}{\partial t} \cdot D$$

$$= j\omega D$$

$$= j\omega \epsilon E$$

$$\Rightarrow E = \frac{1}{j\omega\epsilon} (\nabla \times H)$$

Comparing the H_r, H_θ, H_ϕ of Hertzian dipole & short dipole we observe that

$$H_r = 0 \quad , H_\theta = 0$$

Hertzian dipole

$$H_\phi = \frac{jk I_0 dl \sin\theta}{4\pi} \frac{e^{-jkr}}{r} \left(1 + \frac{1}{jkr}\right).$$ [Eqn (13) of Hertzian dipole]

Short dipole

$$H_\phi = \frac{jk I_0 l}{8\pi} \frac{e^{-jkr}}{r} \left(1 + \frac{1}{jkr}\right) \sin\theta$$ [Eqn (21) of Short dipole]

$$\therefore \left(H_\phi\right)_{Short} = \frac{1}{2} \times \left(H_\phi\right)_{Hertzian}.$$

[Repeating the same procedure to find E_r, E_θ, E_ϕ]

~~$\therefore \left(H_\phi\right)_{Short}$~~

$$E_r = \frac{1}{2} \times \left(E_r\right)_{Hertzian}$$

$$E_r = \frac{1}{2} \times \frac{\eta \times I_0 \, l \cos\theta}{2\pi r} \frac{e^{-jkr}}{r} \left(1 + \frac{1}{jkr}\right) \quad \text{—— (24)}$$

Similarly

$$E_\theta = \frac{j\eta}{2} \frac{K I_0 \, l \sin\theta}{4\pi} \cdot \frac{e^{-jkr}}{r} \left(1 + \frac{1}{jkr} - \frac{1}{(kr)^2}\right) \quad \text{— (25)}$$

(26)

$$E_\phi = 0$$

For far field

$$E_r = \frac{1}{2} \times \frac{\eta \, I_0 \, l \cos\theta}{2\pi} \cdot e^{-jkr} \left[\frac{1}{r^2} + \frac{1}{jkr^3}\right]$$

∴ For far field $\boxed{E_r = 0}$

$\because \frac{1}{r^2} \to 0$

$\frac{1}{r^3} \to 0$

Similarly

$$E_\theta = \frac{j\eta}{2} \frac{K I_0 l \sin\theta}{4\pi} e^{-jKr} \left[\frac{1}{r}\right] \quad \text{--- } 26(A)$$

$E_\phi = 0$

$$\therefore \boxed{E = a_\theta \; j\eta \frac{K I_0 l}{8\pi} \cdot \frac{e^{-jKr}}{r} \sin\theta} \quad \text{--- } 26(B)$$

Similar to Hertzian Dipole, For Short dipole,

$\frac{E_\theta}{H_\phi} = \frac{Eq^n\,[26(A)]}{Eq^n\,(22)} = \eta$ = Intrinsic impedance of the medium.

$$\therefore \frac{E_\theta}{H_\phi} = \eta \quad \text{--- } 26(C)$$

In the far-field of the dipole the electric & magnetic field intensities are transverse ($\perp$) to each other as well as to the direction of propagation. Thus E_θ, H_ϕ, and the direction of propagation, $\hat{a}_r$, form right handed system. The expression for the electric and magnetic field intensities are related to the magnetic vector potential by the following eq^{n}s

$$E = -j\omega A_t \quad \text{--- } 27$$

$$H = \frac{-j\omega}{\eta} a_r \times A_t \quad \text{--- } 28$$

Where A_t represents the transverse component of the magnetic vector potential given by

$$A_t = a_\theta A_\theta + a_\phi A_\phi \quad \text{——— (29)}$$

<u>Verification :-</u>

(i) $E = -jwA_t$

$= -jw\,(a_\theta A_\theta + a_\phi A_\phi)$

$= -jw\,(a_\theta A_\theta + 0) \qquad (\because A_\phi = 0)$

$= -jw\,a_\theta \left(-\frac{\mu}{4\pi} I_0 \frac{e^{-jkr}}{r} \cdot \frac{l}{2} \sin\theta\right)$

$= j\,(\mu w)\, \frac{I_0 l}{8\pi} \frac{e^{-jkr}}{r} \cdot \sin\theta \; a_\theta$

$\downarrow$

$E = j(\eta k)\, \frac{I_0 l}{8\pi} \frac{e^{-jkr}}{r} \sin\theta\, a_\theta$

Which is equal to eqn (26-B)

$\therefore \boxed{E = -jwA_t}$ (Proved).

$\because$

$\eta \times k$

$= \sqrt{\frac{\mu}{\epsilon}} \times w\sqrt{\mu\epsilon}$

$= \frac{\sqrt{\mu}}{\cancel{\sqrt{\epsilon}}} \times w \times \sqrt{\mu} \times \cancel{\sqrt{\epsilon}}$

$= w\mu$

(ii) $H = \frac{-jw}{\eta}(a_r \times A_t)$

$= \frac{-jw}{\eta} (a_r) \times (a_\theta A_\theta + a_\phi \cancel{A_\phi}^{\,0})$

$$H = \frac{-jw}{\eta} \times (a_r \times a_\theta) A_\theta$$

$$= \frac{-jw}{\eta} \times a_\phi \; A_\theta \qquad \because \hat{x} \times \hat{y} = \hat{z}$$

$$= \frac{-jw}{\eta} \times a_\phi \left[\frac{-\mu}{4\pi} I_0 \cdot \frac{e^{-jkr}}{r} \cdot \frac{l}{2} \sin\theta \right]$$

$$= j \left(\frac{\mu \omega}{\eta}\right) \cdot \frac{I_0 l}{8\pi} \frac{e^{-jkr}}{r} \cdot \sin\theta \; a_\phi$$

$$H = j \frac{k I_0 l}{8\pi} \frac{e^{-jkr}}{r} \cdot \sin\theta \; a_\phi$$

which is equal to eqn (22).

$$\therefore \boxed{H = \frac{-jw}{\eta} (a_r \times A_t)} \text{ (Proved)}$$

$$\because \frac{\mu \omega}{\eta} = \frac{\mu w}{\sqrt{\frac{\mu}{\epsilon}}} = \frac{\sqrt{\mu}\,\cancel{\mu} w \times \sqrt{\epsilon}}{\sqrt{\cancel{\mu}}} = w\sqrt{\mu \epsilon} = k$$

Radiation Resistance & Directivity of Short Dipole

Since for a dipole oriented along the Z-direction, only E_θ and H_ϕ exist in the far-field region, the average power density, S, is given by

$$S = \frac{1}{2} Re \left(a_\theta E_\theta \times a_\phi H_\phi^* \right)$$

$$\Rightarrow S = a_r \ \frac{1}{2} \ Re\left(E_\theta . H_\phi^*\right) \text{ — (30)}$$

$\because a_\theta \times a_\phi = a_r$

Using the relationship $\frac{E_\theta}{H_\phi} = \eta$, → From eqn 26(A)

the eqn (30) becomes,

$$S = a_r \frac{1}{2} Re\left(E_\theta . \frac{E_\theta^*}{\eta}\right) = a_r \frac{1}{2} \frac{Re|E_\theta|^2}{\eta}$$

$$S = a_r \frac{1}{2\eta} |E_\theta|^2$$

$$= a_r \times \frac{1}{2\eta} \times \left|\eta\, j \frac{K I_0 l}{8\pi} \frac{e^{-jkr}}{r} \sin\theta\right|^2$$

Using eqn (26)-A

$$= a_r \times \frac{1}{2\cancel{\eta}} \times \eta^{\cancel{2}} \left|\frac{K I_0 l}{8\pi}\right|^2 \frac{\sin^2\theta}{r^2}$$

$\because |j^2| = 1$

$|e^{-jkr}|$

$= ~~|\sin kr - j\cos|~~$

$= |\cos kr - j\sin kr|$

$= \sqrt{\cos^2 kr + \sin^2 kr}$

$= 1$

$$S = a_r . \frac{\eta}{2} . \left|\frac{K I_0 l}{8\pi}\right|^2 \frac{\sin^2\theta}{r^2} \text{ — (31)}$$

Radiation Intensity $U(\theta, \phi) = r^2 . S(r, \theta, \phi)\, a_r$ — (32)

Total Power radiated, P_{rad} is obtained by integrating the radiation ~~intensity~~ intensity over a sphere of radius r. (As done earlier for Hertzian dipole)

$$P_{rad} = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} S . a_r\, r^2 . \sin\theta \, d\theta \, d\phi$$

$$\Rightarrow P_{rad} = \int_0^{2\pi}\int_0^{\pi} \frac{\eta}{2} \left|\frac{KI_o l}{8\pi}\right|^2 \cdot \frac{\sin^2\theta}{\cancel{r^2}} \cdot \cancel{r^2} \cdot \sin\theta \, d\theta \, d\phi$$

$$= \left[\int_0^{\pi} \sin^3\theta \, d\theta\right] \times (2\pi) \times \left[\frac{\eta}{2} \times \left|\frac{KI_o l}{8\pi}\right|^2\right]$$

$$\left(\because \int_0^{2\pi} d\phi = 2\pi\right)$$

$$P_{rad} = \frac{4}{3} \times \cancel{2}\pi \times \frac{\eta}{\cancel{2}} \left|\frac{KI_o l}{8\pi}\right|^2 \quad \text{--- } 32(A)$$

$$\because \int_0^{\pi} \sin^3\theta \, d\theta = \frac{4}{3}$$

As derived earlier for Hertzian dipole.

$$= \frac{4}{3} \times \pi \times 120\pi \times \left|\frac{\cancel{2\pi}}{\lambda} \cdot \frac{I_o l}{\cancel{8\pi}_4}\right|^2$$

$$\because K = \frac{2\pi}{\lambda} = \text{Phase const.}$$

$$= \frac{\cancel{4}}{3} \times \pi \times \eta \times |I_o|^2 \times \left(\frac{l}{\lambda}\right)^2 \times \frac{1}{\cancel{16}_4}$$

$$= \eta \times \frac{\pi}{12} |I_o|^2 \left(\frac{l}{\lambda}\right)^2$$

$$\Rightarrow \boxed{P_{rad} = \eta \times \frac{\pi}{12} \times |I_o|^2 \times \left(\frac{l}{\lambda}\right)^2} \quad \text{--- } (33)$$

To obtain the radiation resistance, the total radiated power is equated to the power absorbed by an equivalent resistance carrying the same i/p current, I_o.

$$\therefore P_{rad} = \frac{1}{2} |I_o|^2 R_{rad} \quad \text{--- } (34)$$

Equating 33 & 34

$$\cancel{\frac{1}{2}} \cancel{|I_0|^2} R_{rad} = \eta \times \frac{\pi}{\cancel{12}\,6} \times \cancel{|I_0|^2} \left(\frac{l}{\lambda}\right)^2$$

$$\Rightarrow R_{rad} = \cancel{120}^{20}\pi \times \frac{\pi}{\cancel{6}} \times \left(\frac{l}{\lambda}\right)^2$$

$$\Rightarrow \boxed{R_{rad} = 20\,\pi^2 \left(\frac{l}{\lambda}\right)^2 \text{ ohm}} \quad \text{——— (35)}$$

Note: For Hertzian Dipole

$$\boxed{R_{rad} = 80\,\pi^2 \left(\frac{dl}{\lambda}\right)^2 \text{ ohm}}$$

Directivity :-

Radiation intensity, $U(\theta,\phi) = r^2 S$

$$= \cancel{r^2} \times \frac{\eta}{2} \times \left|\frac{K I_0 l}{8\pi}\right|^2 \frac{\sin^2\theta}{\cancel{r^2}}$$

$$U(\theta,\phi) = \frac{\eta}{2} \left|\frac{K I_0 l}{8\pi}\right|^2 \sin^2\theta \quad \text{——— (36)}$$

$$\text{Directivity } (D) = \frac{U(\theta,\phi)}{\left(\frac{P_{rad}}{4\pi}\right)} = \frac{4\pi\, U(\theta,\phi)}{P_{rad}}$$

$$= \frac{4\pi \times \frac{\eta}{2} \cancel{\left|\frac{K I_0 l}{8\pi}\right|^2} \sin^2\theta}{\frac{4\pi}{3} \times \eta \times \cancel{\left|\frac{K I_0 l}{8\pi}\right|^2}}$$

(Using eqn (36) & 32 (A))

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$$\Rightarrow D = \frac{\cancel{4\pi} \times \cancel{\eta}\ \sin^2\theta}{2} \times \frac{3}{\cancel{4\pi}\ \cancel{\eta}}$$

$$\boxed{D = \frac{3}{2}\sin^2\theta} \quad \text{——— (37)}$$

It may be noted that the Directivity is same that of Hertzian dipole.

$$\left.\begin{aligned} \text{Maximum Directivity } (D_0) &= \frac{3}{2} = 1.5 \\ (D_0)_{dB} &= 10 \log 1.5 = 1.76\ dB \end{aligned}\right\} \text{—— (38)}$$

It occurs at $\theta = 90°$, $\because \sin^2\theta = 1$.

The normalized radiation intensity expressed in dB is given by

$$U_{dB}(\theta, \phi) = 10 \log_{10} (\sin^2\theta)\ dB \quad \text{——— (39)}$$

$$\left(\because U(\theta, \phi) = \frac{\eta}{2}\left|\frac{K I_0 l}{8\pi}\right|^2 \sin^2\theta \quad \Big| \text{ From eq}^n \text{ (36)}\right.$$

$$\left\{\begin{matrix}\text{Normalized} \\ \text{radiation} \\ \text{intensity}\end{matrix}\right\} \rightarrow U_n(\theta, \phi) = \frac{U(\theta,\phi)}{U(\theta,\phi)_{max}} = \frac{\cancel{\frac{\eta}{2}} \cdot \cancel{\left|\frac{K I_0 l}{8\pi}\right|^2} \cdot \sin^2\theta}{\cancel{\frac{\eta}{2}} \cdot \cancel{\left|\frac{K I_0 l}{8\pi}\right|^2}}$$

$$\left. U_n(\theta, \phi) = \sin^2\theta \right)$$

Eq^n (39) can be plotted w.r.to the elevation angle (θ). This called E-plane pattern.

The polar plot of E-plane pattern is shown in figure 1. The radiation pattern of the dipole has null along the axis

(Since at the axis $\theta = 0°$, $\sin^2\theta = 0$.)

and a maximum in the $\theta = 90°$ plane. The radiation pattern is independent of ϕ. The 3D pattern is obtained by rotating the right half of the pattern about the axis of the dipole. Such a pattern is called an omni-directional pattern. An Omni-directional pattern has a non-directional pattern in one plane (e.g. here H-plane XY-Plane) and directional pattern in any plane orthogonal to it (e.g here E-plane (XZ plane))

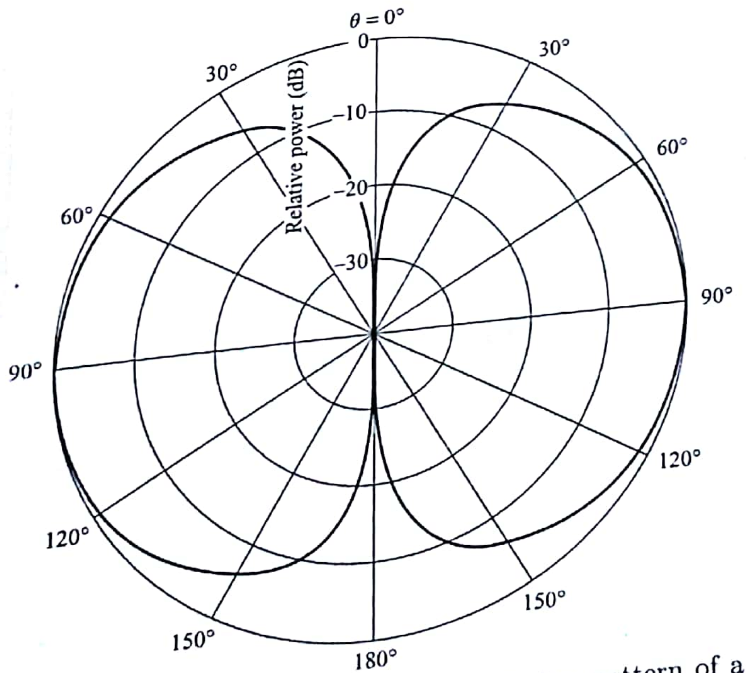


Fig. 1 The x-z plane cut of the radiation pattern of a z-directed short dipole

Ex:-

19) A short dipole with a triangular current distribution radiates Prad watts into free space. Show that the magnitude of maximum electric field at a distance 'r' is given by

$$E_\theta = \frac{\sqrt{90 P_{rad}}}{r} \ \frac{V}{m}.$$

Ans:- The maxm value of the electric field given by eqn 26(A) occures along $\theta = 90°$ and is given by

$$|E_\theta| = \eta \frac{K|I_0|l}{8\pi} \cdot \frac{1}{r} \quad \text{——} (1)$$

The radiated power can be expressed as eqn (33),

$$P_{rad} = \eta \times \frac{\pi}{12} \cdot |I_0|^2 \cdot \left(\frac{l}{\lambda}\right)^2 \quad \text{——} (2)$$

$$\Rightarrow |I_0|^2 = \frac{12 \times P_{rad}}{\eta \times \pi \times \left(\frac{l}{\lambda}\right)^2}$$

$$\Rightarrow I_0 = \sqrt{\frac{12 \times P_{rad} \times \lambda^2}{\eta \times \pi \times l^2}} \quad \text{——} (3)$$

Putting eqn (3), in eqn (1), we have

$$|E_\theta| = \eta \times \frac{K}{8\pi} \times \cancel{l} \times \frac{1}{r} \times \left(\frac{\lambda}{\cancel{l}} \times \sqrt{\frac{12 \times P_{rad}}{\eta \times \pi}}\right)$$

$$= \cancel{120\pi}^{30\pi} \times \frac{2\pi}{\cancel{\lambda}} \cdot \frac{1}{\cancel{8\pi}_{4}} \times \frac{1}{r} \times \cancel{\lambda} \times \sqrt{\frac{\cancel{12} \times P_{rad}}{\cancel{120}_{10} \pi \times \pi}}$$

$$= 30\cancel{\pi} \times \frac{1}{r} \times \sqrt{\frac{P_{rad}}{10}} \times \frac{1}{\cancel{\pi}} = \sqrt{\frac{900 P_{rad}}{10}} \times \frac{1}{r}$$

$$|E_\theta| = \frac{\sqrt{90 P_{rad}}}{r} \text{ Volt/meter. (Proved)}$$

20) A short dipole of length 0.1λ is kept symmetrically about the origin, oriented along the Z-direction and radiating 1 KW power into free space. Calculate the power density at $r = 1$ Km along $\theta = 45°$ and $\phi = 90°$.

Ans:- Given $l = 0.1\lambda$, $P_{rad} = 1$ KW

$r = 1$ km, $\theta = 45°$, $\phi = 90°$

Approach 1:- The radiation resistance of short dipole is

$$R_{rad} = 20\pi^2 \left(\frac{l}{\lambda}\right)^2$$

$$= 20\pi^2 \left(\frac{0.1\cancel{\lambda}}{\cancel{\lambda}}\right)^2$$

$$= 20\pi^2 (0.1)^2$$

$$\boxed{R_{rad} = 1.974\,\Omega}$$

From the relationship

$$P_{rad} = \frac{1}{2} \cdot I_0^2 \cdot R_{rad}$$

$$\Rightarrow 1000 = \frac{1}{2} \times I_0^2 \times 1.974$$

$$\Rightarrow I_0 = \sqrt{\frac{2000}{1.974}} = 31.83 \text{ Amp}$$

From eqn (31),

$$S = \frac{\eta}{2} \times \left|\frac{K I_0 l}{8\pi}\right|^2 \times \frac{\sin^2\theta}{r^2} \qquad \because K = \frac{2\pi}{\lambda}$$

$$= \frac{120\pi}{2} \times \left|\frac{2\cancel{\pi}}{\cancel{\lambda}} \cdot \frac{31.83 \times 0.1\cancel{\lambda}}{\cancel{8}\cancel{\pi}_4}\right|^2 \times \frac{\sin^2 45°}{1000^2}$$

$$S = \frac{120\times\pi}{2} \times 0.6332 \times \frac{\left(\frac{1}{\sqrt{2}}\right)^2}{1000^2}$$

$$= \frac{120\times\pi\times 0.6332}{2\times 2\times 1000^2}$$

$$S = 59.679\times 10^{-6}\ W/m^2$$

$\therefore$ Power density $(S) = 5.968\times 10^{-5}\ \frac{watt}{m^2}$ (Ans).

