Chapter-8 - Rectangular Waveguide

A transmission line can be used to guide EM energy from one point (generator) to another (load). A waveguide is another means of achieving the same goal.

In general, a waveguide consists of a hollow metallic tube of a rectangular or circular shape, used to guide an electromagnetic wave. Waveguides are used primarily at frequencies in the microwave range. A waveguide can operate only above a certain frequency called the cutoff frequency and therefore acts as a high-pass filter.

Q) Differentiate Transmission Line & Wave guide

Transmission line	Wave guide
1) It is operated on ~~(.....)~~ range of frequency i.e it may operate from d.c (f=0) to a very high frequency.	1) It is used only in very high frequency, it can't transmit d.c.
2) It acts as one type of Low pass filter	2) It operates after certain cutoff frequency. So it act as a High pass filter.
3) It supports only TEM mode.	3) It doesn't support TEM mode but supports TE & TM mode
4) Not capable of handling large power	4) Capable of handling large power.

5) T.L become inefficient as a result of skin effect & dielectric losses.

6) In this metal conductor are used

5) The power loss in radiation. Dielectric loss is negligible, since guides are normally air filled. Small power loss as heat in the walls of guides, but loss is very small.

6) Metal hollow tubes are used to avoid loss.

Skin effect :- [In transmission line]

As the Em (H) wave travels in a conducting medium, because of ohmic losses present, its amplitude is attenuated by a factor $e^{-\alpha z}$. [α = attenuation constant]

The distance 'δ', through which the wave amplitude decrease to a factor e^{-1} (about 37% of the original value) is called skin depth or penetration depth.

$$\boxed{\delta = \frac{1}{\alpha} = \frac{1}{\sqrt{\pi f \mu_0 \sigma}}}$$

$\tan\delta$ is called loss tangent.

→ The skin depth is a measure of depth to which Em wave can penetrate the medium.

Types of wave guides.

Wave guide can be of different types

(i) Rectangular wave guide
(ii) Cylindrical / circular waveguide.
(iii) Elliptical wave guide
(iv) Parallel Plate Waveguide

Rectangular waveguide

Consider a rectangular waveguide containing lossless dielectric material and having the walls perfectly conducting ($\sigma_c = \infty$)

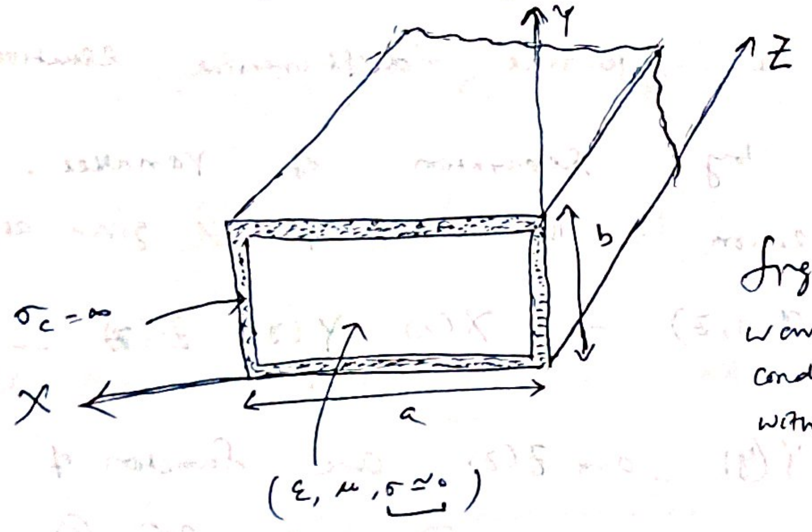

fig : 22 :- A rectangular waveguide with perfectly conducting walls filled with lossless material.

The Maxwell's equations for lossless dielectric medium are

$$\nabla^2 E_s + K^2 E_s = 0 \quad \text{--- (1)}$$

$$\nabla^2 H_s + K^2 H_s = 0 \quad \text{--- (2)}$$

where $K = \omega\sqrt{\mu\epsilon}$

K = wave number.

ω = Angular freq.

μ = Permeability of the medium

ϵ = Permittivity of the medium

Derivation Not required

$\because \nabla \times E = -\frac{\partial B}{\partial t}$ --- (i)

$\nabla \times H = \frac{\partial D}{\partial t}$ --- (ii)

$\nabla \times (\nabla \times \vec{E}) = \nabla \times \left(-\frac{\partial B}{\partial t}\right)$

$= \nabla \times (-j\omega\mu H)$

$= -j\omega\mu(\nabla \times H)$

$= -j\omega\mu\left(\frac{\partial D}{\partial t}\right)$

$= (-j\omega\mu)(j\omega)(\epsilon E)$

$= \omega^2\mu\epsilon E = K^2 E$

$\Rightarrow \nabla(\nabla \cdot \vec{E}) - \nabla^2 E = K^2 E$

$\Rightarrow 0 - \nabla^2 E = K^2 E$

($\because \nabla \cdot \vec{E} = 0$ in a source-free region)

$\Rightarrow \nabla^2 E + K^2 E = 0$

expanding eqn (1),

$$\left(\frac{\partial^2 E_s}{\partial x^2} + \frac{\partial^2 E_s}{\partial y^2} + \frac{\partial^2 E_s}{\partial z^2}\right) + K^2 E_s = 0 \quad \text{--- (3)}$$

If we let

$$E_s = (E_{xs}, E_{ys}, E_{zs}) \quad \text{and} \quad H_s = (H_{xs}, H_{ys}, H_{zs}) \quad \text{--- (4)}$$

$\downarrow$ Phasor form of 'E'