Approach 2:-

We know, Directivity $= \frac{U(\theta,\phi)}{\left(\frac{P_{rad}}{4\pi}\right)} = \frac{S(\theta,\phi)\times r^2}{\left(\frac{P_{rad}}{4\pi}\right)}$

$U(\theta,\phi) \rightarrow$ Radiation intensity; $S(\theta,\phi) \rightarrow$ Power density

$$\Rightarrow D_t(\theta,\phi) = \frac{4\pi S(\theta,\phi)\times r^2}{P_{rad}} \quad \text{——— (1)}$$

The Directivity along (θ,ϕ) is

$$D_t(\theta,\phi) = 1.5\sin^2\theta = 1.5\sin^2 45^\circ$$
$$= 1.5\times\left(\frac{1}{\sqrt{2}}\right)^2$$
$$= 1.5\times\frac{1}{2}$$
$$D_t(\theta,\phi) = 0.75 \quad \text{——— (2)}$$

Putting eqn (2) in eqn (1)

$$0.75 = \frac{4\pi\times S\times \cancel{1000^2}}{(\cancel{1000})}$$

$r = 1\ km$

$S = 1\ kW$

$$\Rightarrow S = \frac{0.75}{4\pi\times 1000} = 5.968\times 10^{-5}\ \frac{watt}{m^2}$$

$\therefore$ Power density $(S) = 5.968\times 10^{-5}\ \frac{watt}{m^2}$ (Ans)

Half-wave Dipole :— $\left(l = \frac{\lambda}{2}\right)$

The current distribution on a thin (radius, $a << \lambda$) wire dipole depends on its length. For a very short dipole ($l < 0.1\lambda$) it is approximately a triangular distribution [As discussed earlier]. As the length of the dipole approaches a significant fraction of the wavelength, it is found that the current distribution is closer to a sinusoidal distribution than a triangular distribution. For a center-fed dipole of length (l), symmetrically placed about the origin with its axis along the Z-axis, [Fig 1] the current on the dipole has only a Z-component and is given by

$$I(z') = a_z I_z(z')$$

$$= \begin{cases} a_z I_0 \sin\left[K\left(\frac{l}{2} - z'\right)\right], & 0 \le z' \le \frac{l}{2} \\ a_z I_0 \sin\left[K\left(\frac{l}{2} + z'\right)\right], & -\frac{l}{2} \le z' \le 0 \end{cases} \quad -(1)$$

where I_0 is the amplitude of the current distribution and K is the propagation constant, [As shown in figure 2]

The technique to compute the radiation characteristics of $\frac{\lambda}{2}$ dipole is very similar to that presented in for short dipole. First, compute the magnetic vector potential in the far-field region of the antenna and

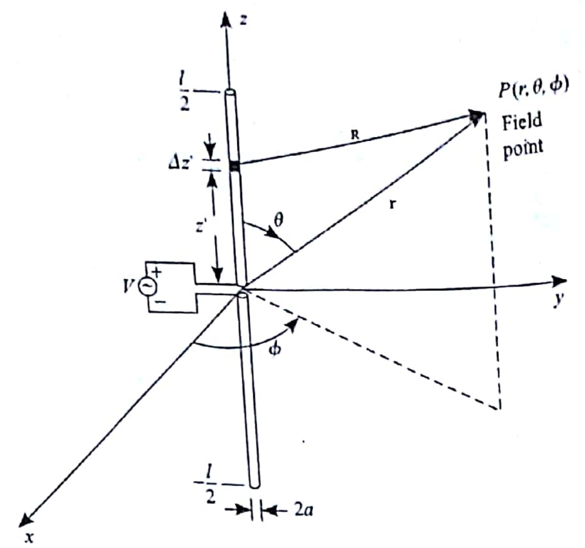


Fig. 1 Geometry of a thin wire dipole $\left(l = \frac{\lambda}{2}\right)$

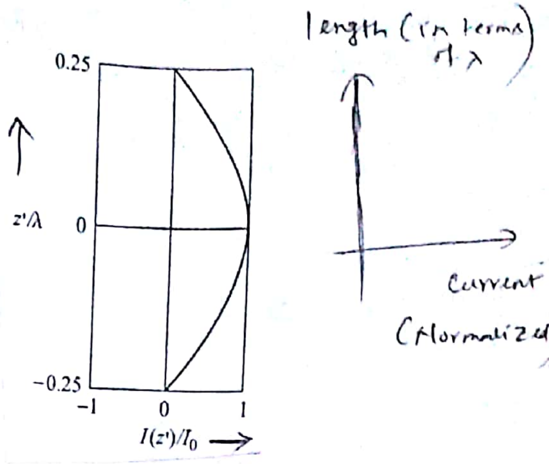


Fig 2:- Current distribution for a $\frac{\lambda}{2}$ dipole antenna.

and then determine the $\underline{E}$ and $\underline{H}$ fields from it.

Since the current has only a Z-component, A also has only the A_z - Component.

$$A_z = \frac{\mu}{4\pi} \frac{e^{-jkr}}{r} \int_{-l/2}^{l/2} I_z(z')\, e^{jkz'\cos\theta}\, dz \quad \text{--- (2)}$$

[Refer eqn (12), of Short dipole]

Substituting eqn (1) [Value of $I_z(z')$] in eqn (2),

we have,

$$A_z = \frac{\mu}{4\pi} \cdot \frac{e^{-jkr}}{r} \cdot I_0 \left[\int_{-l/2}^{0} \sin\left[k\left(\frac{l}{2} + z'\right)\right] e^{jkz'\cos\theta}\, dz' + \int_{0}^{l/2} \sin\left[k\left(\frac{l}{2} - z'\right)\right] \cdot e^{jkz'\cos\theta}\, dz' \right] \quad \text{--- (3)}$$

Integrating this w.r.to z' and substituting appropriate limits, the vector potential expression is reduced to

$$A_z = \frac{\mu I_0}{2\pi} \cdot \frac{e^{-jkr}}{r} \left[\frac{\cos\left(\frac{kl}{2}\cos\theta\right) - \cos\left(\frac{kl}{2}\right)}{k \sin^2\theta} \right] \quad \text{--- (4)}$$

Note :- For evaluating the integration

$$\int u.v = u\int v - \int u'.\int v$$

ILATE   T → Trigonometry
E → Exponential

$$\therefore \int \sin\left[k\left(\frac{l}{2}+z'\right)\right] . e^{jkz'\cos\theta} dz' = \int u.v \; dz'$$

Decomposing A_z into components along r and θ directions, we have

$$A_r = A_z \cos\theta \quad \text{--- (5)}$$

$$A_\theta = -A_z \sin\theta \quad \text{--- (6)}$$

$$A_\phi = 0 \quad \text{--- (7)}$$

As discussed in short dipole, the electric and magnetic field intensities can be calculated from magnetic potential as follows,

$$E = -j\omega A_t \quad \text{--- (8)}$$

$$H = \frac{-j\omega}{\eta} a_r \times A_t \quad \text{--- (9)}$$

Refer eqn (27) & (28) of short dipole.

where A_t = Transverse component of the magnetic vector potential

$$\therefore A_t = a_\theta A_\theta + a_\phi A_\phi \quad \text{--- (10)}$$

Since $A_\phi = 0$, Eq^n (10) becomes

$$A_t = a_\theta A_\theta \quad \text{———— (11)}$$

$\therefore$ Eq^n (8) becomes, [Putting eq^n (11) in eq^n (8)]

$$E = -jw\, a_\theta A_\theta \quad \text{———— (12)}$$

$$= -jw\, a_\theta \left[-A_z \sin\theta\right]$$

$\because$ From eq^n (6)
$A_\theta = -A_z \sin\theta$

$$= jw\, a_\theta \left[\frac{\mu I_0}{2\pi} \cdot \frac{e^{-jkr}}{r} \left[\frac{\cos\left(\frac{kl}{2}\cos\theta\right) - \cos\left(\frac{kl}{2}\right)}{k \cancel{\sin^2\theta}\ \sin\theta} \right] \right] \times \cancel{\sin\theta}$$

$$\therefore E = j\cancel{w}\, a_\theta\, \frac{\cancel{\mu}^{\sqrt{\mu}} I_0}{2\pi} \times \frac{e^{-jkr}}{r} \times \frac{\left[\cos\left(\frac{kl}{2}\cos\theta\right) - \cos\left(\frac{kl}{2}\right)\right]}{\cancel{w}\sqrt{\cancel{\mu}\epsilon}\ \sin\theta}$$

$\because$ Putting the value of A_z from eq^n (4)
$\because$ $K = w\sqrt{\mu\epsilon}$

$$= j\, a_\theta \sqrt{\frac{\mu}{\epsilon}} \times \frac{I_0}{2\pi} \cdot \frac{e^{-jkr}}{r} \frac{\left[\cos\left(\frac{kl}{2}\cos\theta\right) - \cos\left(\frac{kl}{2}\right)\right]}{\sin\theta}$$

$$\therefore E = a_\theta\, j\, \eta \frac{I_0}{2\pi} \frac{e^{-jkr}}{r} \frac{\left[\cos\left(\frac{kl}{2}\cos\theta\right) - \cos\left(\frac{kl}{2}\right)\right]}{\sin\theta} \quad \text{—— (13)}$$

$\because$ $\eta = \sqrt{\frac{\mu}{\epsilon}}$

Similarly, From eq^n (9),

$$H = -\frac{jw}{\eta}\, a_r \times a_\theta A_\theta$$

$$H = -\frac{jw}{\eta}\, a_\phi A_\theta \quad \text{———— (14)}$$

$\because$ $a_r \times a_\theta = a_\phi$

[Carefully observing eq^n (12) & (14)
$|H| = \frac{|E|}{\eta}$ with a 'a_ϕ' component, so H can be directly found from eq^n (13).

$$H = a_\phi\, j \frac{I_0}{2\pi} \frac{e^{-jkr}}{r} \frac{\left[\cos\left(\frac{kl}{2}\cos\theta\right) - \cos\left(\frac{kl}{2}\right)\right]}{\sin\theta} \quad \text{--- (15)}$$

Calculation of 'D' & 'R_r' :-

The radiation intensity is given by,

$$U(\theta) = r^2 \cdot S(\theta)$$ | $S(\theta)$ = Power density

$$= r^2 \times \frac{1}{2} Re(E \times H^*)$$

$$= r^2 \times \frac{1}{2} Re\left[E \times \frac{E^*}{\eta}\right]$$ | $\because H^* = \frac{E^*}{\eta}$

$$= r^2 \times \frac{1}{2} \times \frac{|E|^2}{\eta}$$

$$U(\theta) = r^2 \times \frac{1}{2\eta} \cdot |E_\theta|^2$$

$$= r^2 \times \frac{1}{2\cancel{\eta}} \times \eta^{\cancel{2}} \times \left|\frac{I_0}{2\pi}\right|^2 \cdot \frac{1}{r^2} \left[\frac{\cos\left(\frac{kl}{2}\cos\theta\right) - \cos\left(\frac{kl}{2}\right)}{\sin^2\theta}\right]^2$$

$\because$ Using eqn (13),

$|j| = 1$

$|a_\theta| = 1$

$|e^{-jkr}| = 1$

$$\therefore U(\theta) = \frac{\eta}{2} \times \left|\frac{I_0}{2\pi}\right|^2 \times \frac{\left[\cos\left(\frac{kl}{2}\cos\theta\right) - \cos\left(\frac{kl}{2}\right)\right]^2}{\sin^2\theta} \quad \text{--- (16)}$$

For a half-wave dipole $l = \frac{\lambda}{2}$ --- (17)

putting eqn (17) in eqn (16), we have

$$U = \frac{\eta}{2} \times \left|\frac{I_0}{2\pi}\right|^2 \frac{\left[\cos\left(\frac{2\pi}{\cancel{\lambda}} \cdot \frac{\cancel{\lambda}}{2} \cdot \frac{1}{2}\cos\theta\right) - \cos\left(\frac{2\pi}{\cancel{\lambda}} \times \frac{\cancel{\lambda}}{2} \times \frac{1}{2}\right)\right]^2}{\sin^2\theta}$$

$$U = \frac{\eta}{2} \times \left|\frac{I_0}{2\pi}\right|^2 \left[\frac{\cos\left(\frac{\pi}{2}\cos\theta\right) - \cancel{\cos\left(\frac{\pi}{2}\right)}^{\,0}}{\sin^2\theta}\right]^2$$

$$U = \frac{\eta}{2} \times \left|\frac{I_0}{2\pi}\right|^2 \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta} \quad \text{—— (18)}$$

$$P_{rad} = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} U \sin\theta \, d\theta \, d\phi$$

$$= \left[\int_{\theta=0}^{\pi} \frac{\eta}{2} \times \left|\frac{I_0}{2\pi}\right|^2 \times \cancel{\sin\theta} \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin^{\cancel{2}}\theta} d\theta\right] \times 2\pi$$

$$\left|\ \because \int_0^{2\pi} d\phi = 2\pi\right.$$

$$= \frac{\eta}{\cancel{2}} \times \frac{|I_0|^2}{|2\pi|} \times \cancel{2}\pi \int_{\theta=0}^{\pi} \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} d\theta$$

$$\Rightarrow P_{rad} = \frac{\eta}{2} \times \pi \times \left|\frac{I_0}{2\pi}\right|^2 \times 1.2179$$

$$\Rightarrow P_{rad} = \cancel{120}^{30}\cancel{\pi} \times \cancel{\pi} \times \frac{|I_0|^2}{\cancel{4}\cancel{\pi^2}} \times 1.2179$$

$$\Rightarrow P_{rad} = (30 \times 1.2179) |I_0|^2$$

$$\Rightarrow P_{rad} = 36.54 |I_0|^2 \quad \text{—— (19)}$$

$\because \int_{\theta=0}^{\pi} \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} d\theta = 1.2179$

For proof refer subsequent pages of this note. [After 2 pages]

The radiation resistance is computed by equating the average radiated power to the power dissipated in an equivalent resistance carrying the same i/p current (85)

$$P_{rad} = 36.54\,|I_0|^2 = \frac{1}{2}\,|I_0|^2 \times R_{rad}$$

$$\Rightarrow \boxed{R_{rad} = 73.08\,\Omega} \quad \longleftarrow (20)$$

So one of the most commonly used antennas is half wavelength $\left(\frac{\lambda}{2}\right)$ dipole. Because its radiation resistance is 73 ohm, which is very near the 75 ohm, characteristic impedance of some transmission line practically available.

Directivity : $D = \dfrac{4\pi\,U(\theta,\phi)}{P_{rad}}$

$$\therefore \quad D = 4\pi \times \frac{\text{Eq}^n\,(18)}{\text{Eq}^n\,(19)}$$

$$= \frac{4\pi \times \frac{\eta}{2} \times \left|\frac{I_0}{2\pi}\right|^2 \times \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}}{36.54\,|I_0|^2}$$

$$= \frac{\cancel{4\pi} \times \frac{\overset{60}{\cancel{120\pi}}}{\cancel{2}} \times \frac{\cancel{|I_0|^2}}{\cancel{4\pi^2}} \quad \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}}{36.54\,\cancel{|I_0|^2}}$$

$$\therefore \quad D = \frac{60}{36.54} \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}$$

$$D = 1.642 \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta} \quad \text{—— (21)}$$

The Maximum value of directivity occurs along $\theta = \frac{\pi}{2}$, and is equal to $\underline{1.642}$.

$$\frac{\cos^2\left(\frac{\pi}{2}\cos\frac{\pi}{2}\right)}{\sin^2\frac{\pi}{2}}$$

$$= \frac{\cos^2(0)}{\sin^2\frac{\pi}{2}}$$

$$= \frac{1}{1}$$

$$= 1$$

$$\boxed{D_0 = 1.642} \quad \text{—— (22)}$$

$$\boxed{D_{dB} = 10\log 1.642 = 2.15\text{ dB}} \quad \text{—— (23)}$$

Ex:- Prove that

$$\int_{\theta=0}^{\pi} \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\, d\theta = 1.2179$$

Solution: Let

$$I = \int_{\theta=0}^{\pi} \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} d\theta$$
$$= \frac{1}{2}\int_{\theta=0}^{\pi} \frac{1+\cos(\pi\cos\theta)}{\sin\theta} d\theta$$

Substituting $u = \cos\theta$ and $du = -\sin\theta d\theta$, and interchanging the limits of integration

$$I = \frac{1}{2}\int_{-1}^{1} \frac{1+\cos(\pi u)}{1-u^2} du$$

Using the relation

$$\frac{1}{1-u^2} = \frac{1}{2}\left(\frac{1}{1+u} + \frac{1}{1-u}\right)$$

we can write

$$I = \frac{1}{4}\left(\int_{-1}^{1} \frac{1+\cos(\pi u)}{1-u} du + \int_{-1}^{1} \frac{1+\cos(\pi u)}{1+u} du\right)$$

Substituting $u = -t$ in the first integral and interchanging the limits

$$\int_{-1}^{1} \frac{1+\cos(\pi u)}{1-u} du = \int_{-1}^{1} \frac{1+\cos(\pi t)}{1+t} dt$$

Therefore, we can now write

$$I = \frac{1}{2}\int_{-1}^{1} \frac{1+\cos(\pi u)}{1+u} du$$

We make another substitution, $\pi u = y - \pi$, to get

$$I = \frac{1}{2}\int_{0}^{2\pi} \frac{1-\cos y}{y} dy$$

The relation $\cos(y-\pi) = -\cos y$ has been used to arrive at the above expression. The Taylor series expansion of $\cos y$ is

$$\cos y = 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + \cdots$$

This can be used to rewrite the integral as

$$I = \frac{1}{2}\int_0^{2\pi} \left(\frac{y}{2!} - \frac{y^3}{4!} + \frac{y^5}{6!} - \frac{y^7}{8!} + \cdots \right) dy$$

On performing termwise integration and substituting the limits, we get

$$I = \frac{1}{2}(9.8696 - 16.235 + 14.2428 - 7.5306 + 2.6426 - 0.6586 + 0.1225$$
$$- 0.01763 + \cdots) = 1.2179$$

Note :- The current distributions and radiation patterns of thin wire dipoles of different lengths are shown in figure 1. As the dipole length increases from 0.5λ to 1.2λ, the main beam becomes narrower. The 10dB beamwidth for 0.5λ long dipole is 134.4°; it reduces to 85.7° for a λ long dipole and goes down to 60.2° for $l = 1.2\lambda$. [See the ✓ mark to locate −10dB Beam Width]