For Z-component, eqn (3) becomes

$$\frac{\partial^2 E_{zs}}{\partial x^2} + \frac{\partial^2 E_{zs}}{\partial y^2} + \frac{\partial^2 E_{zs}}{\partial z^2} + k^2 E_{zs} = 0 \quad \text{——— (5)}$$

Which is a partial differential equation. It can be solved by separation of variables. So let the solution to the above eqn is given as,

$$E_{zs}(x, y, z) = X(x)\, Y(y)\, Z(z) \quad \text{——— (6)}$$

Where $X(x)$, $Y(y)$ and $Z(z)$ are function of x, y, z respectively. Substituting eqn (6) in eqn (5) and dividing by XYZ, we have

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = -k^2 \quad \text{——— (7)}$$

$\therefore$

$$\frac{\partial^2}{\partial x^2} XYZ + \frac{\partial^2}{\partial y^2} XYZ + \frac{\partial^2}{\partial z^2} XYZ + k^2 XYZ = 0$$

$$YZ \frac{\partial^2 X}{\partial x^2} + XZ \frac{\partial^2 Y}{\partial y^2} + XY \frac{\partial^2 Z}{\partial z^2} + k^2 XYZ = 0$$

Dividing by XYZ both the sides

$$\Rightarrow \frac{1}{X} \cdot X'' + \frac{1}{Y} \cdot Y'' + \frac{1}{Z} \cdot Z'' + k^2 = 0$$

Since ~~the~~ k has 3 components along x, y, z direction,

$$\Rightarrow \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = -k_x^2 - k_y^2 - k_z^2 \quad \text{——— (8)}$$

Equating the coefficients,

$$\frac{X''}{X} = -k_x^2 \quad \text{——— (9)}$$

$$\frac{Y''}{Y} = -k_y^2 \quad \text{——— (10)}$$

$$\frac{Z''}{Z} = -k_z^2 \quad \text{——— (11)}$$

Since the guided wave propagates along the Z-direction the solution along Z-axis is given as

$$Z(z) = e^{-\gamma z}$$

$$Z'' = \gamma^2 e^{-\gamma z} = \gamma^2 Z$$

$\because$ $Z = e^{-\gamma z}$

$\frac{\partial Z}{\partial z} = -\gamma \cdot e^{-\gamma z}$

$\frac{\partial^2 Z}{\partial z} = \gamma^2 e^{-\gamma z}$

$$\Rightarrow \frac{Z''}{Z} = \gamma^2 \quad \text{--- (12)}$$

$\therefore$ Putting eqn (12) in eqn (11), we have

$$\gamma^2 = -K_z^2 \quad \text{--- (13)}$$

So eqn (11) becomes,

$$\frac{Z''}{Z} = \gamma^2 \quad \text{--- (14)}$$

From eqn (9), (10) & (14), we can write

$$X'' + X K_x^2 = 0 \quad \text{--- (15)}$$

$$Y'' + Y K_y^2 = 0 \quad \text{--- (16)}$$

$$Z'' - Z\gamma^2 = 0 \quad \text{--- (17)}$$

Solution to (15), (16) & (17) are in the form

$$X(x) = C_1 \cos K_x x + C_2 \sin K_x x \quad \text{--- (18)}$$

$$Y(y) = C_3 \cos K_y y + C_4 \sin K_y y \quad \text{--- (19)}$$

$$Z(z) = C_5 e^{\gamma z} + C_6 e^{-\gamma z} \quad \text{--- (20)}$$

Since we have assumed that wave propagates along the waveguide in the +z direction, the multiplicative

Constant $C_5 = 0$. Eq^n (20) becomes

$$Z(z) = C_6 e^{-\gamma z} \quad \text{—— (21)}$$

Putting eq^n (18), (19), (21) in eq^n (6), we have

$$E_{zs}(x,y,z) = (C_1 \cos K_x x + C_2 \sin K_x x)(C_3 \cos K_y y + C_4 \sin K_y y) C_6 e^{-\gamma z}$$

$$= (C_1 C_6 \cos K_x x + C_2 C_6 \sin K_x x)\left[C_3 \cos K_y y + C_4 \sin K_y y\right] e^{-\gamma z}$$

$$\Rightarrow \boxed{E_{zs}(x,y,z) = (A_1 \cos K_x x + A_2 \sin K_x x)(A_3 \cos K_y y + A_4 \sin K_y y)\, e^{-\gamma z}} \quad (22)$$

where $A_1 = C_1 C_6$, $A_2 = C_2 C_6$, $A_3 = C_3$, $A_4 = C_4$

By taking the similar steps, we get the solution of z-component of eq^n (2) as,

$$\boxed{H_{zs}(x,y,z) = (B_1 \cos K_x x + B_2 \sin K_x x)(B_3 \cos K_y y + B_4 \sin K_y y)\, e^{-\gamma z}} \quad (23)$$

→ Instead of solving for other field components $E_{xs}, E_{ys}, H_{xs}, H_{ys}$ in eq^n (1) & (2) in the same manner, it is more convenient to use Maxwell eq^n to determine them from E_{zs} and H_{zs}.

Determination of E_{xs}, E_{ys}, H_{xs}, H_{ys}

From Maxwell's eq^n

$$\nabla \times E_s = -j\omega\mu H_s \quad \text{— (24)}$$

$$\nabla \times H_s = j\omega \epsilon E_s \quad \text{—— (25)}$$

∵ $\nabla \times E = -\frac{\partial B}{\partial t}$

$= -(j\omega) \cdot B$

$= -(j\omega)(\mu H)$

$= -j\omega\mu H$

∵ $\frac{\partial}{\partial t} = j\omega$

$B = \mu H$

Similarly

$\nabla \times H = \frac{\partial D}{\partial t}$

$= j\omega \cdot (\epsilon E)$

$= j\omega\epsilon E$

From eq^n (24), expanding the curl

$$\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_{xs} & E_{ys} & E_{zs} \end{vmatrix} = -j\omega\mu\left[H_{xs}\hat{a}_x + H_{ys}\hat{a}_y + H_{zs}\hat{a}_z\right]$$

$$\Rightarrow \hat{a}_x\left[\frac{\partial}{\partial y}E_{zs} - \frac{\partial}{\partial z}E_{ys}\right] + \hat{a}_y\left[\frac{\partial}{\partial x}E_{zs} - \frac{\partial}{\partial z}E_{xs}\right]$$

$$+ \hat{a}_z\left[\frac{\partial}{\partial x}E_{ys} - \frac{\partial}{\partial y}E_{xs}\right]$$

$$= -j\omega\mu H_{xs}\hat{a}_x - j\omega\mu H_{ys}\hat{a}_y - j\omega\mu H_{zs}\hat{a}_z$$

Comparing the Coefficients of unit vector, both the sides.

$$\frac{\partial}{\partial y}E_{zs} - \frac{\partial}{\partial z}E_{ys} = -j\omega\mu H_{xs} \quad \text{———} \quad (26)$$

$$\frac{\partial}{\partial z}E_{xs} - \frac{\partial}{\partial x}E_{zs} = -j\omega\mu H_{ys} \quad \text{———} \quad (27)$$

$$\frac{\partial}{\partial x}E_{ys} - \frac{\partial}{\partial y}E_{xs} = -j\omega\mu H_{zs} \quad \text{———} \quad (28)$$

Similarly from eq^n (25)

$$\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_{xs} & H_{ys} & H_{zs} \end{vmatrix} = j\omega\epsilon\left[E_{xs}\hat{a}_x + E_{ys}\hat{a}_y + E_{zs}\hat{a}_z\right]$$

Expanding & equating the Coefficients, we have.

$$\frac{\partial}{\partial y}H_{zs} - \frac{\partial}{\partial z}H_{ys} = j\omega\epsilon E_{xs} \quad \text{———} \quad (29)$$

$$\frac{\partial}{\partial z}H_{xs} - \frac{\partial}{\partial x}H_{zs} = j\omega\epsilon E_{ys} \quad \text{———} \quad (20)$$

$$\frac{\partial}{\partial x}H_{ys} - \frac{\partial}{\partial y}H_{xs} = j\omega\epsilon E_{zs} \quad \text{———} \quad (21)$$

We will now express E_{xs}, E_{ys}, H_{xs} & H_{ys} in terms of E_{zs} and H_{zs}.

~~Putting eqn~~ From eqn (29), we have

$$j\omega\epsilon E_{xs} = \frac{\partial}{\partial y} H_{zs} - \frac{\partial}{\partial z} H_{ys}$$

$$= \frac{\partial}{\partial y} H_{zs} - \frac{\partial}{\partial z}\left[\frac{-1}{j\omega\mu}\left(\frac{\partial}{\partial z} E_{xs} - \frac{\partial}{\partial x} E_{zs}\right)\right]$$

[using eqn 27]

$$\Rightarrow j\omega\epsilon E_{xs} = \frac{\partial}{\partial y} H_{zs} + \frac{1}{j\omega\mu} \frac{\partial}{\partial z}\cdot\left[\frac{\partial}{\partial z} E_{xs} - \frac{\partial}{\partial x} E_{zs}\right]$$

$$\Rightarrow j\omega\epsilon E_{xs} = \frac{\partial}{\partial y} H_{zs} + \frac{1}{j\omega\mu}\left(\frac{\partial^2}{\partial z^2} E_{xs} - \frac{\partial^2 E_{zs}}{\partial x \partial z}\right) \quad \text{——— (32)}$$

From eqn (22) & (23), it is clear that all field components vary with z according to $e^{-\gamma z}$,

Let $$E_{zs} = E_p\, e^{-\gamma z}$$

$$\frac{\partial E_{zs}}{\partial z} = (-\gamma).\, \underline{E_p\, e^{-\gamma z}} = (-\gamma).\, E_{zs}$$

$$\therefore \frac{\partial E_{zs}}{\partial z} = -\gamma\, E_{zs} \quad \text{——— (33)}$$

$$\frac{\partial^2 E_{zs}}{\partial z^2} = \frac{\partial}{\partial z}\cdot\frac{\partial}{\partial z}\left[E_p\cdot e^{-\gamma z}\right] = \frac{\partial}{\partial z}\cdot\left[-\gamma\cdot E_p\, e^{-\gamma z}\right]$$

$$= (-\gamma)\cdot(-\gamma)\cdot E_p\, e^{-\gamma z}$$

$$\frac{\partial^2 E_{zs}}{\partial z^2} = \gamma^2 E_{zs} \quad \text{——— (34)}$$

Similarly
$$\frac{\partial^2 E_{xs}}{\partial z^2} = \gamma^2 E_{xs} \quad \text{--- (35)}$$

So from eqn (32),

$$\Rightarrow j\omega\epsilon E_{xs} = \frac{\partial}{\partial y} H_{zs} + \frac{1}{j\omega\mu}\left[\frac{\partial^2}{\partial z^2} E_{xs} - \frac{\partial}{\partial x}\left(\frac{\partial E_{zs}}{\partial z}\right)\right]$$