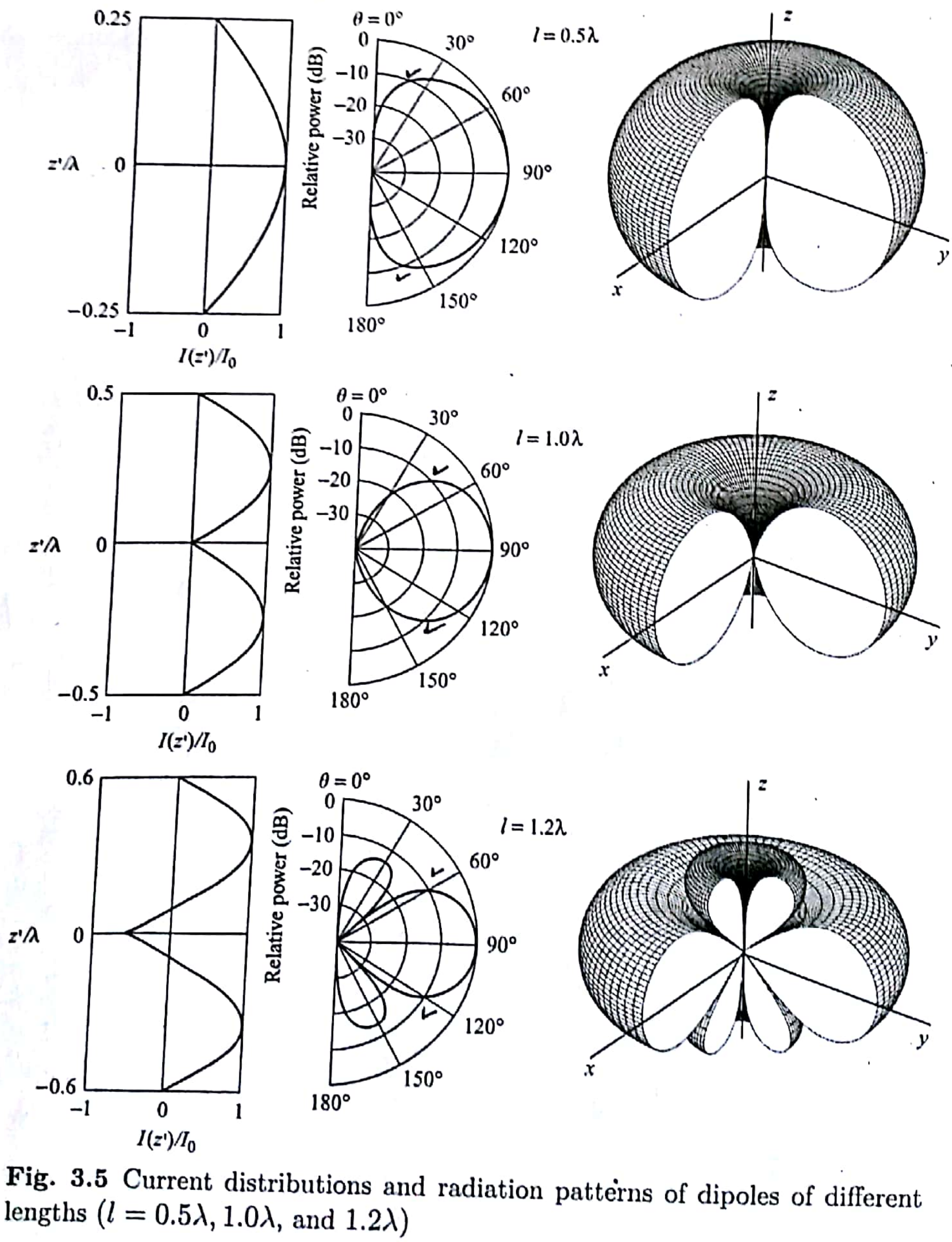


Fig. 3.5 Current distributions and radiation patterns of dipoles of different lengths ($l = 0.5\lambda, 1.0\lambda$, and 1.2λ)

Ex :- 21) A 6 cm long Z-directed dipole (88)
Carries a Current of 1 A at 2.4 GHz. Calculate the electric and magnetic field strengths at a distance of 50 cm along $\theta = 60°$.

Ans :- Given

$l = 6\ cm$, $I_0 = 1\ Amp$, $f = 2.4\ GHz$

$r = 50\ cm = 0.5$ meter, $\theta = 60° = \frac{\pi}{3}$

$E = ?$, $H = ?$

→ The Wavelength (λ) at 2.4 GHz is

$$\lambda = \frac{c}{f} = \frac{3 \times 10^{8}}{2.4 \times 10^{9}} = \frac{3}{24} = 0.125\ m.$$

∴ For the given antenna,

$$\frac{l}{\lambda} = \frac{6\ cm}{0.125\ m} = \frac{6 \times 10^{-2}}{0.125} = \frac{0.06}{0.125} = \frac{60}{125} = 0.48$$

$$\therefore \boxed{\frac{l}{\lambda} = 0.48} \Rightarrow \boxed{l = 0.48\lambda}$$

→ For this length we can assume a sinusoidal current distribution on the dipole.

Since $r = 0.5$, Lets check whether it belongs to far-field region or not.

$$r > \frac{2D^2}{\lambda}$$

where D = Highest dimension of the antenna.

$$\therefore \frac{2D^2}{\lambda} = \frac{2 \times l^2}{\lambda} = \frac{2 \times (0.06)^2}{0.125} = 0.0576\ m.$$

$\therefore$ $\underline{r = 0.5\,m} > \underline{0.0576\,m}$, the field point is in the far field of the antenna. Therefore, we can use the electric field eqn of $\frac{\lambda}{2}$ antenna [As $l = 0.48\lambda$]

$\therefore$ From eqn (3) of $\frac{\lambda}{2}$ (Half-wave) dipole,

$$E = j\eta \frac{I_0}{2\pi} \frac{e^{-jkr}}{r} \frac{\left[\cos\left(\frac{kl}{2}\cos\theta\right) - \cos\left(\frac{kl}{2}\right)\right]}{\sin\theta}$$

$$kr = \frac{2\pi}{\lambda} \times r = \frac{2\pi}{(\cancel{0.125})} \times (\cancel{0.5})^{4} = 8\pi$$

$$\frac{kl}{2} = \frac{2\pi}{\lambda} \times \frac{0.06}{2} = \frac{\cancel{2}\pi}{0.125} \times \frac{0.06}{\cancel{2}} = 0.48\pi$$

Putting these values, in eqn of E, we have

$$E = j \times \cancel{120\pi}^{60} \times \frac{1}{\cancel{2\pi}} \times \frac{e^{-j8\pi}}{0.5} \frac{\left[\cos\left(0.48\pi \cos\left(\frac{\pi}{3}\right)\right)\right] - \cos(0.48\pi)]}{\sin\left(\frac{\pi}{3}\right)}$$

$$= j \times 120 \times e^{-j8\pi} \frac{[\cos(0.24\pi)^c - \cos(0.48\pi)^c}{\left(\frac{\sqrt{3}}{2}\right)}$$

$$= j \times 120 \times e^{-j8\pi} \times \frac{2}{\sqrt{3}} \times [\cos(0.7539)^c - \cos(1.50)^c]$$

$$= j \times 138.56 \times e^{-j8\pi} [0.6582]$$

$$\boxed{E_\theta = j\, 91.21 \; \frac{V}{m}}$$

$\because$ $\cos(-8\pi) + j\sin(8\pi) = 1$

(0.24π) in radian

(0.48π) in radian.

$$H = a_\phi \frac{E}{\eta}$$

$$\therefore H_\phi = a_\phi \frac{E_\theta}{\eta} = \frac{j\,91.21}{120\pi}$$

$$\therefore \boxed{H_\phi = j\,0.2419 \ \frac{A}{m}}$$ (Ans)

Monopole :-

Dipole antennas for HF & VHF (High freq
& very High freq i.e 3-30 MHz & 30-300 MHz)
applications tend to be several meters long.
Constructing a dipole to radiate vertically polarized
(electric field orientation is perpendicular to
the surface of the earth) e-m waves poses
some real challenges due to size of the
antenna and the presence of the earth itself.

From the image theory, we know
that the fields due to a vertical
electric current element kept above an infinitely
large perfect electrical conductor (also known as
the ground plane) are the same as the fields
radiated by the element and its image.

Therefore, it is possible to virtually
create a half-wave ($\frac{\lambda}{2}$) dipole by placing a
quarter wavelength ($\frac{\lambda}{4}$) long wire (called monopole)
vertically above an infinitely large ground plane.

Consider a monopole of length $\left(\frac{l}{2}\right)$, fed at its base and kept above the ground plane as shown in figure 1. By image theory, this structure is equivalent to a dipole of length (l) radiating into free space. Therefore, the electric and magnetic fields in the far-field region are given by

$$E_\theta = a_\theta \, j \eta \frac{I_0}{2\pi} \frac{e^{-jkr}}{r} \left[\frac{\cos\left(\frac{kl}{2}\cos\theta\right) - \cos\left(\frac{kl}{2}\right)}{\sin\theta} \right] \quad \text{--- (1)}$$

$$H_\phi = a_\phi \, j \frac{I_0}{2\pi} \frac{e^{-jkr}}{r} \left[\frac{\cos\left(\frac{kl}{2}\cos\theta\right) - \cos\left(\frac{kl}{2}\right)}{\sin\theta} \right] \quad \text{--- (2)}$$

[Refer eqn (13) & (15) of Half wave dipole]

For a monopole of quarter wavelength $\left(\frac{\lambda}{4}\right)$ long, the field expression reduces to

$$E_\theta = j\eta \frac{I_0}{2\pi} \cdot \frac{e^{-jkr}}{r} \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} \quad \text{--- (3)}$$

$$H_\phi = j \frac{I_0}{2\pi} \cdot \frac{e^{-jkr}}{r} \cdot \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} \quad \text{--- (4)}$$

($\because$ $\frac{\lambda}{4}$ monopole with a ground plane behave as a $\frac{\lambda}{2}$ dipole.

& $\frac{kl}{2} = \frac{2\pi}{\lambda} \cdot \frac{\lambda}{2} \times \frac{1}{2} = \frac{\pi}{2}$ and $\cos\left(\frac{kl}{2}\right) = \cos\frac{\pi}{2} = 0$

As derived for Halfwave dipole)

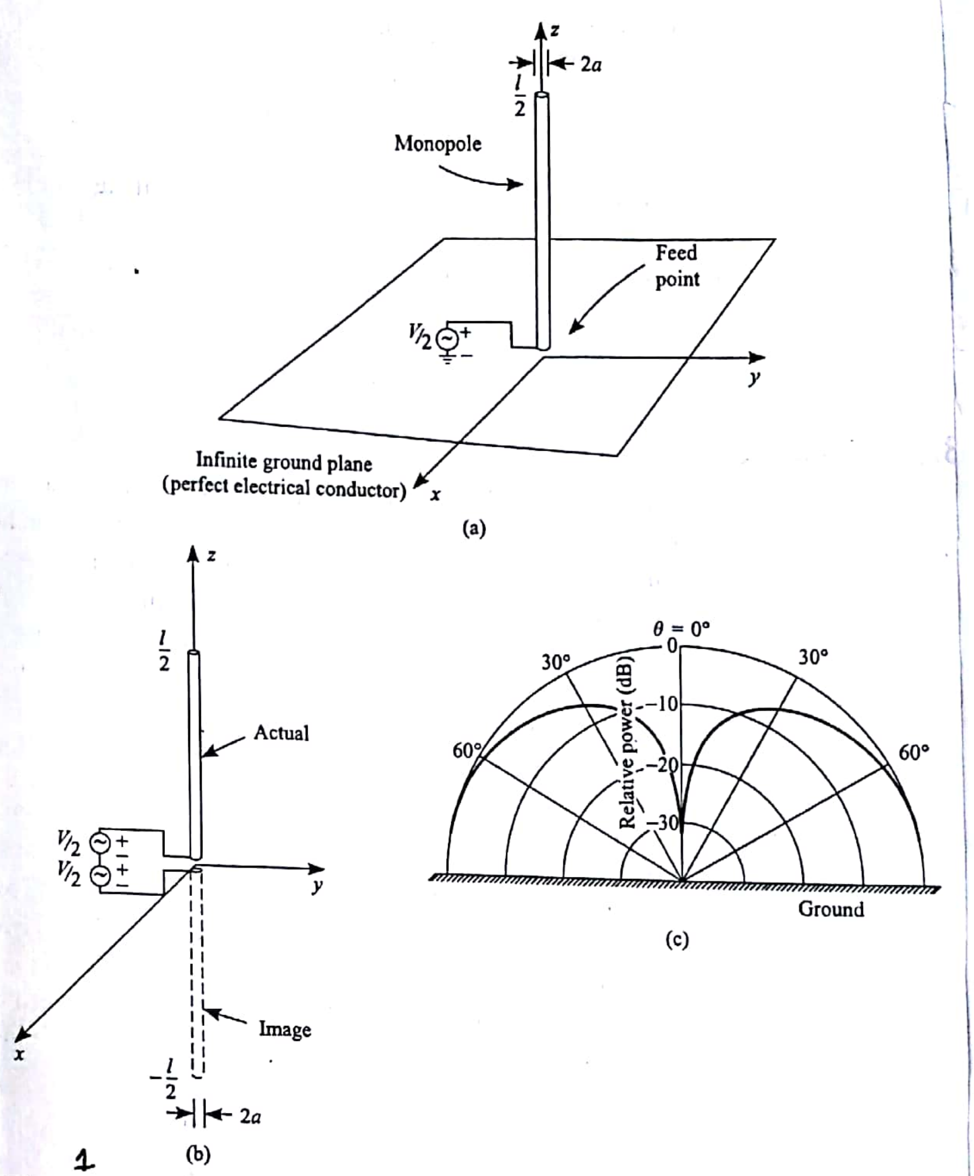


Fig. 1 (a) Geometry of a monopole above an infinite perfect electrical conductor, (b) its dipole equivalent, and (c) the radiation pattern for $l = \lambda/2$

The original problem has an infinitely large ground plane and there are no fields below the ground plane. Therefore eqn (3) & (4) are evaluated only in the upper hemisphere. i.e for $0 \leq \theta \leq \frac{\pi}{2}$ and $0 \leq \phi \leq 2\pi$. The total radiated power is obtained by integrating the radiation intensity over the upper hemisphere.

$$P_{rad} = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} U(\theta,\phi) \sin\theta \, d\theta \, d\phi \quad \text{——} (5)$$

Repeating the same procedure as that of $\frac{\lambda}{2}$ dipole [From eqn (15) to eqn (18)]

$$P_{rad} = \frac{1}{2} \times |I_0|^2 \times 36.54 \quad \text{——— (6)}$$

[Note: only change is $\int_{\theta=0}^{\pi/2}$ instead of $\int_{\theta=0}^{\pi}$ as that of $\frac{\lambda}{2}$ dipole.

i.e. why $\underset{\substack{\lambda/4 \\ \text{dipole}}}{(P_{rad})} = \frac{1}{2} \times \underset{\substack{\lambda/2 \\ \text{dipole}}}{(P_{rad})}$]

Equating this to the power dissipated in an equivalent resistor, the radiation resistance of a monopole is

$$\cancel{\frac{1}{2}} \cancel{|I_0|^2} R_{rad} = \cancel{\frac{1}{2}} \times \cancel{|I_0|^2} \times (36.54)$$

$$\Rightarrow \boxed{R_{rad} = 36.54 \text{ ohm}} \quad \text{——— (7)}$$

which is half of that of $\frac{\lambda}{2}$ dipole.

From eqn (18) of Half-wave dipole,

$$U = \frac{\eta}{2} \times \left|\frac{I_0}{2\pi}\right|^2 \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta} \quad \text{— (8)}$$

$$U_{max} = \frac{\eta}{2} \left|\frac{I_0}{2\pi}\right|^2 \quad \text{——— (9)}$$

[Which occurs at $\theta = \frac{\pi}{2}$]

Directivity $(D) = \dfrac{4\pi U}{P_{rad}}$

$$= \frac{4\pi \times \frac{\eta}{2} \times \left|\frac{I_0}{2\pi}\right|^2 \times \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}}{\frac{1}{2} \times |I_0|^2 \times 36.5}$$

$$= \cancel{2} \times \cancel{4\pi} \times \frac{\eta}{\cancel{2}} \times \frac{\cancel{|I_0|^2}}{\cancel{4\pi^2}} \times \frac{1}{36.5\,\cancel{|I_0|^2}} \times \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}$$

$$= 120\cancel{\pi} \times \frac{1}{\cancel{\pi}} \times \frac{1}{36.5} \times \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}$$

$$D = \frac{120}{36.5} \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}$$

$$D = 3.287 \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta} \quad \text{——— (9)}$$

$$D_{max} = D_0 = 3.287 \quad \text{——— (10)}$$

$$(D_{max})_{dB} = 10 \log(3.287) = 5.16 \text{ dB} \quad \text{——— (11)}$$

Thus from eqn (10), it is found that the directivity of a quarter wave monopole above a ground plane is equal to twice that of a half-wave dipole radiating in free space.

The maximum directivity occurs along the ground plane $(\theta = \frac{\pi}{2})$ and radiation is vertically polarized.

—o—

Loop antenna

→ Simple, inexpensive and very versatile antenna type is the loop antenna.

→ Loop antennas take many different forms such as rectangular, square, triangle, ellipse, circle and many other configurations.

→ Because of the simplicity in analysis and construction, the circular loop is the most popular and has received the widest attention.

→ Small loop is equivalent to an infinitesimal magnetic dipole whose axis is ⊥ to the plane of the loop.

Small circular loop :-

Let the circular loop is ~~placed~~ positioned symmetrically on the X-Y plane, at $Z=0$. The wire is assumed to be very thin and current distribution is given by

$$I_\phi = I_0 \, a_\phi \quad \text{--- (1)}$$

Where I_0 is a constant.

Radiated field :-

To find the fields radiated by the loop, the same procedure is followed as for the linear dipole. The potential function 'A' is given by

$$A = \frac{\mu}{4\pi} \int_C I_e(x', y', z') \frac{e^{-jkR}}{R} dl' \quad \text{--- (2)}$$

Note: Small loop ≃ Infinitesimal magnetic dipole of length 'l' and const. magnetic current I_m

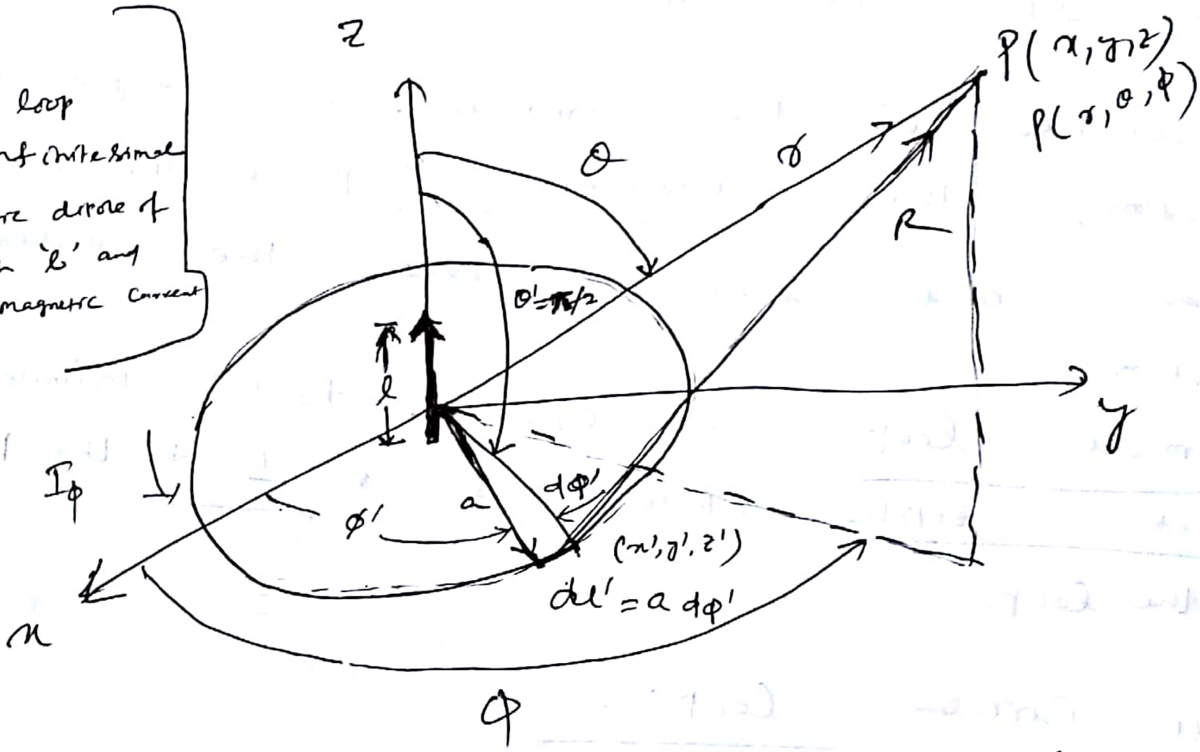


fig 1:- Geometry for circular loop.