Using eqn (33) & (35), we have

$$\Rightarrow j\omega\epsilon E_{xs} = \frac{\partial}{\partial y} H_{zs} + \frac{1}{j\omega\mu}\left[\gamma^2 E_{xs} - \frac{\partial}{\partial x}(-\gamma E_{zs})\right]$$

$$\Rightarrow j\omega\epsilon E_{xs} = \frac{\partial}{\partial y} H_{zs} + \frac{1}{j\omega\mu}\left[\gamma^2 E_{xs} + \gamma \frac{\partial E_{zs}}{\partial x}\right]$$

$$\Rightarrow j\omega\epsilon E_{xs} - \frac{\gamma^2 E_{xs}}{j\omega\mu} = \frac{\gamma}{j\omega\mu}\frac{\partial}{\partial x} E_{zs} + \frac{\partial}{\partial y} H_{zs}$$

$$\Rightarrow \frac{-1}{j\omega\mu}\left[\gamma^2 + \omega^2\mu\epsilon\right] E_{xs} = \frac{\gamma}{j\omega\mu}\frac{\partial E_{zs}}{\partial x} + \frac{\partial}{\partial y} H_{zs}$$

Let $h^2 = \gamma^2 + \omega^2\mu\epsilon = \gamma^2 + k^2$ $\quad (\because k = \omega\sqrt{\mu\epsilon}$ in eqn ① & ②$)$ --- 35(a)

Then above eqn becomes,

$$\Rightarrow \frac{-1}{j\omega\mu}(h^2) . E_{xs} = \frac{\gamma}{j\omega\mu}\frac{\partial}{\partial x} E_{zs} + \frac{\partial}{\partial y} H_{zs}$$

$$\Rightarrow E_{xs} = \left(\frac{\cancel{j\omega\mu}}{-h^2}\right) \times \left(\frac{\gamma}{\cancel{j\omega\mu}}\right)\frac{\partial}{\partial x} E_{zs} - \frac{j\omega\mu}{h^2}\frac{\partial}{\partial y} H_{zs}$$

$$\Rightarrow \boxed{E_{xs} = -\frac{\gamma}{h^2}\frac{\partial E_{zs}}{\partial x} - \frac{j\omega\mu}{h^2}\frac{\partial}{\partial y} H_{zs}} \quad \text{--- (36)}$$

$\therefore$ E_{xs} is now expressed in terms of $\underline{E_{zs}}$ & $\underline{H_{zs}}$.

Similar manipulation of eq^ns $\underline{26 \text{ to } -31}$, yield expression for E_{ys}, H_{xs} and H_{ys} in terms of E_{zs} & H_{zs}. Thus

$$
\boxed{
\begin{aligned}
E_{xs} &= \frac{-\gamma}{h^2}\frac{\partial E_{zs}}{\partial x} - \frac{j\omega\mu}{h^2}\frac{\partial H_{zs}}{\partial y} && \text{— 37(a)}\\
E_{ys} &= \frac{-\gamma}{h^2}\frac{\partial E_{zs}}{\partial y} + \frac{j\omega\mu}{h^2}\frac{\partial H_{zs}}{\partial x} && \text{— 37(b)}\\
H_{xs} &= \frac{j\omega\epsilon}{h^2}\frac{\partial E_{zs}}{\partial y} - \frac{\gamma}{h^2}\frac{\partial H_{zs}}{\partial x} && \text{— 37(c)}\\
H_{ys} &= \frac{-j\omega\epsilon}{h^2}\frac{\partial E_{zs}}{\partial x} - \frac{\gamma}{h^2}\frac{\partial H_{zs}}{\partial y} && \text{— 37(d)}
\end{aligned}
}
\qquad (37)
$$

Where $h^2 = \gamma^2 + k^2 = \gamma^2 + k_x^2 + k_y^2 + k_z^2 = -\cancel{k_z^2} + k_x^2 + k_y^2 + \cancel{k_z^2}$

$= \gamma^2 + \omega^2\mu\epsilon$

(From eq^n 13)

$\therefore$ $\underline{h^2 = \gamma^2 + k^2}$ or $\underline{h^2 = k_x^2 + k_y^2}$

From eq^n (22), (23) & (37), we notice that the field pattern or configuration comes in $\underline{\text{different types}}$. Each of these distinct field patterns is called a $\underline{\text{mode}}$. $\underline{\text{Four different}}$ mode categories can exist.

Namely:

1) $E_{zs} = 0 = H_{zs}$ ($\underline{\text{TEM mode}}$) :- In the Transverse Electromagnetic mode, both E & H fields are

transverse to the direction of wave propagation.

From eqn (37) [All the field Components E_{xs}, E_{ys}, H_{xs}, H_{ys} Vanishes because E_{zs} & $H_{zs} = 0$]. So, we conclude that a rectangular waveguide can't support TEM mode. Imp BPUT exam

2) TE Mode ($E_{zs} = 0$, $H_{zs} \neq 0$)

For this case, the remaining components (E_{xs} & E_{ys}) of electric field are transverse to the direction of propagation a_z. Under this condition, fields are said to be in transverse electric (TE) mode.

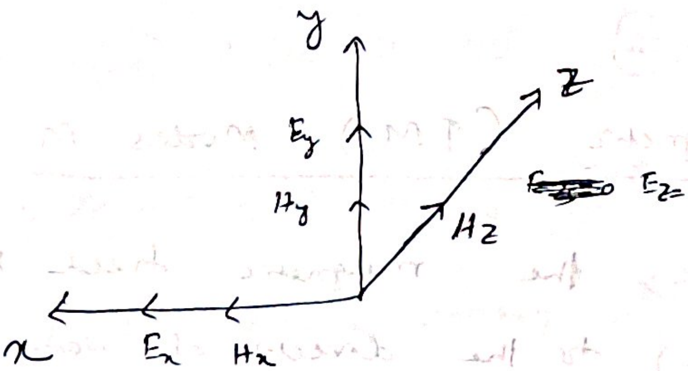

fig-23:- Component of EM fields on a rectangular waveguide: (a) TE mode ($E_z = 0$)

3) TM Mode ($E_{zs} \neq 0$, $H_{zs} = 0$)

In this case, the H field is transverse to the direction of wave propagation. Thus we have transverse magnetic (TM) mode. [See fig 24]

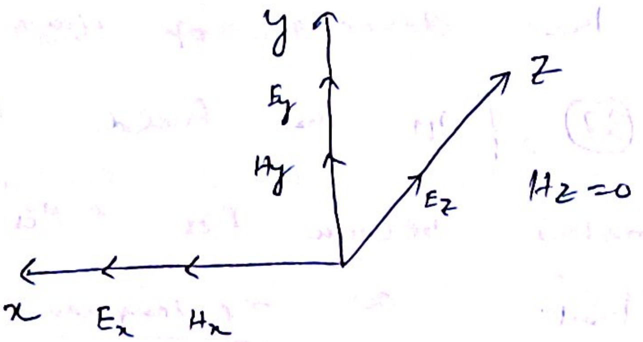

fig :- 24 :- Components of EM fields in a rectangular wave guide : (b) TM mode, $H_z = 0$

4(c) <u>Hybrid mode: (HE mode)</u> : ($E_{zs} \neq 0$, $H_{zs} \neq 0$)

In this case <u>neither</u> the E nor <u>the H</u> field is transverse to the direction of wave propagation. Sometimes these modes are referred to as hybrid modes.

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Transverse Magnetic (TM) Modes in Rectangular waveguide

For TM mode, the magnetic field has its components <u>transverse</u> (or <u>normal</u>) to the <u>direction of</u> wave propagation.

(Direction of Prop → z →)

∴ H_z (Component <u>along z direction</u>) = 0.

If H_z exist it will be parallel. So only H_x, H_y exist.

At the walls (perfect conductor) of the waveguide, the tangential components of E field must be continuous.

i.e

$E_{zs} = 0$ at $y = 0$ (bottom wall) — 38(a).

$E_{zs} = 0$ at $y = b$ (top wall) — 38(b)

$E_{zs} = 0$ at $x = 0$ (Left wall) — 38(c)

$E_{zs} = 0$ at $x = a$ (Right wall) — 38(d)

[Refer figure 22.]

From eq^n (22),

$$E_{zs} = (A_1 \cos k_x x + A_2 \sin k_x x)(A_3 \cos k_y y + A_4 \sin k_y y) \cdot e^{-\gamma z} \quad \text{--- (22)}$$

Putting the boundary condition, at $y=0$, $E_z=0$

$$\Rightarrow 0 = (A_1 \cos k_x x + A_2 \sin k_x x)(A_3)$$

$$\Rightarrow \boxed{A_3 = 0} \quad \text{--- (39)}$$

At $x=0$, $E_z=0$

$$0 = (A_1)(A_3 \cos k_y y + A_4 \sin k_y y) \cdot e^{-\gamma z}$$

$$\Rightarrow \boxed{A_1 = 0} \quad \text{--- (40)}$$

Putting eq^n (39) & (40) in eq^n (22)

$$E_{zs} = A_2 \sin k_x x \cdot A_4 \sin k_y y \cdot e^{-\gamma z}$$

$$E_{zs} = A_2 A_4 \sin k_x x \sin k_y y \; e^{-\gamma z}$$

$$E_{zs} = E_0 \sin k_x x \sin k_y y \; e^{-\gamma z} \quad \text{--- (41)}$$

Where $E_0 = A_2 A_4$.

Again putting the boundary cond^n, in eq^n (41)

At $y=b$, $E_z=0$

$$0 = E_0 \sin k_x x \cdot \sin k_y b \; e^{-\gamma z}$$

$$\therefore \sin k_y b = 0$$

$$\Rightarrow k_y b = n\pi \quad , n = 1, 2, 3 \ldots$$

$$k_y = \frac{n\pi}{b} \quad , \quad n = 1, 2, 3 \; --- \qquad (42)$$

Similarly, At $x = a$, $E_z = 0$, using eqn (41)

$$0 = E_0 \cdot \sin k_x a \; \sin k_y b \cdot e^{-\gamma z}$$

$$\Rightarrow \sin k_x a = 0$$

$$\Rightarrow k_x a = m\pi \quad , \quad m = 1, 2, 3 \; ---$$

$$\Rightarrow k_x = \frac{m\pi}{a} \quad , \quad m = 1, 2, 3 \; --- \qquad (43)$$

Using Eqn (42) & (43), in eqn (41), we have

$$\boxed{E_{zs} = E_0 \sin\left(\frac{m\pi}{a}\right)x \; \sin\left(\frac{n\pi}{b}\right)y \cdot e^{-\gamma z}} \qquad (44)$$

where m & n denotes number of half cycles along x-axis & y-axis respectively.