Referring to the fig above, R is the distance from any point on the loop to the observation point and dl' is an infinitesimal section of the loop antenna. In general, the current distribution $I_e(x', y', z')$ can be written as

$$I_e(x', y', z') = \hat{a}_x I_x(x', y', z') + \hat{a}_y I_y(x', y', z') + \hat{a}_z I_z(x', y', z') \quad \text{--- (3)}$$

For the circular loop antenna, whose current is directed along a circular path, it would be more

Convenient to write the rectangular current components of eq^n (3) in terms of the cylindrical components using transformation.

$$\begin{bmatrix} I_x \\ I_y \\ I_z \end{bmatrix} = \begin{bmatrix} \cos\phi' & -\sin\phi' & 0 \\ \sin\phi' & \cos\phi' & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} I_\rho \\ I_\phi \\ I_z \end{bmatrix} \quad -(4)$$

Expanding

$$\left.\begin{aligned} I_x &= I_\rho \cos\phi' - I_\phi \sin\phi' \\ I_y &= I_\rho \sin\phi' + I_\phi \cos\phi' \\ I_z &= I_z \end{aligned}\right\} -(5)$$

Since the radiated fields are usually determined in spherical components, the rectangular unit vectors of (3), are transformed to spherical unit vectors using the transformation matrix i.e

$$\left.\begin{aligned} \hat{a}_x &= \hat{a}_r \sin\theta\cos\phi + \hat{a}_\theta \cos\theta\cos\phi - \hat{a}_\phi \sin\phi \\ \hat{a}_y &= \hat{a}_r \sin\theta\sin\phi + \hat{a}_\theta \cos\theta\sin\phi + \hat{a}_\phi \cos\phi \\ \hat{a}_z &= \hat{a}_r \cos\theta - \hat{a}_\theta \sin\theta \end{aligned}\right\} -(6)$$

Substituting (6) and (5) into (3), we have

$$I_e = \hat{a}_r \left[I_\rho \sin\theta \cos(\phi-\phi') + I_\phi \sin\theta \sin(\phi-\phi') + I_z \cos\theta \right]$$

$$+ \hat{a}_\theta \left[I_\rho \cos\theta \cos(\phi-\phi') + I_\phi \cos\theta \sin(\phi-\phi') - I_z \sin\theta \right]$$

$$+ \hat{a}_\phi \left[-I_\rho \sin(\phi-\phi') + I_\phi \cos(\phi-\phi') \right] \quad \text{--- (7)}$$

→ For the circular loop, the current is flowing in the ϕ direction (I_ϕ) so eqn (7) reduces to [take only I_ϕ components from eqn (7)]

$$I_e = \hat{a}_r I_\phi \sin\theta \sin(\phi-\phi') + \hat{a}_\theta I_\phi \cos\theta \sin(\phi-\phi')$$

$$+ \hat{a}_\phi I_\phi \cos(\phi-\phi') \quad \text{--- (8)}$$

The distance $\underline{\underline{R}}$, from any point on the loop to the observation point, can be written as

$$R = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2} \quad \text{--- (9)}$$

Since

$$\left.\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}\right] \quad [\text{Rect} \rightarrow \text{Spherical}]$$

$$\left.\begin{aligned} x^2+y^2+z^2 &= r^2 \\ x' &= a\cos\phi' \\ y' &= a\sin\phi' \\ z' &= 0 \end{aligned}\right] \quad \text{--- (10)}$$

and $x'^2 + y'^2 + z'^2 = a^2$

So Putting eqn (10), in eqn (9), we have.

$$R = \sqrt{(r\sin\theta\cos\phi - a\cos\phi')^2 + (r\sin\theta\sin\phi - a\sin\phi')^2 + (r\cos\theta - 0)^2}$$

$$= \sqrt{r^2\sin^2\theta\cos^2\phi + a^2\cos^2\phi' - 2ar\sin\theta\cos\phi\cos\phi' + r^2\sin^2\theta\sin^2\phi + a^2\sin^2\phi' - 2ar\sin\theta\sin\phi\sin\phi' + r^2\cos^2\theta}$$

$$R = \sqrt{r^2 + a^2 - 2ar\sin\theta\cos(\phi - \phi')} \quad \text{--- (11)}$$

Referring to fig (2),

$$dl' = a\,d\phi' \quad \text{--- (12)} \qquad \left(\because \theta = \frac{l}{r} \Rightarrow l = r\theta \Rightarrow l = a\,d\phi'\right)$$

Using eqn (8), (11) & (12) the ϕ component of eqn (2), can be written as,

$$A_\phi = \frac{\mu a}{4\pi}\int_0^{2\pi} I_\phi \cos(\phi - \phi')\cdot\frac{e^{-jk\sqrt{r^2 + a^2 - 2ar\sin\theta\cos(\phi-\phi')}}}{\sqrt{r^2 + a^2 - 2ar\sin\theta\cos(\phi-\phi')}}\,d\phi' \quad \text{--- (13)}$$

Since the current I_ϕ as given in eqn (1), is constant, the field radiated by the loop will not be a function of the observation angle ϕ.

Thus any observation angle ϕ can be chosen; for simplicity $\phi = 0$; Therefore eq (13), can be written as

$$A_\phi = \frac{a\mu I_0}{4\pi} \int_0^{2\pi} \cos\phi' \frac{e^{-jK\sqrt{r^2+a^2-2ar\sin\theta\cos(\phi')}}}{\sqrt{r^2+a^2-2ar\sin\theta\cos\phi'}} d\phi' \quad (14)$$

~~For Sr~~ The integration can n't be carried out without any approximations. For small loops, the f^n

$$f(a) = \frac{e^{-jK\sqrt{r^2+a^2-2ar\sin\theta\cos\phi'}}}{\sqrt{r^2+a^2-2ar\sin\theta\cos\phi'}} \quad (15)$$

can be expanded in Maclaurin's series in 'a' using.

$$f = f(0) + f'(0).a + \frac{f'(0)a^2}{2!} \cdots + \frac{1}{(n-1)!} f^{(n-1)} a^{n-1} + \cdots \quad 15(a)$$

where

$$f'(0) = \left.\frac{\partial f}{\partial a}\right|_{a=0}, \quad f''(0) = \left.\frac{\partial^2 f}{\partial a^2}\right|_{a=0} \quad \text{and so forth.}$$

Taking into account only the first two terms of 15 (a), we have

$$f(0) = \frac{e^{-jKr}}{r} \quad 15(b)$$

$$f'(0) = \left(\frac{jk}{r} + \frac{1}{r^2}\right) e^{-jkr} \sin\theta \cos\phi' \quad \text{—— } 15(c)$$

putting 15(b) & (c) into eq^n 15(a)

$$\therefore f \simeq \left[\frac{1}{r} + a\left(\frac{jk}{r} + \frac{1}{r^2}\right) \sin\theta \cos\phi'\right] e^{-jkr} \quad \text{—— } 15(d)$$

So eq^n (14) becomes,

$$A_\phi \simeq \frac{a\mu I_0}{4\pi} \int_0^{2\pi} \cos\phi' \left[\frac{1}{r} + a\left(\frac{jk}{r} + \frac{1}{r^2}\right) \sin\theta \cos\phi'\right] e^{-jkr} d\phi'$$

Simplifying (only variable is ϕ' rest are const.)

$$A_\phi \simeq \frac{a^2 \mu I_0}{4} e^{-jkr} \left(\frac{jk}{r} + \frac{1}{r^2}\right) \sin\theta \quad \text{—— } (16)$$

$$\left(\because \int_0^{2\pi} \cos^2\phi' \, d\phi' = \pi\right)$$

In a similar manner, the r- and θ components of (2), can be written as

$$A_r \simeq \frac{a\mu I_0}{4\pi} \sin\theta \int_0^{2\pi} \sin\phi' \left[\frac{1}{r} + a\left(\frac{jk}{r} + \frac{1}{r^2}\right) \sin\theta \cos\phi'\right] e^{-jkr} d\phi' \quad \text{—— } 16(a)$$

$$A_\theta \simeq \frac{a\mu I_0}{4\pi} \cos\theta \int_0^{2\pi} \sin\phi' \left[\frac{1}{r} + a\left(\frac{jk}{r} + \frac{1}{r^2}\right) \sin\theta \cos\phi'\right] e^{-jkr} d\phi' \quad \text{—— } 16(b)$$

which when integrated reduce to zero.

(For proof - $A_r = A_\theta = 0$, refer after 2 pages)

Thus

$$A \simeq \hat{a}_\phi A_\phi = \hat{a}_\phi \frac{a^2 \mu I_0}{4} e^{-jkr} \left[\frac{jk}{r} + \frac{1}{r^2}\right] \sin\theta$$

$$A = \hat{a}_\phi \frac{jk\mu a^2 I_0 \sin\theta}{4r} \left[1 + \frac{1}{jkr}\right] e^{-jkr} \quad \text{—— } (17)$$

From eqn (17),

$$A_\phi = a_\phi \frac{jk\mu a^2 I_0 \sin\theta}{4r}\left[1+\frac{1}{jkr}\right]e^{-jkr}$$

$$= a_\phi \frac{\mu}{4} I_0 jka^2 \sin\theta \left[\frac{1}{r}+\frac{1}{jkr^2}\right]e^{-jkr}$$

Neglecting the $\frac{1}{r^2}$ term for far-field,

we have

$$A_\phi = \frac{\mu}{4} I_0 jka^2 \sin\theta \cdot \frac{e^{-jkr}}{r} a_\phi \quad \text{——— (18)}$$

Since the vector potential has only a_ϕ component Using eqn (8), (9) & (10) of Halfwave dipole, i.e

$$E = -jwA_t \quad \text{——— (19)}$$

$$H = \frac{-jw}{\eta} a_r \times A_t \quad \text{——— (20)}$$

&

$$A_t = a_\theta A_\theta + a_\phi A_\phi \quad \text{——— (21)}$$

We have

$$E = -jwA_t$$

$$= -jw(a_\theta \cancel{A_\theta}^{0} + a_\phi A_\phi)$$

Using eqn (18)

$$E = -jw\, a_\phi A_\phi \quad \text{—(22)}$$

$$= -jw\, a_\phi \left[\frac{\mu}{4} \cdot I_0 k \cdot j\, a^2 \sin\theta \frac{e^{-jkr}}{r}\right]$$

$$\Rightarrow E = \omega \times \frac{\mu}{4} \times I_0 \times k \times a^2 \sin\theta \frac{-e^{-jkr}}{r} a_\phi$$

$$= \frac{k}{\sqrt{\mu\epsilon}} \times \frac{\sqrt{\mu}\,\cancel{\mu}}{4} \times I_0 \times k \times a^2 \sin\theta \frac{e^{-jkr}}{r} a_\phi$$

$$= k^2 \times \sqrt{\frac{\mu}{\epsilon}} \times \frac{1}{4} \times I_0 \, a^2 \sin\theta \frac{e^{-jkr}}{r} a_\phi$$

$$E_\phi = \eta \times \frac{a^2 k^2}{4} I_0 \frac{e^{-jkr}}{r} \sin\theta \quad \text{— (23)}$$

$\because k = \omega\sqrt{\mu\epsilon}$

$\Rightarrow \omega = \frac{k}{\sqrt{\mu\epsilon}}$

$\because \sqrt{\frac{\mu}{\epsilon}} = \eta$

Similarly,

$$H = \frac{-j\omega}{\eta} \, a_r \times A_t$$

$$= \frac{-j\omega}{\eta} (a_r \times a_\phi A_\phi)$$

$$= \frac{-j\omega}{\eta} \times (-a_\theta) A_\phi$$

$$= -a_\theta \left(\frac{-j\omega}{\eta} A_\phi \right) \quad \text{— (24)}$$

$r \to \theta \to \phi$

$\because a_r \times a_\phi = \hat{a}_\theta$

Comparing eq^n (22) & (24)

$$H = -\frac{E}{\eta}, \hat{a}_\theta$$

$$\therefore H_\theta = -\frac{1}{\cancel{\eta}} \times \cancel{\eta} \times \frac{a^2 k^2}{4} I_0 \frac{e^{-jkr}}{r} \sin\theta.$$

$$H_\theta = -\frac{a^2 k^2}{4} I_0 \frac{e^{-jkr}}{r} \sin\theta \quad \text{— (24)}$$

$\therefore$ The radiation pattern has a null along the axis of loop ($\theta = 0°$) and max^m in $\theta = 90°$ plane. The fields have the <u>same power pattern</u> as that of <u>Hertzian dipole</u>.

1) Prove that for a small loop

$$A_r = \frac{a \mu I_0}{4\pi} \sin\theta \int_0^{2\pi} \sin\phi' \left[\frac{1}{r} + a\left(\frac{jk}{r} + \frac{1}{r^2}\right) \sin\theta \cos\phi'\right] \cdot e^{-jkr} \, d\phi'$$

$$= 0$$

Ans : Here, ϕ' is the only variable rest can be treated as constants.

$$A_r = \frac{a \mu I_0}{4\pi} \sin\theta \left[\int_0^{2\pi} \frac{\sin\phi'}{r} e^{-jkr} \, d\phi' + \int_0^{2\pi} a\left(\frac{jk}{r} + \frac{1}{r^2}\right) \sin\theta \sin\phi' \cos\phi' \cdot e^{-jkr} \, d\phi'\right]$$

$$= \frac{a \mu I_0}{4\pi} \sin\theta \left[\frac{e^{-jkr}}{r} \int_0^{2\pi} \sin\phi' \, d\phi' + \frac{a\left(\frac{jk}{r} + \frac{1}{r^2}\right) \sin\theta \cdot e^{-jkr}}{2} \int_0^{2\pi} 2 \sin\phi' \cos\phi' \, d\phi'\right]$$

$$= \frac{a \mu I_0}{4\pi} \sin\theta \left[\frac{e^{-jkr}}{r} \cdot \left[-\cos\phi'\right]_0^{2\pi} + \frac{a\left(\frac{jk}{r} + \frac{1}{r^2}\right) \sin\theta \, e^{-jkr}}{2} \times \left[-\frac{\cos 2\phi'}{2}\right]_0^{2\pi}\right]$$

$$= 0 + 0$$

$\therefore A_r = 0$

(Proved)

$\because$

$\cos\phi' \Big|_0^{2\pi} = 1 - 1 = 0$

$\cos 2\phi' \Big|_0^{2\pi} = 1 - 1 = 0$

$\because$

$2 \sin\phi' \cos\phi' = \sin 2\phi'$

$\int \sin 2\phi' \, d\phi' = \frac{\cos 2\phi'}{2}$

Radiation Resistance & Directivity Calculation of Small loop

The time averaged power density given by

$$S = \frac{1}{2} \operatorname{Re}[E \times H^*]$$

$$= \frac{1}{2} \operatorname{Re}\left[E \times \frac{E^*}{\eta}\right]$$

$$= \frac{1}{2\eta} \cdot |E_\phi|^2$$

$\because$

Putting the value of $|E_\phi|$ from eqn (23), we have

$$S = \frac{1}{2\eta} \times \left| \eta \times \frac{a^2k^2}{4} I_0 \cdot \frac{e^{-jkr}}{r} \sin\theta \right|^2$$

$$S = \frac{1}{2\cancel{\eta}} \times \eta^{\cancel{2}} \times \left| \frac{a^2k^2}{4} \cdot \frac{I_0}{r} \right|^2 \cdot \sin^2\theta \qquad \because |e^{-jkr}| = 1$$

$$\Rightarrow \quad S = \frac{\eta}{2} \left(\frac{a^2 k^2 |I_0|}{4r} \right)^2 \sin^2\theta \ \frac{\text{Watt}}{m^2} \quad \text{——— (25)}$$

The radiation intensity

$$U(\theta, \phi) = r^2 S = \cancel{r^2} \times \frac{\eta}{2} \left(\frac{a^2k^2 |I_0|}{4} \right)^2 \times \frac{1}{\cancel{r^2}} \sin^2\theta$$

$$\Rightarrow U(\theta, \phi) = \frac{\eta}{2} \left(\frac{a^2k^2 |I_0|}{4} \right)^2 \sin^2\theta \ \frac{\text{Watt}}{Sr} \quad \text{—(26)}$$

$$P_{rad} = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} U(\theta, \phi) \sin\theta \, d\theta \, d\phi$$

99

$$\therefore P_{rad} = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \frac{\eta}{2} \times \left(\frac{a^2k^2|I_0|}{4}\right)^2 \sin^2\theta \cdot \sin\theta \, d\theta \, d\phi$$

$$= 2\pi \times \frac{\eta}{2} \left(\frac{a^2k^2|I_0|}{4}\right)^2 \int_0^{\pi} \sin^3\theta \, d\theta \qquad \Bigg| \; \because \int_0^{2\pi} d\phi = 2\pi$$

$$= \cancel{2}\pi \times \frac{\eta}{\cancel{2}} \left(\frac{a^2k^2|I_0|}{4}\right)^2 \times \frac{4}{3}$$

$$\Rightarrow P_{rad} = \eta\pi \frac{(ka)^4 |I_0|^2}{\cancel{4}\times 4} \times \frac{\cancel{4}}{3}$$

$$\Rightarrow \boxed{P_{rad} = \eta \cdot \frac{\pi}{12} \times (ka)^4 |I_0|^2} \quad \text{—(26)}$$

$$P_{rad} = \cancel{120}^{10}\pi \times \frac{\pi}{\cancel{12}} \times a^4k^4 |I_0|^2$$

$$\Rightarrow \boxed{P_{rad} = 10\pi^2 a^4 k^4 |I_0|^2} \quad \text{—(27)}$$

To find $\int_0^{\pi} \sin^3\theta \, d\theta$, we know

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$$

$$\Rightarrow \sin^3\theta = \frac{3\sin\theta - \sin 3\theta}{4}$$

$$\therefore \int_0^{\pi} \frac{3\sin\theta - \sin 3\theta}{4} \, d\theta$$

$$= \frac{1}{4}\left[-3\cos\theta + \frac{\cos 3\theta}{3}\right]_0^{\pi}$$

$$= \frac{1}{4}\left[\left(3 - \frac{1}{3}\right) - \left(-3 + \frac{1}{3}\right)\right]$$

$$= \frac{1}{4}\left[6 - \frac{2}{3}\right]$$

$$= \frac{1}{4}\left[\frac{16}{3}\right]$$

$$= \frac{4}{3}$$

Equating this power to the power dissipated in an equivalent resistance carrying the same current

$$\frac{1}{2} \times \cancel{|I_0|^2} R_{rad} = 10\pi^2 a^4 k^4 \cancel{|I_0|^2}$$

$$\Rightarrow \boxed{R_{rad} = 20\pi^2 a^4 k^4} \quad \text{—(28)}$$

Let $L_A = \pi a^2$ be the area of the loop and

$L_C = 2\pi a$ be the circumference of the loop.

$$\therefore \quad R_{rad} = 20\pi^2 \, a^4 \times \left(\frac{2\pi}{\lambda}\right)^4 \qquad \because K = \frac{2\pi}{\lambda}$$

$$R_{rad} = 20\pi^2 \left(\frac{2\pi a}{\lambda}\right)^4$$

$$\Rightarrow \boxed{R_{rad} = 20\pi^2 \left(\frac{L_c}{\lambda}\right)^4} \quad \text{--- (29)}$$

Another way

$$R_{rad} = 20\pi^2 \, a^4 \times \left(\frac{2\pi}{\lambda}\right)^4$$

$$= 20\pi^2 \times a^4 \times \frac{16 \times \pi^4}{\lambda^4}$$

$$\boxed{R_{rad} = 320\,\pi^6 \left(\frac{a}{\lambda}\right)^4} \quad \text{--- (30)}$$

Another way

$$R_{rad} = 20\pi^2 \, a^4 \times \left(\frac{2\pi}{\lambda}\right)^4$$

$$= 20 \times (\pi a^2)^2 \times \frac{16\pi^4}{\lambda^4}$$

$$\boxed{R_{rad} = 31171 \, \frac{L_A^2}{\lambda^4}} \quad \text{--- (31)}$$

The radiation resistance of a small loop is generally very small and is difficult to match to the source. The radiation resistance can be increased by having more turns in the loop.

If the loop is made of N turns and is carrying the same i/p current, Io, the loop current would by N Io. Hence the field strength of multi-turn loop would be N times that of a single turn loop. Replacing I_0 by $N I_0$ in the eqn (27), we have

$$\frac{1}{2} \times \cancel{|I_0|^2} \times R_{rad} = 10\pi^2 a^4 K^4 N^2 \cancel{|I_0|^2}$$

$$\Rightarrow \quad R_{rad} = 20\pi^2 N^2 a^4 K^4 \quad \text{—— (32)}$$

$$\therefore \quad R_{rad} = 20\pi^2 N^2 \left(\frac{Lc}{\lambda}\right)^2 \quad \text{—— (33)}$$

$$R_{rad} = 31171 \, N^2 \, \frac{L_A^2}{\lambda^4} \quad \text{—— (34)}$$

$$R_{rad} = 320\,\pi^6 \, N^2 \left(\frac{a}{\lambda}\right)^4 \quad \text{—— (35)}$$

Ex:- 21) What is the total power radiated by a small circular loop of radius 0.5m carrying a current of 10A at 15 MHZ? If the loop is symmetrically placed at the origin and in the x-y plane, calculate the magnitude of the electric field intensity in the xy plane at a distance of 10 km.

Ans:- The wavelength of 15 MHZ e.m wave propagation in free space is

$$\lambda = \frac{c}{f} = \frac{\cancel{3} \times 10^{\cancel{8}\,2}}{\cancel{15}\,5 \times 10^{\cancel{6}}} = \frac{100}{5} = 20\text{m}.$$

$$\text{Propagation constant } (K) = \frac{2\pi}{\lambda} = \frac{\cancel{2}\pi}{\cancel{20}\,10} = \frac{\pi}{10} \ \frac{\text{rad}}{\text{m}}$$

Substituting

$a = 0.5\,m, \quad I_0 = 10\,A, \quad k = \frac{\pi}{10}\ \frac{rad}{m}.$

$$P_{rad} = 10\pi^2 a^4 k^4 |I_0|^2$$

$$= 10\pi^2 \times (0.5)^4 \left(\frac{\pi}{10}\right)^4 \times 10^2$$

(i) $\boxed{P_{rad} = 6.01 \text{ watt}}$ (Ans)

(ii) $$E_\phi\Big|_{\theta=90^\circ} = \eta \times \frac{a^2k^2}{4} I_0 \frac{e^{-jkr}}{r} \cdot \sin\theta$$

$$|E_\phi|_{\theta=90^\circ} = \frac{\eta \times a^2k^2}{4r} \times I_0$$

$$= \frac{120\pi \times 0.5^2 \times \left(\frac{\pi}{10}\right)^2 \times 10}{4 \times 10 \times 10^3}$$

$$\boxed{E_\phi\Big|_{\theta=90^\circ} = 2.32 \ \frac{\text{mVolt}}{\text{meter}}}$$ (Ans)

Antenna Arrays :—

→ Usually the radiation pattern of a single element is relatively wide, and each element provides low values of directivity (gain). In many applications it is necessary to design antennas with very directive characteristics (very high gain) to meet the demand of long distance communication. This can only be accomplished by increasing the electrical size of the antenna.

→ Another way to achieve directional characteristics without increasing the size of the individual elements, is to form an assembly of radiating elements in an electrical and geometrical configuration. This new antenna, formed by multielements is referred to as an array.

→ The total field of the array is determined by the vector addition of the fields radiated by the individual elements.

→ To provide very directive patterns, it is necessary that the fields from the elements of the array interfere constructively (add) in the desired directions and interfere destructively (cancel each other) in the remaining space.

→ In an array of identical elements, there are at least five controls that can be used to shape the overall pattern of the antenna. (104)

1. The geometrical configuration of the overall array (linear, circular, rectangular, spherical, etc.)
2. The relative displacement between the elements.
3. The excitation amplitude of individual elements.
4. The excitation phase of individual elements.
5. The relative pattern of the individual elements.

→ There are a plethora of antenna arrays used for personal, commercial, and military appl^ns utilizing different element including dipoles, loops, apertures, microstrips, horns, reflectors and so on.

Linear Array:-

Consider an infinitesimal dipole (Hertzian dipole) of length dl kept at a point $(0, 0, z_1')$ in free space. Let the z-directed current in the dipole be I_1. The fields produced by the the dipole are computed using vector potential approach. [Refer fig 5.1]

Since the dipole current is z-directed, the vector potential also has only a z-component which is given by

$$A_z = \frac{\mu}{4\pi} I_1 \, dl \, \frac{e^{-jkr_1}}{r_1} \quad \text{——— (1)}$$

[Refer eqn (2) in Hertzian dipole]

Where r_1 is the distance from the center of the current element to the field point (x, y, z). When the field point is at large distance, we can approximate r_1 to

$$r_1 = r \quad (\text{for Amplitude}) \quad \text{---} (2)$$

$$r_1 = r - z' \cos\theta \quad (\text{for Phase}) \quad \text{---} (3)$$

[∵ Refer for field approximation in short dipole eqn (7) & (9)]

Using the vector potential approach with these far field approximations, we get the electric field radiated by the dipole as

$$E_{\theta_1} = j\eta \frac{k I_1 dl}{4\pi} \sin\theta \cdot \frac{e^{-jkr}}{r} \cdot e^{jkz' \cos\theta} \quad \text{---} (4)$$

[∵ Refer eqn (10) in short dipole]

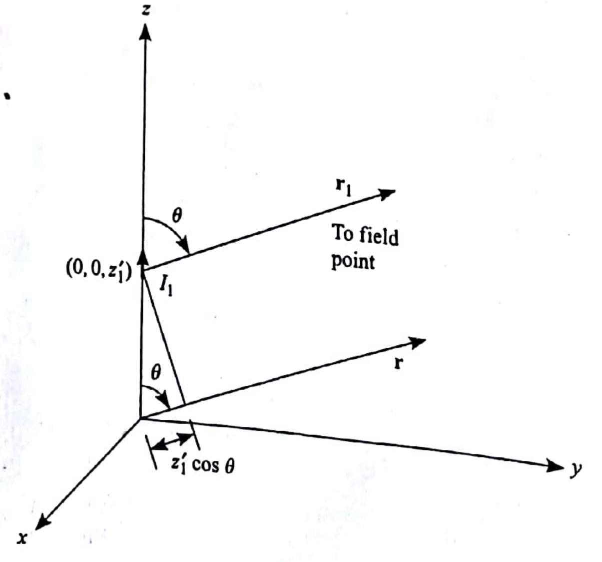


Fig. 5.1 Geometry of a z-directed, infinitesimal dipole radiating into free space

Let us now consider N such infinitesimal (106)
z-directed current elements kept along the z-axis
at points z_1', z_2', ------, z_N'. Let the currents
in these dipoles be I_1, I_2, ---- I_N, respectively.
(See fig 5.2). It is implied that all the
currents have the same frequency. Using
superposition, the field at any point can be
written as the sum of the fields due to
each of elements.

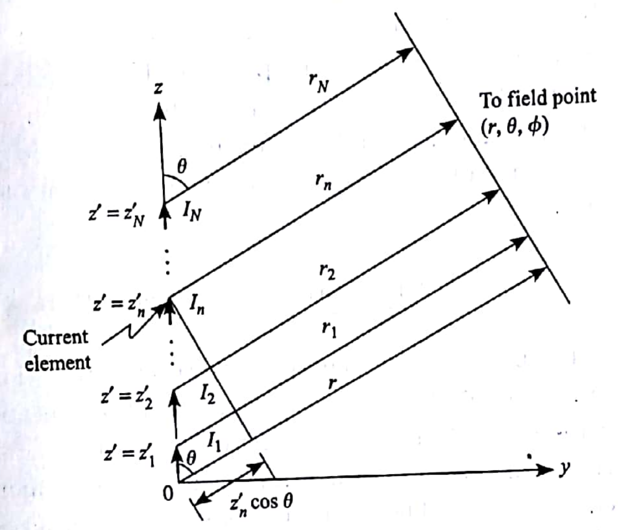


Fig. 5.2 Array of N z-directed, infinitesimal dipoles radiating into free space

So,

$$E_\theta = E_{\theta 1} + E_{\theta 2} + E_{\theta 3} + \cdots + E_{\theta N} \quad \text{—— (5)}$$

$$= j\eta k \frac{dl \sin\theta}{4\pi} \left[I_1 \frac{e^{-jkr_1}}{r_1} + I_2 \frac{e^{-jkr_2}}{r_2} + \cdots + I_N \frac{e^{-jkr_N}}{r_N} \right] \quad (6)$$

(without far field approximation)

where $r_1, r_2, \ldots, r_N$ are respectively the distances
from the dipoles 1, 2, ---, N to the field point.
In the far field region of these dipoles, the
distance from the nth dipole to the field point, r_n,

is approximated to

$$r_n \simeq r \;;\; n = 1, 2, 3, \ldots N \quad \text{for amplitude} \quad (7)$$

$$r_n \simeq r - z_n' \cos\theta \;;\; n = 1, 2, 3 \ldots N \quad \text{for Phase} \quad (8)$$

Where z_n' is location of the nth dipole. The $r_1, r_2, \ldots r_N$ in the denominator (amplitude) of eqn (6) are replaced by $\underline{\underline{r}}$ and in the exponent (phase term) they are replaced by eqn (8).

∴ Eqn (6) becomes

$$E_\theta = \frac{j\eta k\, dl \sin\theta}{4\pi} \left[I_1 \cdot \frac{e^{-jk(r - z_1' \cos\theta)}}{r} + I_2 \frac{e^{-jk(r - z_2' \cos\theta)}}{r} + \ldots + I_N \frac{e^{-jk(r - z_N' \cos\theta)}}{r} \right] \quad (8A)$$

$$\Rightarrow E_\theta = \underbrace{\frac{j\eta k\, dl \sin\theta}{4\pi} \cdot \frac{e^{-jkr}}{r}}_{\text{Element Pattern}} \; \underbrace{\sum_{n=1}^{N} I_n \, e^{jk z_n' \cos\theta}}_{\text{Array factor}} \quad (9)$$

The term outside the summation corresponds to the electric field produced by an infinitesimal dipole excited by a unit current kept at the origin and is known as the element pattern.

The remaining portion of the equation is called the array factor. Thus the radiation (108)

pattern of an array of equioriented identical antenna elements is given by the product of the element pattern and the array factor. This is known as pattern multiplication theorem.

$$\boxed{\text{Array Pattern} = \text{Element Pattern} \times \text{Array factor}} \quad (10)$$

It can be shown that the pattern multiplication theorem is applicable to any array of identical, equi-oriented antenna elements. The elements can be arranged to form a linear, 2D (Planar) or 3D array.

It is assumed that there is no interaction between the elements, which may result in altering the individual radiation patterns.

The overall pattern of of array of elements mainly controlled by the array factor. The array factor (AF) is given by

$$\boxed{AF = \sum_{n=1}^{N} I_n \, e^{jKz_n' \cos\theta}} \quad (11)$$

The array factor depends on the excitation currents (both amplitude and phase) and position of the elements. So, it is possible to achieve a wide variety of patterns having interesting characteristics by adjusting the excitation amplitudes, phases and the element positions.

Two-element Array :-

Consider two infinitesimal z-directed current elements placed symmetrically about the origin along z-axis. (Fig 5.3).

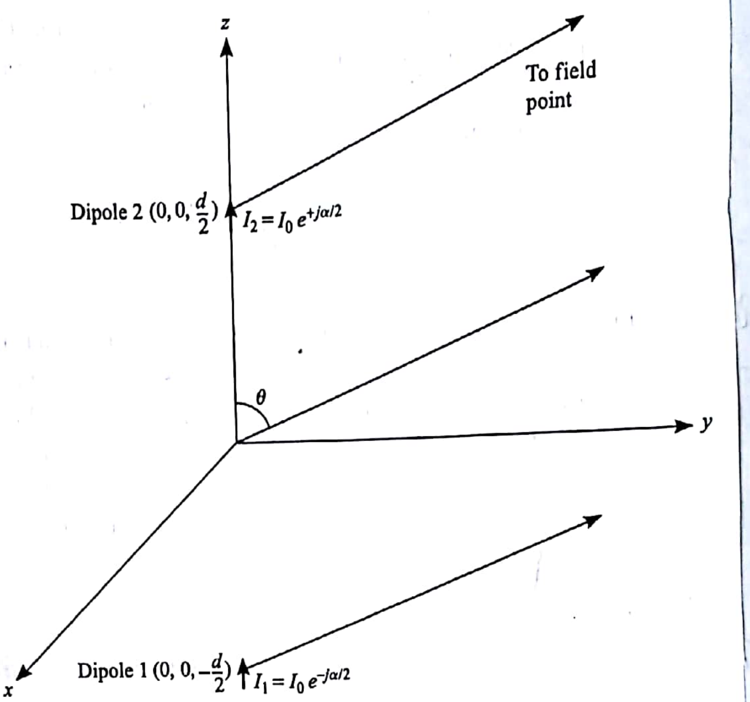


Fig. 5.3 Geometry of two z-directed, infinitesimal dipoles radiating into free space

Let dipole 1 be kept at $z_1' = -\frac{d}{2}$ and carry a current $I_1 = I_0 e^{-j\alpha/2}$ and dipole 2 be at $z_2' = \frac{d}{2}$ with a current $I_2 = I_0 e^{j\alpha/2}$. 'α' is known as relative phase shift. For +ve values of α, the current in dipole 2 leads the current in dipole 1. The electric field of the two-element array can be computed using eqn $\boxed{8A}$ with $N=2$, which is

$$E_\theta = j\eta \frac{k\,dl}{4\pi} \frac{\sin\theta}{r} \left[I_0 \cdot e^{-j\alpha/2} e^{-jk\left(r - \left(-\frac{d}{2}\right)\cos\theta\right)} + I_0 e^{+j\alpha/2} e^{-jk\left(r - \frac{d}{2}\cos\theta\right)} \right] \quad (12)$$

$$\Rightarrow E_\theta = j\eta \cdot \frac{k\,dl}{4\pi}\,\frac{\sin\theta}{r}\cdot e^{-jkr}\left[I_0\, e^{-j\alpha/2} \cdot e^{-jk\frac{d}{2}\cos\theta} + I_0\, e^{j\alpha/2}\cdot e^{jk\frac{d}{2}\cos\theta} \right] \quad (11D)$$

$$= j\eta\,\frac{k\,dl}{4\pi}\,\frac{e^{-jkr}}{r}\cdot \sin\theta \cdot I_0\left[e^{j\left(\frac{\alpha}{2}+\frac{kd}{2}\cos\theta\right)} + e^{-j\left(\frac{\alpha}{2}+\frac{kd}{2}\cos\theta\right)} \right] \quad 12(A)$$

$$E_\theta = \underbrace{j\eta\,\frac{k\,dl}{4\pi}\,\frac{e^{-jkr}}{r}\,\sin\theta}_{\text{Element Pattern}}\;\underbrace{\left[2 I_0 \cos\left(\frac{kd}{2}\cos\theta + \frac{\alpha}{2}\right)\right]}_{\text{Array Factor}} \quad (13)$$

$$\because \frac{e^{j\theta}+e^{-j\theta}}{2} = \cos\theta$$

$$\Rightarrow e^{j\theta}+e^{-j\theta} = 2\cos\theta$$

Consider a situation where the two currents are in phase with each other, (i.e $\alpha = 0$), The array-factor of the two elements array reduces to

$$AF = 2 I_0 \cos\left(\frac{kd}{2}\cos\theta\right) \quad (14)$$

The array factor of a two-element array for different element spacings from 0.25λ to 2λ are shown in fig 5.4. As the element spacings increases from 0.25λ to 0.5λ, the main beam gets narrower. At $d = 0.5\lambda$, two nulls (along $\theta = 0°$ and $180°$) appear in the pattern. Further increase in the spacing results in the appearance of multiple lobes in the pattern.

We will now derive the expressions for the

directions of the maxima and nulls of the array (11)
factor. The maxima of the array factor occur when the argument of the cosine function is equal to an integer multiple of π.

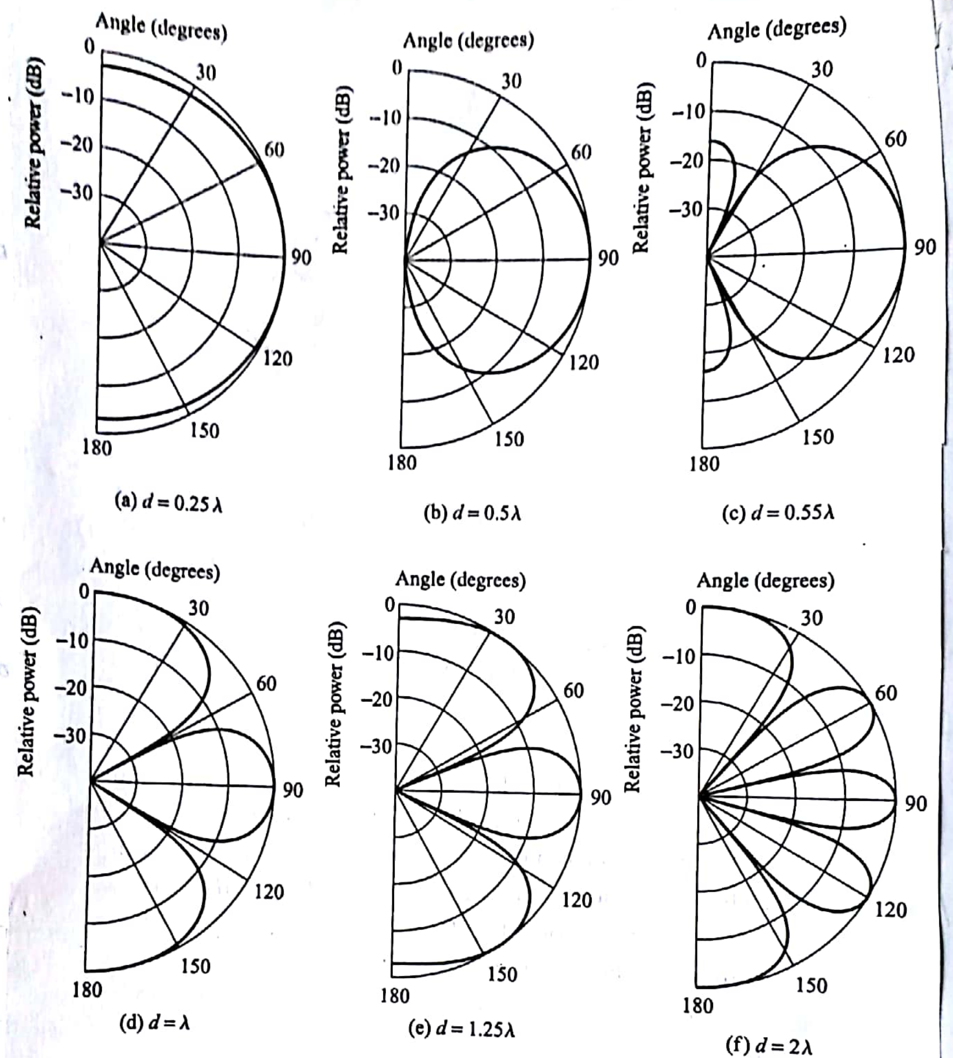


Fig. 5.4 Array factors of a two-element array with $\alpha = 0$ for some selected element spacings

when $\frac{Kd}{2}\cos\theta_m = \pm m\pi$; $m = 0, 1, 2, \ldots$ — (15)

$$\cos\theta_m = \frac{\pm 2m\pi}{Kd} = \pm\frac{2m\pi}{\left(\frac{2\pi}{\lambda}\right)d} = \pm\frac{\cancel{2}m\cancel{\pi}\times\lambda}{\cancel{2\pi}\,d}$$

$$\cos\theta_m = \pm\frac{m\lambda}{d}$$

$$\Rightarrow \theta_m = \cos^{-1}\left(\pm \frac{m\lambda}{d}\right) \qquad m = 0, 1, 2 \ldots \quad (112)$$

— (16)

where θ_m are the directions of the maxima.