The other field components can be obtained, using eqn (37)

$$E_{xs} = \frac{-\gamma}{h^2} \frac{\partial}{\partial x} E_{zs} \qquad [\because H_{zs} = 0 \text{ for TM mode}]$$

$$= \frac{-\gamma}{h^2} \cdot \frac{\partial}{\partial x}\left[E_0 \sin\left(\frac{m\pi}{a}\right)x \; \sin\left(\frac{n\pi}{b}\right)y \cdot e^{-\gamma z}\right]$$

$$= \left(\frac{-\gamma}{h^2} E_0 \sin\left(\frac{n\pi}{b}\right)y \cdot e^{-\gamma z}\right)\left(\cos\left(\frac{m\pi}{a}\right)x\right) \cdot \left(\frac{m\pi}{a}\right)$$

$$\boxed{E_{xs} = -\frac{\gamma}{h^2}\left(\frac{m\pi}{a}\right) E_0 \cos\left(\frac{m\pi}{a}\right)x \; \sin\left(\frac{n\pi}{b}\right)y \cdot e^{-\gamma z}} \qquad (45)$$

Similarly, Putting $H_{zs} = 0$ and taking the derivative we have,

$$E_{ys} = \frac{-\gamma}{h^2} \frac{\partial}{\partial y}\left[E_0 \sin\left(\frac{m\pi}{a}\right)x \sin\left(\frac{n\pi}{b}\right)y\, e^{-\gamma z}\right]$$

$$\boxed{E_{ys} = \frac{-\gamma}{h^2}\left(\frac{n\pi}{b}\right) E_0 \sin\left(\frac{m\pi}{a}\right)x \cdot \cos\left(\frac{n\pi}{b}\right)y\, e^{-\gamma z}} \quad (46)$$

$$H_{xs} = \frac{j\omega\epsilon}{h^2} \frac{\partial}{\partial y}\left[E_0 \sin\left(\frac{m\pi}{a}\right)x \sin\left(\frac{n\pi}{b}\right)y\, e^{-\gamma z}\right]$$

$$\boxed{H_{xs} = \frac{j\omega\epsilon}{h^2} \cdot \left(\frac{n\pi}{b}\right) \cdot E_0 \sin\left(\frac{m\pi}{a}\right)x \cdot \cos\left(\frac{n\pi}{b}\right)y\, e^{-\gamma z}} \quad (47)$$

$$H_{ys} = \frac{-j\omega\epsilon}{h^2} \frac{\partial}{\partial x}\left[E_0 \sin\left(\frac{m\pi}{a}\right)x \sin\left(\frac{n\pi}{b}\right)y \cdot e^{-\gamma z}\right]$$

$$\Rightarrow \boxed{H_{ys} = \frac{-j\omega\epsilon}{h^2}\left(\frac{m\pi}{a}\right) \cdot E_0 \cos\left(\frac{m\pi}{a}\right)x \sin\left(\frac{n\pi}{b}\right)y \cdot e^{-\gamma z}} \quad (48)$$

Where $h^2 = K_x^2 + K_y^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$ — (49)

From eqn 35 (a), $h^2 = \gamma^2 + k^2$

$$\Rightarrow \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 = \gamma^2 + k^2$$

$$\Rightarrow \gamma = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2} \quad (50)$$

Note :-

1) For each set of integers m & n gives a different field pattern or mode, referred to as TM_{mn} mode, in the waveguide.

2) Integer 'm' equals to <u>number of half-cycle</u> variations in the x-direction and integer 'n' is the number of half-cycle variation in the y-direction.

<u>Case-I</u> (cutoff) $\begin{pmatrix} \gamma=0 \\ \alpha=0 \\ \beta=0 \end{pmatrix}$ $\left[k^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \right]$

Since waveguide is a HPF we have to determine cutoff freq (f_c), it is the minm freq after which propagation occurs inside waveguide.

From eqn (50), we have

$$\gamma = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}$$

At $f = f_c$, $\gamma = 0$ [ie $\alpha = 0$, $\beta = 0$] [No propagation at this freq, more than f_c propagation takes place]

Since $\gamma = 0$

$$\Rightarrow \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 = k^2$$

$$\Rightarrow \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 = \omega_c^2 \mu \epsilon \qquad \left(\because k = \omega\sqrt{\mu\epsilon}\right)$$

$$\Rightarrow \omega_c^2 = \frac{1}{\mu\epsilon}\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]$$

$$\Rightarrow \boxed{\omega_c = \frac{1}{\sqrt{\mu\epsilon}}\sqrt{\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]}} \quad \text{— (51)}$$

$$\Rightarrow 2\pi f_c = \frac{1}{\sqrt{\mu\epsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

$$\Rightarrow f_c = \frac{1}{\sqrt{\mu\epsilon}} \cdot \frac{1}{2\cancel{\pi}} \cdot \cancel{\pi} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

$$\Rightarrow f_c = \left(\frac{\frac{1}{\sqrt{\mu\epsilon}}}{2}\right) \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

$$\Rightarrow \boxed{f_c = \frac{u'}{2} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}} \quad \text{—— (52)}$$

where $u' = \frac{1}{\sqrt{\mu\epsilon}}$ = Phase velocity of EM wave in lossless dielectric medium $(\sigma = 0, \mu, \epsilon)$ in absence of wave guide.

Imp

$$\therefore \boxed{f_c = \frac{1}{2\sqrt{\mu\epsilon}} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}} \quad \text{—— (53)}$$

~~The~~ ∴ The cutoff freq is the operating freq below which attenuation occures and above which propagation takes place.

Cutoff Wavelength (λ_c)

$\lambda_c = \frac{u'}{f_c}$, from eqn (52),

$$\boxed{\lambda_c = \frac{2}{\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}}} \quad \text{—— (54)}$$

Case - 2 (Evanscent)

If $k^2 < \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$

i.e $\omega^2\mu\epsilon < \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$

$\boxed{\gamma = \alpha}$, $\beta = 0$

→ No wave propagation at all.
< Attenuation occurs >

→ It is called Evanscent or attenuating mode or non-propagation mode due to $\underline{\gamma = \alpha}$ (attenuation const).