If $m = 0$,

$$\theta_m = \cos^{-1}(0) = \frac{\pi}{2}.$$

$\therefore$ There is always a maximum at $\theta = \frac{\pi}{2}$.

The array factor always has a maximum along $\theta = 90^\circ$ direction, or the broadside direction, hence the array is known as a broadside array.

[Note :-

If the maximum radiation of an array directed normal to the axis of the array, it is known as broadside array.

If the maximum radiation of an array directed along the axis of the array, it is known as End-fire array.]

→ For maxima to occur along the real angles, the argument of the cosine inverse function must be between -1 and $+1$.

$$\left|\frac{m\lambda}{d}\right| \leq 1 \text{ or } m \leq \frac{d}{\lambda} \quad — (17)$$

For null

The Array factor satisfy the condition

$$\cos\left(\frac{Kd}{2}\cos\theta\right) = 0 \quad \text{—(18)}$$

$$\therefore \frac{Kd}{2}\cos\theta = \pm(2n-1)\frac{\pi}{2} \quad , n = 1, 2, 3 \ldots$$

$$\frac{2\cancel{\pi}}{\lambda}\cdot\frac{d}{\cancel{2}}\cos\theta = \pm(2n-1)\frac{\cancel{\pi}}{\cancel{2}}$$

$$\Rightarrow \cos\theta = \pm\frac{(2n-1)\lambda}{2d}$$

$$\Rightarrow \theta_n = \cos^{-1}\left[\pm\frac{(2n-1)\lambda}{2d}\right] \quad \text{—(19)}$$

θ_n = Directions of nulls.

For nulls to occur along the real angles, we must have

$$-1 \leq \frac{(2n-1)\lambda}{2d} \leq 1$$

or

$$\left|\frac{(2n-1)\lambda}{2d}\right| \leq 1$$

$$\Rightarrow (2n-1)\lambda \leq 2d$$

$$\Rightarrow (2n-1) \leq \frac{2d}{\lambda}$$

$$\Rightarrow 2n \leq \frac{2d}{\lambda} + 1$$

$$\Rightarrow \boxed{n \leq \frac{d}{\lambda} + \frac{1}{2}} \quad \text{--- (20)}$$

e.g. For $d = \frac{\lambda}{2}$ and $n = 1$ } Eqn (19) becomes,

$$\theta_n = \cos^{-1}\left[\pm \frac{\lambda}{2 \times \frac{\lambda}{2}}\right]$$

$$\theta_n = \cos^{-1}(\pm 1)$$

$$\theta_n = 0° \text{ or } 180°.$$

i.e why in fig 5.4, at $\theta = 0°$ and $180°$ two nulls appear on the pattern for $d = \frac{\lambda}{2}$.

Excitation with non-zero Phase shift :- $(\alpha \neq 0)$

Case I :- Consider two infinitesimal dipoles carrying current $I_1 = I_0 e^{-j\alpha/2}$ and $I_2 = I_0 e^{+j\alpha/2}$ (current in dipole 2 is leading to current in dipole 1) from eqn (13), AF (Array factor) is given as

$$AF = 2I_0 \cos\left(\frac{kd}{2}\cos\theta + \frac{\alpha}{2}\right) \quad \text{--- (21)}$$

If $\alpha = kd$, then

$$AF = 2I_0 \cos\left(\frac{kd}{2}\cos\theta + \frac{kd}{2}\right)$$

$$AF = 2I_0 \cos\left(\frac{kd}{2}(1 + \cos\theta)\right) \quad \text{--- (22)}$$

The array factor has a maximum when the argument of cosine function is equal to an integral multiple of π.

$$\therefore \quad \frac{Kd}{2}\left(1+\cos\theta_m\right) = \pm m\pi \,, \quad m=0,1,2,\ldots$$

$$\Rightarrow \quad \frac{Kd}{\cancel{2}}(1+\cos\theta_m) = \pm 2m\pi$$

$$\Rightarrow \quad 1+\cos\theta_m = \pm\frac{2m\pi}{Kd}$$

$$\Rightarrow \quad \cos\theta_m = \pm\frac{2m\pi}{Kd} - 1$$

$$\Rightarrow \quad \cos\theta_m = \frac{\pm 2m\pi - Kd}{Kd}$$

$$\Rightarrow \quad \theta_m = \cos^{-1}\left(\frac{\pm 2m\pi - Kd}{Kd}\right) \;; m=0,1,2\ldots \quad (23)$$

If $m=0$, for any value of d (say $d=\frac{\lambda}{2}$)

$$\theta_m = \cos^{-1}(-1) = \pi$$

If $m=1$ & $d=\frac{\lambda}{2}$

$$\theta_m = \cos^{-1}\left(\frac{2\pi - \frac{2\pi}{\lambda}\cdot\frac{\lambda}{2}}{\frac{2\pi}{\lambda}\cdot\frac{\lambda}{2}}\right)$$

$$= \cos^{-1}\left(\frac{2\pi-\pi}{\pi}\right)$$

$$= \cos^{-1}(1)$$

$$\boxed{\theta_m = 0^\circ}$$

∴ For Spacing of $\frac{\lambda}{2}$, between two dipoles the maxima occures at $\theta = 0°$ & $\theta = \pi$.

Case-II Current in dipole 1 is leading to current in dipole 2.

i.e $I_1 = I_0 \cdot e^{+j\alpha/2}$, $I_2 = I_0\, e^{-j\alpha/2}$

Eqn (23) becomes

$$\theta_m = \cos^{-1}\left(\frac{\pm 2m\pi + Kd}{Kd}\right), \quad m = 0, 1, 2, 3 \ldots$$

for $m = 0$ $\quad \theta_m = \cos^{-1}(1) = 0°$, for any d.

for $m = 1$, $d = \frac{\lambda}{2}$ $\quad \theta_m = \cos^{-1}\left(\frac{\pm 2\pi + \frac{2\pi}{\lambda}\cdot\frac{\lambda}{2}}{\frac{2\pi}{\lambda}\cdot\frac{\lambda}{2}}\right)$

$$= \cos^{-1}\left(\frac{-2\pi + \pi}{\pi}\right)$$

$$= \cos^{-1}\left(\frac{-\pi}{\pi}\right)$$

$$= \cos^{-1}(-1)$$

$$\boxed{\theta_m = 180°}$$

Thus, for a two-element array with phase $\alpha = \pm Kd$ in the excitation, the array factor always has maximum along $\theta = 0°$ or $180°$. Which are along the axis of array. Therefore, this array is known as an end fire array.

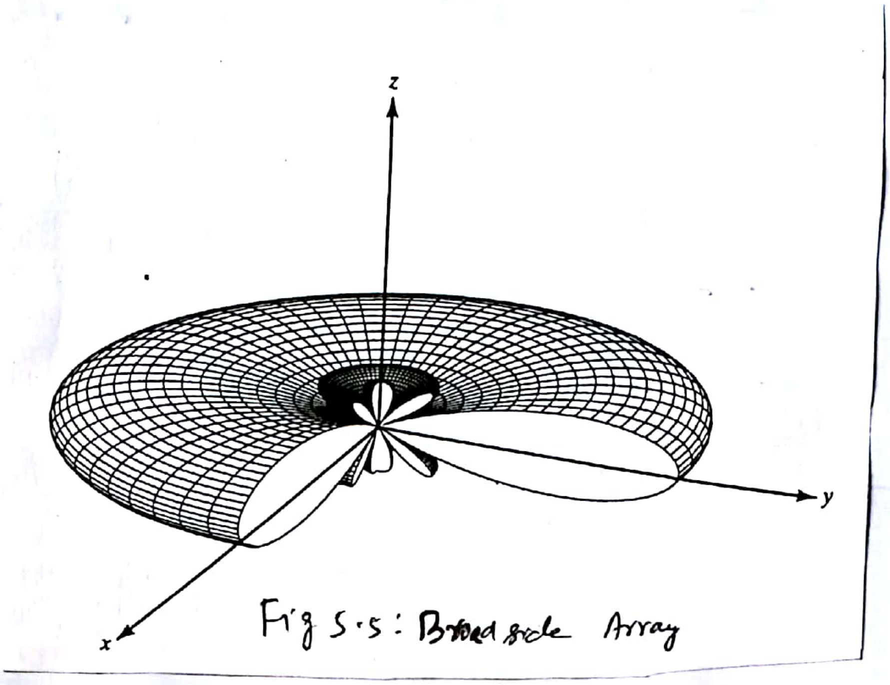


Fig 5.5: Broad side Array

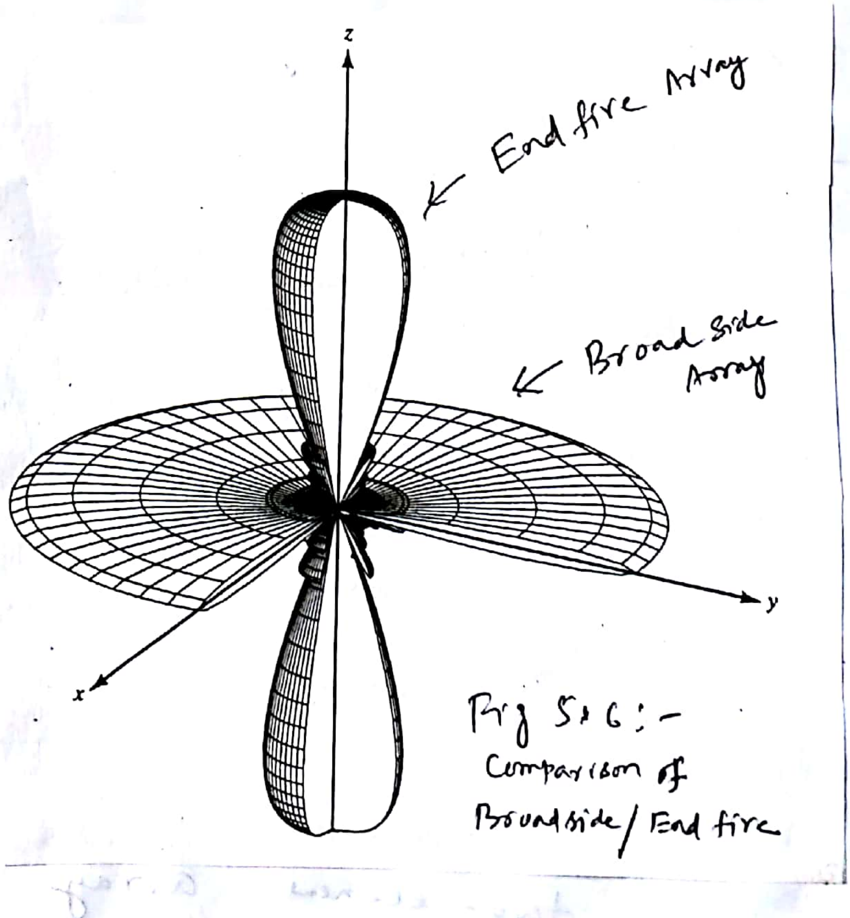
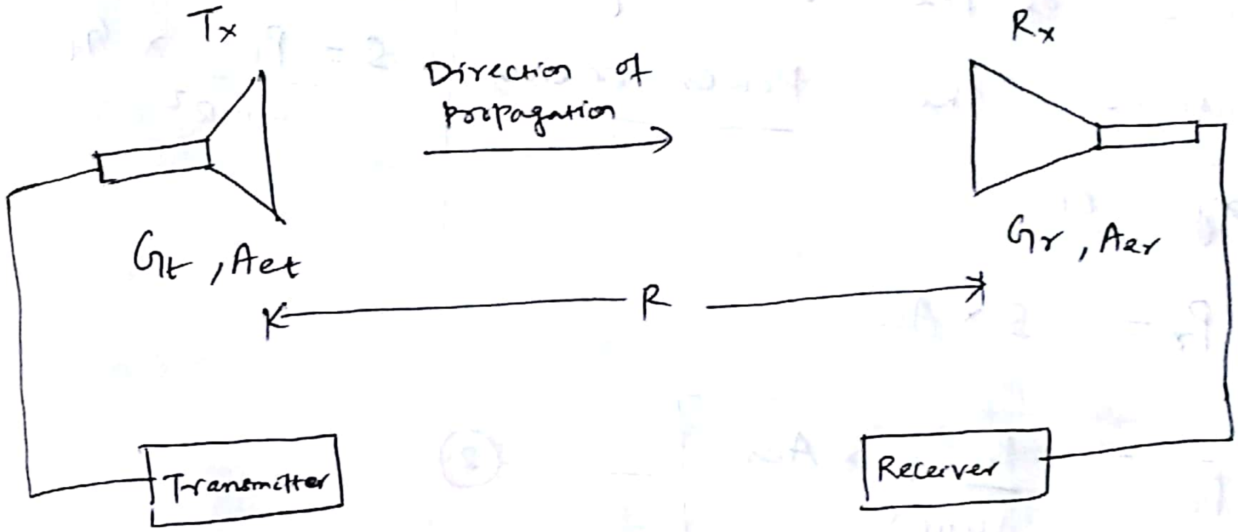


Fig 5.6:- Comparison of Broadside / End fire

Friis Transmission Formula

A wireless system consists of a transmitter connected to an antenna radiating electromagnetic energy into free space and at the other end of the system, another antenna picks up the e.m energy, and delivers it to the receiving system.



→ The received power depends on the transmitted power, gains of the transmit and receive antennas, wavelength of the electromagnetic wave in free space, and the distance between the transmitter and the receive antennas. This relationship is known as Friis transmission formula.

Consider an antenna having gain, G_t, transmitting power P_t into free space. A receive antenna having a gain, G_r, kept at a distance, R, is used to receive the e.m waves. Let 'λ' be the free space wavelength. The power density at a distance R from the transmitting antenna along the main beam direction is given by

$$S = \frac{P_t G_t}{4\pi R^2} \quad \text{--- (1)}$$

∵ For an isotropic source, power density

$$S = \frac{P_t}{4\pi R^2}$$

For an directive antenna gain is multiplied, i.e.,

$$S = \frac{P_t}{4\pi R^2} \times G_t$$

If Aer is the effective area of the receiving antenna, the power received by it

$$P_r = S \times A_{er}$$

$$\boxed{P_r = \frac{P_t G_t}{4\pi R^2} \times A_{er}} \quad \text{--- (2)}$$

If effective aperture is replaced by the following relation,

$$A_{er} = G_r \frac{\lambda^2}{4\pi},$$

∵ $G_r = \frac{4\pi A_{er}}{\lambda^2}$

$\Rightarrow A_{er} = \frac{G_r \cdot \lambda^2}{4\pi}$

We have

$$P_r = \frac{P_t G_t}{4\pi R^2} \times \frac{G_r \lambda^2}{4\pi}$$

$$\Rightarrow \boxed{P_r = P_t G_t G_r \left(\frac{\lambda}{4\pi R}\right)^2 \text{ Watt}} \quad \text{--- (3)}$$

Eq^n ② & ③ are known as Friis transmission formula which can be expressed in dB as,

$$\boxed{P_{r\,dB} = P_{t\,dB} + G_{t\,dB} + G_{r\,dB} + 20\log_{10}\left(\frac{\lambda}{4\pi R}\right) dB}$$

$$\left(\because 10\log\left(\frac{\lambda}{4\pi R}\right)^2 = 20\log\left(\frac{\lambda}{4\pi R}\right) \right)$$

Ex:- 1) In a microwave communication link, two identical antenna operating at 10 GHz are used with power gain of 40 dB. If the transmitter power is 1 watt, find the received power, if the range of the link is 30 km.

Ans: Given

$G_t = G_r = 40\,dB$

$\Rightarrow 10\log_{10} G_t = 40$

$\Rightarrow \log_{10} G_t = 4$

$\Rightarrow G_t = 10^4 = G_r$ —— ①

$f = 10\,GHz$, $\lambda = \frac{c}{f} = \frac{3\times10^8}{10\times10^9} = \frac{3}{100} = .03\,m.$

$P_t = 1\,watt$, $R = 30\,km.$

From Friis transmission formula

$$P_r = P_t G_t G_r \left(\frac{\lambda}{4\pi R}\right)^2$$

$$= 1 \times 10^4 \times 10^4 \left(\frac{0.03}{4\pi \times 30 \times 10^3}\right)^2$$

$$= \frac{10^4 \times 10^4 \times 0.03 \times 0.03}{16\pi^2 \times 30 \times 30 \times 10^6}$$

$$P_r = 0.633 \times 10^{-6}$$

$$\Rightarrow \boxed{P_r = 0.633 \ \mu Watt}$$ (Ans)

2) The radial component of the radiated power density of an antenna is given by

$$W_{rad} = \hat{a}_r W_r = \hat{a}_r A_0 \frac{\sin\theta}{r^2} \quad \left(\frac{Watt}{m^2}\right),$$

where A_0 is the peak value of power density, θ is usual spherical co-ordinate, and $\hat{a}_r$ is the radial unit vector. Determine the total radiated power.

Ans:

$$P_{rad} = \oiint_{\Omega} U \, d\Omega$$

$$P_{rad} = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} U \sin\theta \, d\theta \, d\phi$$

But

$$U = r^2 W_{rad} = W_{rad} \cdot r^2$$

$$\therefore P_{rad} = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \frac{A_0 \sin\theta}{\cancel{r^2}} \times \cancel{r^2} \cdot \sin\theta \, d\theta \, d\phi$$

$$= A_0 \left[\int_{\theta=0}^{\pi} \sin^2\theta \, d\theta \right] \times 2\pi \qquad \Bigg| \quad \int_0^{2\pi} d\phi = 2\pi$$

$$= 2\pi A_0 \left[\int_0^{\pi} \frac{1 - \cos 2\theta}{2} d\theta \right]$$

$$= \frac{2\pi A_0}{2} \left[\theta - \frac{\sin 2\theta}{2} \right]_0^{\pi}$$

$$= \frac{2\pi A_0}{2} \left[(\pi - 0) - (0 - 0) \right]$$

$$= \frac{\cancel{2}\pi A_0}{\cancel{2}} \times \pi$$

$$\boxed{P_{rad} = \pi^2 A_0 \text{ Watt}}$$

3) Calculate the approximate gain and beamwidth of a paraboloidal reflector antenna operating freq 4 GHz, Diameter 20 meters and illumination efficiency 55%.

Ans:-

$$G = \frac{4\pi A_e}{\lambda^2}$$

Here A_e = (illumination efficiency) × (Actual Area)

$$\boxed{A_e = K \times A}$$

$$\text{Actual area} = \pi r^2$$

$$= \pi\left(\frac{D}{2}\right)^2$$

$$= \frac{\pi D^2}{4}$$

$$= \frac{\pi \times 20^2}{4}$$

$$= \frac{\pi \times \cancel{20}^{5} \times 20}{\cancel{4}}$$

$$A = 100\pi$$

$$A_e = K \times A = 0.55 \times 100\pi = 55\pi$$

$$G = \frac{4\pi A_e}{\lambda^2} = \frac{4\pi \times 55\pi}{\left(\frac{c}{f}\right)^2} = \frac{4\pi \times 55\pi}{\left(\frac{3\times 1\cancel{0^8}}{4\times 1\cancel{0^9}_{10}}\right)^2}$$

$$G = \frac{4\pi \times 55\pi \times 4\times 4\times 10\times 10}{3\times 3}$$

$$G = 386011.1944.$$

$$G_{dB} = 10\log(G) = 55.86\ dB.$$

$$\lambda = \frac{c}{f} = \frac{3\times 1\cancel{0^8}}{4\times 1\cancel{0^9}_{10}} = \frac{3}{40}$$

$$\underline{HPBW} = \frac{70\lambda}{D} = \frac{7\cancel{0} \times \frac{3}{4\cancel{0}}}{20} = \frac{21}{80} = 0.2625^\circ$$

$$FNBW/BWFN \simeq 2 \times HPBW = 2 \times 0.2625 = 0.525^\circ$$

∴ Half Power Beam width is $\underline{0.2625\ \text{deg}}$ & First Null Beam width is 0.525 degree.

Front to Back Ratio (FBR)

FBR is defined as the ratio of power radiated in desired direction to the power radiated in the opposite direction i.e

$$\boxed{FBR = \frac{\text{Power radiated in desired direction}}{\text{Power radiated in opposite direction}}}$$

Obviously, higher the FBR, the better it is. The FBR changes if frequency of operation of antenna system shifts. Its value tends to decrease if spacing between elements of antennas increases.

The FBR depends on the tuning conditions or electrical length of parasitic elements. The higher FBR is achieved by diverting the gain of opposite direction (i.e backward response) to the forward or desired direction by adjusting or tuning the length of parasitic elements.

Hence, the higher value of FBR is achieved at the cost of sacrificing gain from opposite direction. In practice, for receiving purposes adjustments are made to get maximum FBR rather than maximum gain.

Uniform Array :-

Consider an array of N point sources placed along Z-axis with first element at the origin. Let the distance between any two consecutive elements be equal to d. The excitation currents of all the elements have equal magnitude and a progressive phase shift of 'α' i.e the current in the nth element lead the current in the $(n-1)^{th}$ element by α.

If the current in the first element, $I_1 = I_0$, the current in the nth element can be written as $I_n = I_0 e^{j(n-1)\alpha}$. Such an array is called uniform array.

The array factor of an N-element linear array along the Z-axis is given by

$$AF = \sum_{n=1}^{N} I_n e^{jkz_n' \cos\theta} \quad — (24)$$

∵ Refer eqn (11) in linear array

Where I_n is the current on the nth element and $z_n' = (n-1)d$ is the location of the nth element. [Fig 5.11]

Substituting the expressions for I_n and z_n' into the array factor, we have

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$$AF = \sum_{n=1}^{N} I_0\, e^{j(n-1)\alpha} \cdot e^{jK(n-1)d\cos\theta} \quad \text{--- (25)}$$

Since 'I_0' is constant [Uniform Array]. It can be taken out of summation & can be multiplied with Element pattern.

(∵ E_θ = Element pattern × AF).

$$\therefore AF = \sum_{n=1}^{N} e^{j(n-1)\alpha} \cdot e^{jK(n-1)d\cos\theta}$$

$$AF = \sum_{n=1}^{N} e^{j(n-1)[\alpha + Kd\cos\theta]}$$

$$\therefore AF = \sum_{n=1}^{N} e^{j(n-1)\psi} \quad \text{--- (26)}$$

where $\psi = \alpha + Kd\cos\theta$.

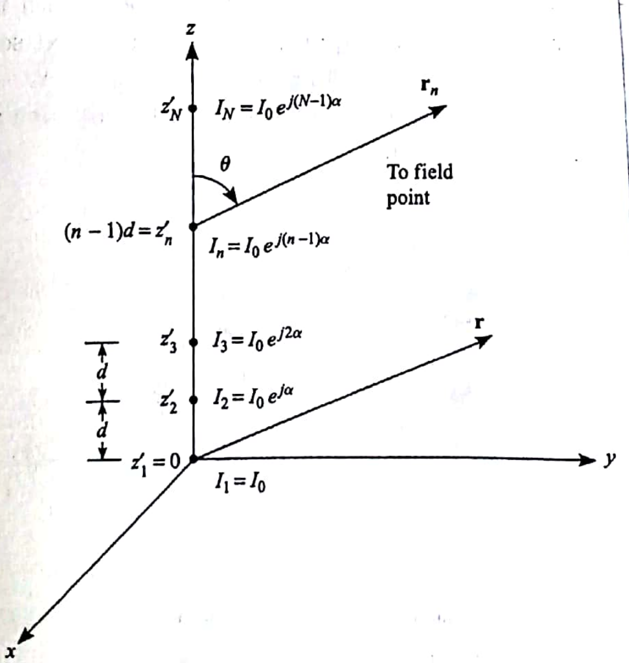

Fig. 5.11 Geometry of a uniform array of point sources radiating into free space

Thus the array factor is summation of $\underline{N}$ phasors which form a geometric series.

$$AF = 1 + e^{j\psi} + e^{j2\psi} + e^{j3\psi} + \cdots\cdots + e^{j(N-1)\psi} \quad \text{—— (27)}$$

The magnitude of the array factor depends on the value of ψ. For a given spacing (d) and progressive phase shift (α), the magnitude of the array factor changes with angle θ.

Multiplying $e^{j\psi}$ on both the sides of eqn (27), we have

$$e^{j\psi} . AF = e^{j\psi} + e^{j2\psi} + e^{j3\psi} \cdots\cdots + e^{jN\psi} \quad \text{—— (28)}$$

Subtracting eqn (27) from eqn (28)

$$e^{j\psi} AF - AF = -1 + e^{jN\psi}$$

$$\Rightarrow AF \left(e^{j\psi} - 1\right) = -1 + e^{jN\psi}$$

$$\Rightarrow AF = \frac{e^{jN\psi} - 1}{e^{j\psi} - 1} \quad \text{—— (29)}$$

$$\Rightarrow AF = \frac{e^{\frac{jN\psi}{2}} \left[e^{\frac{jN\psi}{2}} - \frac{1}{e^{\frac{jN\psi}{2}}} \right]}{e^{\frac{j\psi}{2}} \left[e^{\frac{j\psi}{2}} - \frac{1}{e^{\frac{j\psi}{2}}} \right]}$$

$$\Rightarrow AF = \frac{e^{\frac{j(N-1)\psi}{2}}\left[e^{\frac{jN\psi}{2}} - e^{\frac{-jN\psi}{2}}\right]}{\left[e^{\frac{j\psi}{2}} - e^{\frac{-j\psi}{2}}\right]}$$

$$\Rightarrow AF = \frac{e^{\frac{j(N-1)\psi}{2}} \times \cancel{2} \sin\frac{N\psi}{2}}{\cancel{2} \sin\frac{\psi}{2}}$$

$\because \dfrac{e^{jx} - e^{-jx}}{2} = \sin x$

$$\Rightarrow AF = \frac{e^{\frac{j(N-1)\psi}{2}} \sin\left(\frac{N\psi}{2}\right)}{\sin\left(\frac{\psi}{2}\right)} \quad \text{--- (30)}$$

The magnitude of array factor is given by

$$|AF| = \left|\frac{\sin\left(\frac{N\psi}{2}\right)}{\sin\left(\frac{\psi}{2}\right)}\right| \quad \text{--- (31)}$$

$\because \left|e^{jx}\right| = 1$

For $\psi = 0$, AF has $\frac{0}{0}$ form, Applying L' Hospital Rule,

$$|AF| = \lim_{\psi \to 0} \frac{\cos\left(\frac{N\psi}{2}\right) \cdot \frac{N}{2}}{\cos\left(\frac{\psi}{2}\right) \cdot \frac{1}{2}}$$

$$= \frac{1 \cdot \frac{N}{2}}{1 \cdot \frac{1}{2}}$$

$$|AF| = N \quad \text{--- (32)}$$

Therefore, the normalized array factor is

$$|AF_n| = \frac{\left|\dfrac{\sin \frac{N\psi}{2}}{\sin \frac{\psi}{2}}\right|}{|N|} = \left|\frac{\sin\left(\frac{N\psi}{2}\right)}{N \sin \frac{\psi}{2}}\right| \quad \text{--- (33)}$$

The array factor has a principal maximum if both numerator and denominator simultaneously go to zero, which occurs under the following condition

$$\left.\frac{\psi}{2}\right|_{\psi=\psi_m} = \pm m\pi \quad , m = 0, 1, 2, \text{-----}$$

$$\Rightarrow \psi_m = \pm 2m\pi$$

∴ The principal maxima of the array factor occurs for $\psi_m = \pm 2m\pi$, for $m = 0, 1, 2 \text{---}$ — (34)

The array factor has periodic maxima at interval of 2π [Fig 5.13]. The lobe containing principal maximum corresponding to $m=0$ is the main lobe and all other lobes containing principal maxima are called grating lobes.

Note: Grating lobes are undesired, because the power is wasted in the undesired direction.

→ Between the two principal maxima, the array factor can have several nulls. The nulls of the array factor occur if the numerator alone goes to zero. i.e

$$\left.\frac{N\psi}{2}\right|_{\psi=\psi_Z} = \pm p\pi, \quad p = 1, 2, 3, \ldots \text{ and } p \neq 0, N, 2N, 3N, \ldots \quad (35)$$

Because At $p = 0, N, 2N, 3N$ the array factor has maxima as both numerator & denominator goes to zero, and hence are excluded. Therefore, the zeros are given by

$$\psi_Z = \left(\pm\frac{2p\pi}{N}\right), \quad p = 1, 2, 3 \ldots \text{ and } p \neq 0, N, 2N, \ldots \quad (36)$$

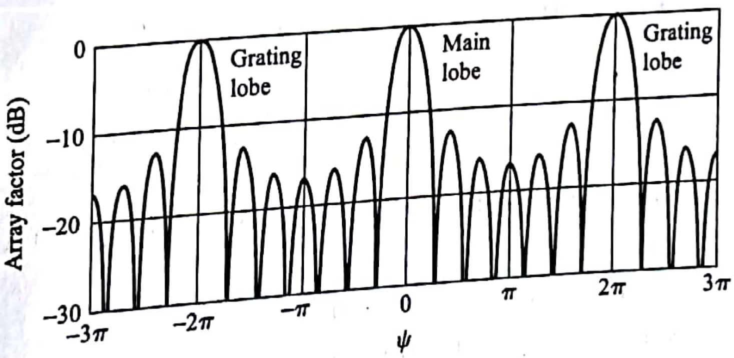

Fig. 5.13 Array factor of a 7-element array

Broadside Array:-

In many applications it is desirable to have maximum radiation of an array directed normal to the axis of the array $[\theta_0 = 90^\circ]$. To optimize the design, the maxima of the single element and of the array factor should both be directed towards $\theta_0 = 90^\circ$.

The requirements of the single elements can be accomplished by the judicious choice of radiators, and those of the array factor by proper separation and excitation of the individual radiators.

Referring eqn (33), the first maximum of array factor occurs when

$$\psi = 0$$

$$\Rightarrow kd\cos\theta + \alpha = 0$$

Since it is desired to have the first maximum directed toward $\theta_0 = 90^\circ$, then

$$kd\cos 90^\circ + \alpha = 0$$

$$\Rightarrow 0 + \alpha = 0$$

$$\Rightarrow \boxed{\alpha = 0} \quad \text{—— (37)}$$

Thus to have maximum of the array factor of a uniform linear array directed broadside

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to the axis of array, it is necessary that all the elements have the same phase excitation (in addition to same amplitude excitation). The separation between the elements can be any value. [Refer fig: 5.5] - As discussed earlier
$\alpha = 0$ & $d = \frac{\lambda}{4}$, $N = 10$

To ensure that there are no principal maxima in other directions, which are referred to as grating lobes, the separation between the elements should not be equal to multiples of a wavelengths ($d \neq n\lambda$, $n = 1, 2, 3$ ---) when $\alpha = 0$. If $d = n\lambda$, $n = 1, 2,$ --- and $\alpha = 0$, then

$$\psi = kd\cos\theta + \alpha \Big|_{\substack{d = n\lambda \\ \alpha = 0 \\ n = 1, 2, 3, \ldots}}$$

$$= \frac{2\pi}{\cancel{\lambda}} \cdot n\cancel{\lambda} \cos\theta + 0$$

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$$\psi = 2\pi n \cos\theta$$

At $\theta = 0$ or 180° $\psi = \text{Max}^m$.

Thus for a uniform array with $\alpha = 0$ and $d = n\lambda$ in addition to having the maxima of array factor directed broadside ($\theta_0 = 90^\circ$) to the axis of the array, there are additional maxima directed along the axis ($\theta_0 = 0^\circ, 180^\circ$) of the array (end fire radiation).
[Refer fig:- 5.6 - As discussed earlier]
Here, $\alpha = 0$, $d = \lambda$, $N = 10$

One of the objective in many designs is to avoid multiple maxima, in addition to main maximum, which are referred to as grating lobes. Often, it may be required to select the largest spacing between elements but with no grating lobes. To avoid any grating lobes, the largest spacing between the elements should be less than one wavelength.

i.e $\boxed{d_{max} < \lambda}$. —— (39)

Ordinary end-fire Array

Instead of having the maximum radiation broadside to the axis of the array, it may be desirable to direct it along the axis of the array (end-fire). As a matter of fact, it may be necessary that it radiates towards only one direction (either $\theta_0 = 0°$ or $\theta_0 = 180°$).

To direct first maximum toward $\theta_0 = 0°$,

$$\psi = Kd\cos\theta + \alpha = 0$$

$$\Rightarrow Kd\cos 0° + \alpha = 0$$

$$\Rightarrow Kd + \alpha = 0$$

$$\Rightarrow \boxed{\alpha = -Kd} \quad \text{—— (40)}$$

If the first maximum is desired toward

$\theta_0 = 180°$, then

$\psi = Kd\cos\theta + \alpha = 0$

$\Rightarrow Kd\cos 180° + \alpha = 0$

$\Rightarrow -Kd + \alpha = 0$

$\Rightarrow \boxed{\alpha = Kd}$ —— (41)

Thus the end fire radiation is accomplished when $\alpha = -Kd$ (for $\theta_0 = 0°$) or $\alpha = Kd$ (for $\theta_0 = 180°$).

(i) If the element separation is $d = \frac{\lambda}{2}$ and $\alpha = 0$

$$\psi = Kd\cos\theta + \alpha = \frac{2\pi}{\lambda} \cdot \frac{\lambda}{2} \cos\theta + 0$$

$$= \pi\cos\theta$$

When $\theta = 0°$ or $180°$, $\psi = \pi$ or $-\pi$ and

Array factor eqn (33) is maxm.

$\therefore$ for $d = \frac{\lambda}{2}$, end fire radiation occures ($\theta = 0°$, $180°$)

(ii) If $d = n\lambda$, and $\alpha = 0$.

$$\psi = Kd\cos\theta = \frac{2\pi}{\lambda} \cdot n\lambda \cos\theta = 2\pi n\cos\theta$$

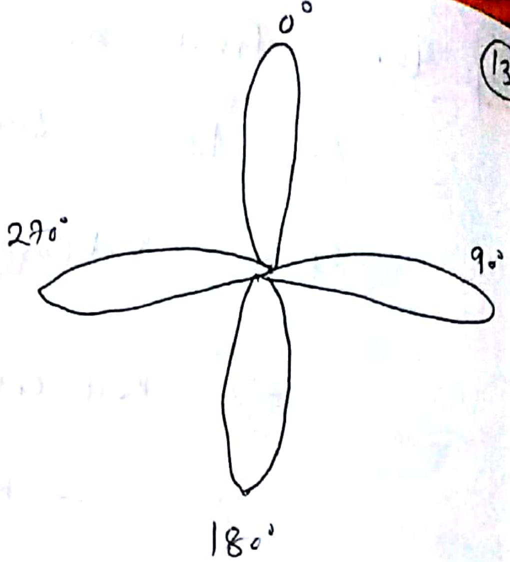
For $n=1$ i.e. $d=\lambda$

If $\theta = 0^\circ$, $\psi = 2\pi$

$\theta = 90^\circ$, $\psi = 0$

$\theta = 180^\circ$, $\psi = -2\pi$

$\theta = 270^\circ$, $\psi = 0$



(135)

Since $\psi = 0$, 2π or -2π

Array factor eqn (33) is maxm.

Thus for $d = n\lambda$, $n = 1, 2, 3$ -- there exist four maxima, two in the broadside directions and two along axis of the array.

To ~~avoid one~~ have only one end-fire maximum and to avoid any grating lobes, the maxm spacing between the elements to be less than $\frac{\lambda}{2}$.

i.e. $$\boxed{d_{max} < \frac{\lambda}{2}} \quad \text{--- (42)}$$

Polynomial Representation :-

The array factor of a uniform array of N-elements kept along z-axis with an inter-element spacing of 'd' is given by eqn (26)

$$AF = \sum_{n=1}^{N} e^{j(n-1)\psi} \quad \text{--- (43)}$$

where $\psi = kd\cos\theta + \alpha$ and 'α'

is the progressive phase shift.

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Let $$Z = e^{j\psi} \quad \text{——— (44)}$$

so eq^n (43) becomes,

$$AF = \sum_{n=1}^{N} Z^{n-1}$$

$$AF = 1 + Z^2 + Z^3 \cdots\cdots + Z^{N-1} \quad \text{— (45)}$$

$\because$
$$AF = \sum_{n=1}^{N} e^{j(n-1)\psi}$$
$$= \sum_{n=1}^{N} \left(e^{j\psi}\right)^{n-1}$$
$$= \sum_{n=1}^{N} Z^{n-1}$$

This is a polynomial of degree (N-1), and therefore, has (N-1) roots, which correspond to the nulls of the array factor.

The nulls of the array factor can easily be computed by writing the array factor on the form given by eq^n (29)

$$AF = \frac{e^{jN\psi} - 1}{e^{j\psi} - 1} = \frac{Z^N - 1}{Z - 1} \quad \text{——— (46)}$$

Equating the numerator of array factor to zero

$$Z^N - 1 = 0 \quad \text{——— (47)}$$

and solving for Z, we get the roots as

$$Z = e^{-j\frac{2\pi}{N}}, e^{-j2.\frac{2\pi}{N}}, e^{-j3.\frac{2\pi}{N}} \cdots\cdots\cdots e^{-j(N-1)\frac{2\pi}{N}}, 1 \quad \text{— (48)}$$

These are the N roots in Complex Plane.
Since the magnitudes of the roots are all equal to unity, all the roots lie on the unit circle and are equally spaced. [Fig 5.17]

That is, they divide the circle into N equal parts. The last root in the above list [eqn 48], i.e $Z=1$, Corresponds to the maximum of the array factor $\left(\lim_{Z\to 1} AF = N\right)$.

$$\left[\because \quad \lim_{Z\to 1} \frac{Z^N-1}{Z-1} \quad \left(\frac{0}{0}\right)\right.$$

$$= \lim_{Z\to 1} \frac{N \cdot Z^{N-1}}{1} \qquad \text{[L'Hospital Rule]}$$

$$\left. = N \right]$$

→ Therefore, except $Z=1$, all other Nth roots of unity Corresponds to the nulls of array factor.

[$AF = \dfrac{Z^N-1}{Z-1}$, ∵ At all other roots numerator is zero and denominator is non zero.]

Thus, the array factor can be written in the factored form as

$$AF = \frac{(Z-Z_1)(Z-Z_2)\cdots(Z-Z_{N-1})\cancel{(Z-1)}}{\cancel{(Z-1)}}$$

where $Z_1 = e^{-j\frac{2\pi}{N}}$ etc.

$Z_2 = e^{-j2\cdot\frac{2\pi}{N}}$ Refer eqn (48)

...

$$\Rightarrow AF = (Z - Z_1)(Z - Z_2) \text{ - - - - - } (Z - Z_{N-1}) \quad (49)$$

where $Z_1, Z_2, \dots$ are the roots given by eqn (48).

[The last root Z_N i.e. $Z=1$ corresponds to the peak of the pattern, as discussed in last page

$$\lim_{Z \to 1} \frac{Z^N - 1}{Z - 1} = N$$]

Thus, for any given Z (or direction ψ) the array factor is product of vectors $(Z-Z_1)(Z-Z_2)$, etc

Consider an equi-spaced, six element array with uniform excitation. The nulls of the array factor are at

$$Z = e^{-j\frac{2\pi}{6}}, \; e^{-j2\times\frac{2\pi}{6}}, \; e^{-j3\times\frac{2\pi}{6}}, \; e^{-j4\times\frac{2\pi}{6}}, \; e^{-j5\times\frac{2\pi}{6}}. \quad (50)$$

[and at $Z=1$, array factor is maximum.]

And these are plotted on the unit circle on Fig [5.17] as hollow circles.