$\because$

$\gamma = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}$

$\therefore$ Since

$k^2 < \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$

γ = +ve real number

No complex part

$\gamma = \alpha + j.(0)$

$\therefore \beta = 0$

Case-3 (Propagation)

$k^2 > \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$

$\gamma = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}$

$\therefore \gamma = \sqrt{-ve}$

$\gamma = j\beta$

$\boxed{\gamma = j\beta}$ ($\alpha = 0$, loss less medium)

$\therefore \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2} = j\beta$

$\Rightarrow \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2 = -\beta^2$

$\Rightarrow \beta^2 = k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2$

$\Rightarrow$ Imp $\boxed{\beta = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}}$ — (55)

This is the only case in which propagation takes place because all field components will have the factor $e^{-\gamma z} = e^{-j\beta z}$.

$$\beta_g = \sqrt{K^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

$$= \sqrt{K^2 \left(1 - \frac{1}{K^2}\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]\right)}$$

$$= K \sqrt{1 - \frac{1}{\omega^2 \mu\epsilon}\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]}$$

$$= K \sqrt{1 - \frac{1}{\omega^2} \cdot \omega_c^2}$$

$$\beta = \omega\sqrt{\mu\epsilon}\sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2}$$

[$\because \omega_c = \frac{1}{\sqrt{\mu\epsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$ eqn (51)]

imp

$$\Rightarrow \boxed{\beta = \beta' \sqrt{1 - \left(\frac{f_c}{f}\right)^2}} \quad \text{--- (56)}$$

β_g

$$\beta' = \omega\sqrt{\mu\epsilon} = 2\pi f' \cdot \frac{1}{U'} = \frac{2\pi}{\left(\frac{U'}{f'}\right)} = \frac{2\pi}{\lambda}$$

where

β' = Phase Constant of EM wave in dielectric medium in absence of wave guide.

U', β' → Absence of wave guide

U, β → Presence of wave guide.

β → Phase Const. in Presence of wave guide.

f_c → Cutoff freq

f → operating freq..

Intrinsic Wave Impedance. (η_{TM}) [The impedance offered by wave guide in TE/TM mode]

$$\eta_{TM} = \frac{E_x}{H_y} = -\frac{E_y}{H_x} \quad \text{--- (57)}$$

(See eqn 45, 46, 47, 48)

From eq^n (45) & (48)

$$\eta_{TM} = \frac{E_{xs}}{H_{ys}} = \frac{\gamma}{j\omega\epsilon} = \frac{j\beta}{j\omega\epsilon} = \frac{\beta'\sqrt{\cancel{1} - \left(\frac{f_c}{f}\right)^2}}{\omega\epsilon}$$

$$= \frac{\cancel{\omega}\sqrt{\mu\epsilon}}{\cancel{\omega}\epsilon}\sqrt{1-\left(\frac{f_c}{f}\right)^2}$$

$$\eta_{TM} = \sqrt{\frac{\mu}{\epsilon}}\sqrt{1-\left(\frac{f_c}{f}\right)^2}$$

Imp.

$$\boxed{\eta_{TM} = \eta'\sqrt{1-\left(\frac{f_c}{f}\right)^2}} \quad \text{—— (58)}$$

where $\eta' = \sqrt{\frac{\mu}{\epsilon}}$ = Intrinsic wave impedance in dielectric medium without waveguide

(Phase) Velocity inside waveguide - (U_p or u)

$$U_p/u = \frac{\omega}{\beta} = \frac{\omega}{\beta'\sqrt{1-\left(\frac{f_c}{f}\right)^2}}$$

$\frac{\omega}{\beta} = \frac{2\pi f}{\frac{2\pi}{\lambda}}$
$= \frac{\cancel{2\pi} f \times \lambda}{\cancel{2\pi}}$
$= f\lambda$
$= u$

from eq^n (56)

$$U_p/u = \frac{\cancel{\omega}}{\cancel{\omega}\sqrt{\mu\epsilon}\sqrt{1-\left(\frac{f_c}{f}\right)^2}}$$

$$U_p/u = \frac{\frac{1}{\sqrt{\mu\epsilon}}}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}}$$

where $u' = \frac{1}{\sqrt{\mu\epsilon}}$ = Velocity in dielectric medium in absence of waveguide

Imp

$$\Rightarrow \boxed{U_p/u = \frac{u'}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}}} \quad \text{—— (59)}$$

If it is free space.

$u' = c$

$$u = \frac{c}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}}$$

f_c is cutoff freq.

f " operating freq.

$f > f_c$.

$\Rightarrow f_c < f$

$\Rightarrow \frac{f_c}{f} < 1$

$\Rightarrow \left(\frac{f_c}{f}\right)^2 < 1$

$\therefore$ The denominator is a fraction.

$$\boxed{u > u'}$$

$\therefore$ Velocity of wave inside waveguide > Velocity of light.

Imp — wavelength in the guide

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\beta' \sqrt{1-\left(\frac{f_c}{f}\right)^2}} \quad \left[\text{from eq}^n\ 56\right]$$

$$\lambda = \frac{\left(\frac{2\pi}{\beta'}\right)}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}}$$

Imp

$$\boxed{\lambda = \frac{\lambda'}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}}} \quad (60)$$

$\because \lambda'$ = wavelength in dielectric med^m in absence of waveguide.