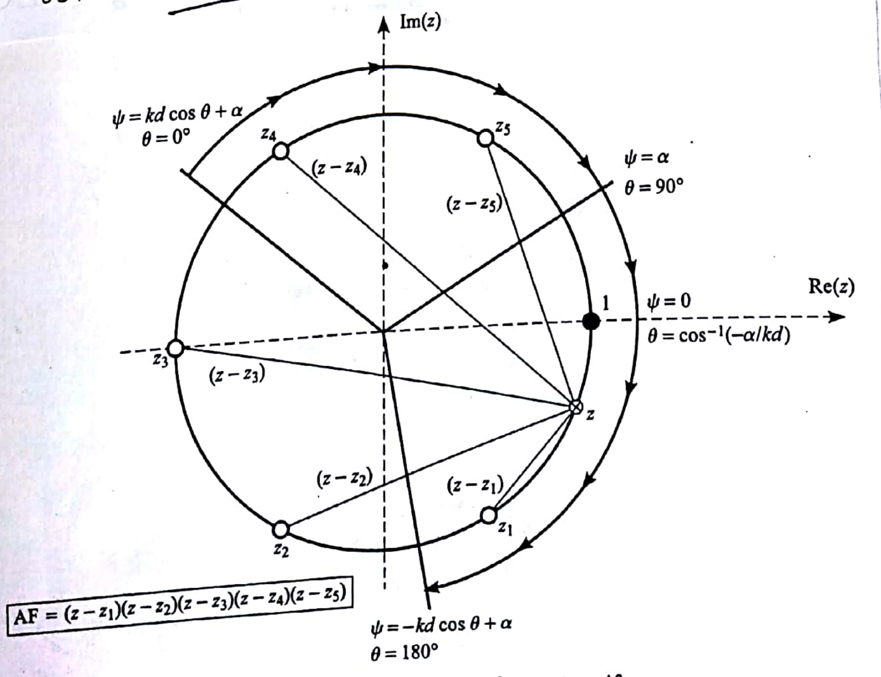


Fig. 5.17 Unit circle representation of a six-element uniform array

The maximum of the array factor, which occurs at $Z=1$, corresponds to $\psi=0$. This is

$$\left[\because \quad AF = \frac{\sin\left(\frac{N\psi}{2}\right)}{\sin\frac{\psi}{2}} \text{ and } \psi=0, \ AF \text{ is max}^m \quad \frac{0}{0} \text{ form}\right]$$

shown as a filled circle in Fig 5-17.

As angle θ varies from $0°$ through $90°$ to $180°$, $\psi = Kd\cos\theta + \alpha$ varies from $(Kd+\alpha)$

$$\left(\begin{array}{ll} \because \theta = 0°, & \psi = Kd + \alpha \\ \theta = 180°, & \psi = -Kd + \alpha \end{array}\right)$$

through $(-Kd+\alpha)$. This represents the visible region. As Z traverses a zero in the visible region, it produces a null in the array factor. The pattern maximum or main beam is at $Z=1$. The direction, θ_0, of the maximum, is obtained from

$$\psi = Kd\cos\theta + \alpha \Big|_{\theta=\theta_0} = 0 \quad \text{——— (51)}$$

$$\Rightarrow \quad Kd\cos\theta_0 + \alpha = 0$$

$$\Rightarrow \quad \cos\theta_0 = -\frac{\alpha}{Kd}$$

$$\Rightarrow \quad \boxed{\theta_0 = \cos^{-1}\left(-\frac{\alpha}{Kd}\right)} \quad \text{——— (52)}$$

Array With Non-Uniform Excitation:-

Some of the characteristics of radiation pattern of an array of uniformly excited isotropic sources can be controlled by changing the number of elements, inter-element spacing, and progressive phase shift.

While spacing (d) affect the extent of ψ and 'α' controls the starting and ending values of ψ keeping the extent constant. However, the expression for the array factor in terms of ψ does not change; only the visible region is decided by d and α.

There are applications where it is required to suppress the side lobes to a much lower level. This can be achieved by changing the excitation amplitudes. It can be shown that it is possible to change the level of the side lobes by proper choice of amplitudes of the array excitation coefficients.

Array factor

An array factor of an even number of isotropic elements 2M (where M is an integer) is positioned symmetrically along z-axis, as shown in fig 6.19(a). The separation between the elements is 'd', and M elements are placed on each side of the origin. (14)

Assume that the amplitude excitation is symmetrical about the origin, the array factor for a non-uniform amplitude broadside array can be written as

$$(AF)_{2M} = a_1 e^{+j\frac{kd\cos\theta}{2}} + a_2 e^{+j3.\frac{kd\cos\theta}{2}} + \dots$$

$$+ a_M e^{+j(2M-1)\frac{kd\cos\theta}{2}}$$

$$+ a_1 e^{-j\frac{kd\cos\theta}{2}} + a_2 e^{-j3\frac{kd\cos\theta}{2}} + \dots$$

$$+ a_M e^{-j(2M-1)\frac{kd\cos\theta}{2}} \quad \text{--- } 52(A)$$

$$(AF)_{2M} = 2\sum_{n=1}^{M} a_n \cos\left[\left(\frac{2n-1}{2}\right) kd\cos\theta\right] \quad \text{--- } (53)$$

Where

a_n's are the excitation coefficients of the array elements.

Eq^n (53), in normalized form reduces to

$\because$

$e^{j\theta} + e^{-j\theta} = 2\cos\theta$

Here, in general

$\theta = \left(\frac{2M-1}{2}\right) kd\cos\theta$

$$(AF)_{2M} = \sum_{n=1}^{M} a_n \cos\left[\left(\frac{2n-1}{2}\right) kd\cos\theta\right] \quad \text{--- } (54)$$

If total number of isotropic elements of the array is odd $\underline{2M+1}$ (where M is an integer), as shown in fig 6.19 (b), the array factor can be written as,

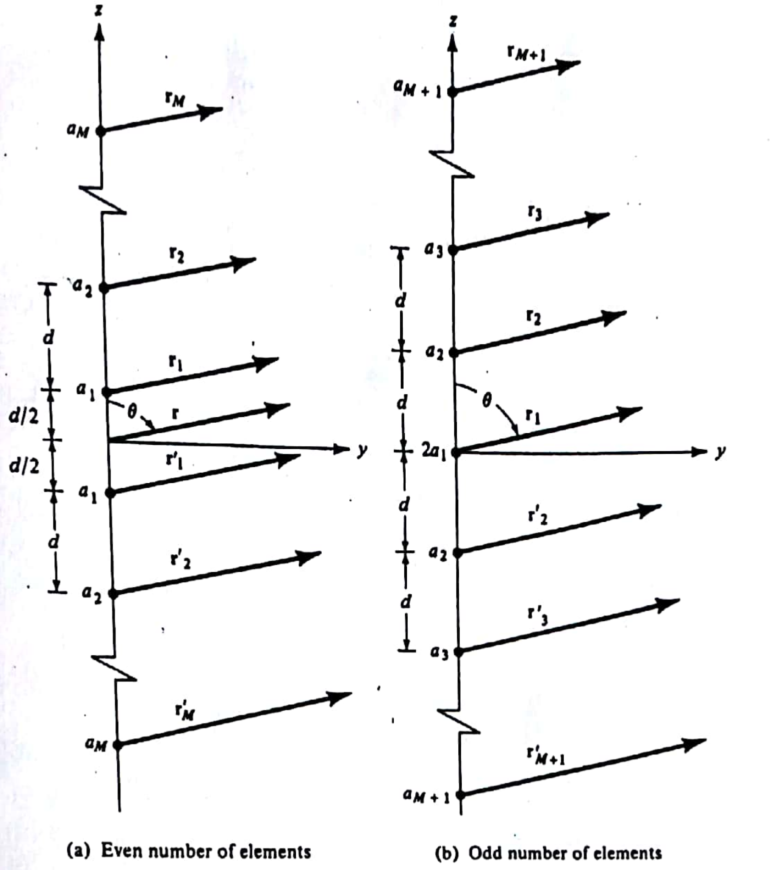


Figure 6.19 Nonuniform amplitude arrays of even and odd number of elements.

Here, the amplitude of excitation of the center element is $\underline{2a_1}$.

$$(AF)_{2M+1} = 2a_1 + a_2 e^{jkd\cos\theta} + a_3 e^{j2kd\cos\theta} \cdots + a_{M+1} e^{jMkd\cos\theta}$$
$$+ a_2 e^{-jkd\cos\theta} + a_3 e^{-j2kd\cos\theta} \cdots + a_{M+1} e^{-jMkd\cos\theta} \quad \text{— 54(A)}$$

$$(AF)_{2M+1} = 2\sum_{n=1}^{M+1} a_n \cos[(n-1)kd\cos\theta] \quad \text{— (55)}$$

Note:- for $\underline{M=0}$ $(AF)_1 = 2a_1$

$M=1$, $(AF)_3 = 2\sum_{n=1}^{2} a_n \cos[(n-1)kd\cos\theta]$

$$= 2[a_1 \cos 0 + a_2 \cos kd\cos\theta]$$

$$(AF)_3 = 2[a_1 + a_2 \cos kd\cos\theta]$$

$$= 2a_1 + 2a_2 \cos kd\cos\theta.$$

$$(AF)_3 = 2a_1 + a_2 e^{jkd\cos\theta} + a_2 e^{-jkd\cos\theta}$$

Verified

Eqⁿ (55) in normalized form reduces to

$$(AF)_{2M+1} = \sum_{n=1}^{M+1} a_n \cos\left[(n-1)kd\cos\theta\right] \quad \text{——} (56)$$

The amplitude excitation of the center element is $\underline{2a_1}$.

The eqⁿ (54) & (56) can be written in normalized form as

$$(AF)_{2M}\ (\text{even}) = \sum_{n=1}^{M} a_n \cos\left[(2n-1)u\right] \quad \text{——} (57)$$

$$(AF)_{2M+1}\ (\text{odd}) = \sum_{n=1}^{M+1} a_n \cos\left[2(n-1)u\right] \quad \text{——} (58)$$

where $u = \frac{\pi d}{\lambda}\cos\theta$ ——— (59)

[Note:-

$$(AF)_{2M} = \sum_{n=1}^{M} a_n \cos\left[\left(\frac{2n-1}{2}\right)kd\cos\theta\right]$$

eqⁿ (54)

$$= \sum_{n=1}^{M} a_n \cos\left[\left(\frac{2n-1}{\cancel{2}}\right)\times \underbrace{\frac{\cancel{2}\pi}{\lambda}\cdot d\cos\theta}_{u}\right]$$

$$(AF)_{2M} = \sum_{n=1}^{M} a_n \cos\left[(2n-1)u\right] \quad \Big|\ \because \frac{\pi d\cos\theta}{\lambda} = u$$

Similarly

$$(AF)_{2M+1} = \sum_{n=1}^{M+1} a_n \cos\left[(n-1)\times \underbrace{\frac{2\pi}{\lambda}\cdot d\cos\theta}_{u}\right]$$

eqⁿ (56)

$$(AF)_{2M+1} = \sum_{n=1}^{M+1} a_n \cos\left[2(n-1)u\right]$$
]

Binomial Array :-

In this array the relative amplitudes of excitation of elements are binomial coefficients.

A. Excitation Coefficients :-

To determine the excitation coefficients of binomial array, J. S. Stone suggested that the function $(1+x)^{m-1}$ be written in series, using binomial expansion as

$$(1+x)^{m-1} = 1 + (m-1).x + \frac{(m-1)(m-2)}{2!}x^2 + \frac{(m-1)(m-2)(m-3)}{3!}x^3 + \cdots \quad (60)$$

The +ve coefficients of the series expansion for different values of m are

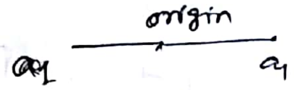
$$\begin{array}{lccccccccccc}
m=1, & & & & & & 1 & & & & & \\
m=2, & & & & & 1 & & 1 & & & & \\
m=3, & & & & 1 & & 2 & & 1 & & & \\
m=4, & & & 1 & & 3 & & 3 & & 1 & & \\
m=5, & & 1 & & 4 & & 6 & & 4 & & 1 & \\
m=6, & 1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array} \quad (61)$$

This above represents Pascal's triangle. If the values of m are represent the number of elements of array, then the coefficients of the expansion represent the relative amplitudes of the elements. Since the coefficient are determined from a binomial series expansion, the array is known as binomial array.

Refering eqn 52(A), 54(A), and (61), the amplitude coefficients for the following array are:

1) Two elements $[2M = 2]$

$2M = 2, \Rightarrow M = 1$



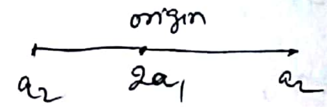
From 52(A),

$(AF)_{2M}$ has 2 components $a_1 e^{+jkd\cos\theta/2} + a_1 e^{-jkd\cos\theta/2}$

From pascal's triangle $a_1 = 1$

2) For Three elements $(2M+1 = 3)$

From eqn 54(A)



$(AF)_{2M+1}$ has 3 components

a_2, $2a_1$, and a_2. They are

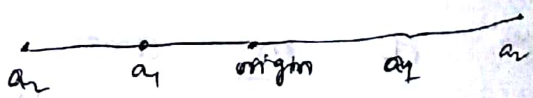
center element

$\langle 1, 2, 1 \rangle$

$a_2 = 1$, $2a_1 = 2 \Rightarrow a_1 = 1$, $\therefore \boxed{a_1 = 1, a_2 = 1}$

3) 4 elements

a_2, a_1, a_1, a_2



From Pascal's triangle

$$1, \quad 3, \quad 3, \quad 1$$

$$a_2, \quad a_1, \quad a_1, \quad a_2$$

5) Five elements

origin

$$a_3 \quad a_2 \quad 2a_1 \quad a_2 \quad a_3$$

i.e

$$1 \quad 4 \quad 6 \quad 4 \quad 1$$

Comparing with Pascal's triangle

$$a_3 = 1, \quad a_2 = 4, \quad a_1 = 3$$

B. Design Procedure

For binomial array the amplitude excitation coefficients are given by eqn (60) or (61).

For a design, the other figures of merit are directivity, HPBW, and side lobe level.

Binomial arrays don't exhibit any minor lobes provided the spacing between the elements is equal or less than one-half of a wavelength.

Approximate closed-form expressions for the HPBW and maxm directivity for $d = \frac{\lambda}{2}$ spacing in terms of number of elements or length of the array, are given by

$$HPBW\left(d = \frac{\lambda}{2}\right) \simeq \frac{1.06}{\sqrt{N-1}} = \frac{1.06}{\sqrt{2L/\lambda}} = \frac{0.75}{\sqrt{L/\lambda}} \quad \text{(62)}$$

$$D_0 = \frac{(2N-2)(2N-4) \cdots (2)}{(2N-3)(2N-5) \cdots (1)}$$

$$D_0 \simeq 1.77\sqrt{N} = 1.77\sqrt{1+\frac{2L}{\lambda}}$$

Where N = number of elements
L = Length of the array

Disadvantages and Advantages

Advantages :-

→ Binomial array have no minor lobes or very low level minor lobes.

→ They exhibit large beamwidth compared to Uniform a Dolph-Tschebyshelf's array,

Disadvantages :-

1) A major practical disadvantage of binomial array is the wide variations between the amplitudes of different elements of an array, especially for an array with a large number of elements.

e.g For a 10 element array, from Pascal's triangle amplitudes of excitation are

1 9 36 84 126 126 84 36 9 1

This leads to very low efficiencies for the feed network, and it makes the method not very desirable in practice.

* Dolph-Tschebyscheff Array [Not in Course]

The method was originally introduced by Dolph and investigated afterward by other. It is primarily a compromise between uniform and binomial arrays. Its excitation coefficients are related to Tschebyscheff polynomials. A D-T array with no side lobes reduces to the binomial design.

Note:- Of the three distributions (Uniform, binomial and Tschebyscheff), a uniform amplitude array yields the smallest half-power beamwidth (HPBW). It is followed, in order, by Dolph-Tschebyscheff and binomial arrays.

In contrast, binomial arrays usually possess the smallest side lobes followed, in order, by D-T and uniform arrays. As a matter of fact, binomial arrays with element spacing equal or less than $\lambda/2$ have no side lobes [Fig 5.26]. It is apparent that the designer must compromise between side lobe level and beamwidth. [Refer Fig 5.26]

Summary :- $(HPBW)_{Uniform} < (HPBW)_{D-T} < (HPBW)_{Binomial}$

$(\text{Side lobe})_{Binomial} < (\text{Side lobe})_{D-T} < (\text{Side lobe})_{Uniform}$

Note :- Uniform Array usually possess largest directivity.

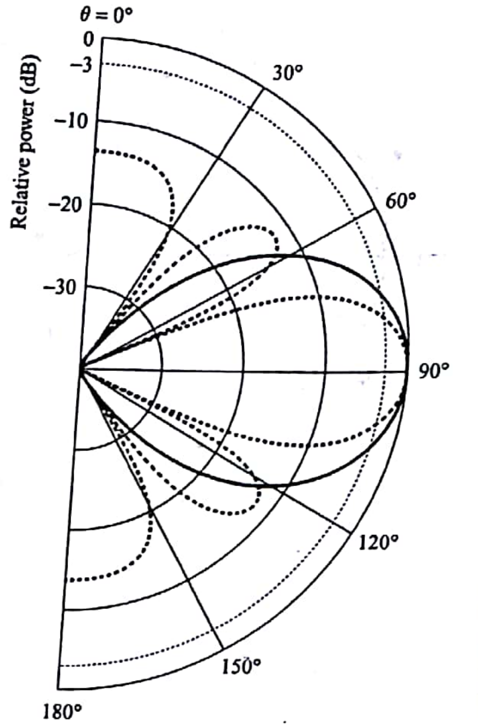


Fig. 5.26 Array factors of a 5-element broadside array with binomial (solid line) and uniform (dashed line) excitation

Ex-1) Show that the direction of maxima of the array factor of two-element array with excitation $I_1 = e^{jKd/2}$ and $I_2 = e^{-jKd/2}$ are given by

$$\theta_m = \cos^{-1}\left(\frac{\pm 2m\pi + Kd}{Kd}\right); \quad m = 0, 1, 2, \dots$$

and array factor has at least one maximum along $\theta = 0$

Ans:- The array factor of the two-element array [Refer eqn 12(A) in Two-element Array] with excitations I_1 & I_2 is given by

$$AF = I_1 e^{-j\frac{Kd}{2}\cos\theta} + I_2 e^{+j\frac{Kd}{2}\cos\theta}$$

$\alpha = 0$ No Phase shift

Substituting $I_1 = e^{j\frac{Kd}{2}}$ & $I_2 = e^{-j\frac{Kd}{2}}$, we get (150)

$$AF = e^{j\frac{Kd}{2}} \cdot e^{-j\frac{Kd}{2}\cos\theta} + e^{-j\frac{Kd}{2}} \cdot e^{+j\frac{Kd}{2}\cos\theta}$$

$$= e^{j\frac{Kd}{2}(1-\cos\theta)} + e^{-j(1-\cos\theta)\frac{Kd}{2}}$$

$$= 2\cos\left[\frac{Kd}{2}(1-\cos\theta)\right] \qquad \left|\; e^{jx}+e^{-jx} = 2\cos x \right.$$

The array factor reaches a maximum when the argument of the cosine function is equal to an integer multiple of π.

$$\therefore \left.\frac{Kd}{2}(1-\cos\theta)\right|_{\theta=\theta_m} = \pm m\pi$$

$$\Rightarrow \frac{Kd}{2}(1-\cos\theta_m) = \pm m\pi$$

$$\Rightarrow 1-\cos\theta_m = \frac{\pm 2m\pi}{Kd}$$

$$\Rightarrow \cos\theta_m = 1 - \frac{\pm 2m\pi}{Kd}$$

$$\Rightarrow \theta_m = \cos^{-1}\left(\frac{Kd \pm 2m\pi}{Kd}\right)$$

$$\Rightarrow \theta_m = \cos^{-1}\left(\frac{\pm 2m\pi + Kd}{Kd}\right), \quad m = 0, 1, 2, \ldots$$

For $m=0$, $\theta_0 = \cos^{-1}(1) = 0^\circ$

$\therefore$ Thus there is always one maximum along $\theta = 0^\circ$.

— o —