$\lambda > \lambda'$

Dominant mode :-

From eq^n 44, 45, 46, 47, 48,

If $m = 0,\ n = 0,$ All field component vanish.

$m = 0,\ n = n,$ " " " " [Either $\sin\left(\frac{m\pi}{a}\right)x = 0$ or Constant $\left(\frac{m\pi}{a}\right)$ is multiplied makes them zero]

$m = m,\ n = 0,$ All field component vanish. [$\sin\left(\frac{n\pi}{b}\right)y = 0$ or $\left(\frac{n\pi}{b}\right) = 0$]

$\therefore$ ~~TM_{00}, TM_{m0}, TM_{0m} i.e TM_{00}, T~~

i.e TM_{00}, TM_{m0}, TM_{0n} e.g TM_{00} TM_{10} TM_{01} --- etc don't exist.

Lowest value of m, n for which TM_{mn} exist ~~for~~ is $m = 1, n = 1$

∴ The mode in which lowest cutoff freq (or largest cutoff wavelength) occurs is known as dominant mode.

∴ $\boxed{TM_{11}}$ is the dominant mode for TM.

So in dominant mode

$$f_c = \frac{u'}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} \quad \text{— (61)}$$

imp

$$\boxed{f_c = \frac{u'}{2}\sqrt{\frac{1}{a^2} + \frac{1}{b^2}} \qquad \lambda_c = \frac{2}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}}}$$

— (62) where $u' = \frac{1}{\sqrt{\mu\epsilon}}$

— (63)

Degenerate modes:-

- Whenever two or more modes have the same cutoff freq, they are said to be degenerate modes.

→ In rectangular guide the corresponding TE_{mn} and TM_{mn} modes are always degenerate.

e.g TE_{11} and TM_{11}, → both have same cutoff freq.

$$f_c = \frac{u'}{2}\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}$$

Similarly $\{TE_{21}, TM_{21}\}$, $\{TE_{12} + TM_{12}\}$

Q) Prove that (Assignment)

$$\frac{1}{\lambda_g^2} = \frac{1}{\lambda_0^2} - \frac{1}{\lambda_c^2}$$

or $$\frac{1}{\lambda^2} = \frac{1}{\lambda_0^2} - \frac{1}{\lambda_c^2}$$ or $$\frac{1}{\lambda^2} = \frac{1}{\lambda'^2} - \frac{1}{\lambda_c^2}$$

$(\lambda' = \lambda_0)$

$\lambda_g / \lambda \rightarrow$ wavelength in the guide

$\lambda' / \lambda_0 \rightarrow$ " , free space / dielectric medium.

$\lambda_c \rightarrow$ cutoff " wavelength.

Ans:

$$\lambda_g = \frac{2\pi}{\beta} = \frac{2\pi}{\sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}}$$ [using eq^n 55]

$$\lambda_g = \frac{2\pi}{\sqrt{\omega^2\mu\epsilon - \omega_c^2\mu\epsilon}}$$

$\because$ $k^2 = \omega^2\mu\epsilon$ and $\omega_c = \frac{1}{\sqrt{\mu\epsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$ [from eq^n (51)]

$$\lambda_g = \frac{2\pi}{\sqrt{\mu\epsilon}\sqrt{\omega^2 - \omega_c^2}}$$

$$\lambda_g = c \times \frac{\cancel{2\pi}}{\cancel{2\pi}\sqrt{f^2 - f_c^2}}$$

$$\lambda_g = \frac{c}{f\sqrt{1 - \left(\frac{f_c}{f}\right)^2}}$$

$$\Rightarrow \lambda_g = \frac{\lambda_0}{\sqrt{1 - \left(\frac{f_c}{f}\right)^2}}$$ [Same as derived in eq^n (60)]

$$\Rightarrow \lambda_g = \frac{\lambda_0}{\sqrt{1 - \left(\frac{c/\lambda_c}{c/\lambda_0}\right)^2}} = \frac{\lambda_0}{\sqrt{1 - \left(\frac{\lambda_0}{\lambda_c}\right)^2}}$$

$\because$ $\left[\frac{c/\lambda_c}{c/\lambda} = \frac{\cancel{c}}{\lambda_c} \cdot \frac{\lambda_0}{\cancel{c}} = \frac{\lambda_0}{\lambda_c}\right]$

Squaring both the sides,

$$\lambda_g^2 = \frac{\lambda_0^2}{\left[1 - \left(\frac{\lambda_0}{\lambda_c}\right)^2\right]}$$

$$\Rightarrow \lambda_g^2 - \lambda_g^2 \cdot \left(\frac{\lambda_0}{\lambda_c}\right)^2 = \lambda_0^2$$

$$\Rightarrow \lambda_g^2 = \lambda_0^2 \left[1 + \frac{\lambda_g^2}{\lambda_c^2}\right]$$

Dividing both the sides by $\lambda_g^2 \cdot \lambda_c^2 \cdot \lambda_0^2$

$$\Rightarrow \frac{\cancel{\lambda_g^2}}{\cancel{\lambda_g^2}\,\cancel{\lambda_c^2}\,\lambda_0^2} = \frac{\cancel{\lambda_0^2}}{\lambda_g^2 \cdot \cancel{\lambda_c^2}\,\cancel{\lambda_0^2}} \left[1 + \frac{\lambda^2}{\lambda_c^2}\right]$$

$$\Rightarrow \frac{1}{\lambda_0^2} = \frac{1}{\lambda_g^2}\left[1 + \frac{\lambda^2}{\lambda_c^2}\right] = \frac{1}{\lambda_g^2} + \frac{1}{\lambda_c^2}$$

$$\Rightarrow \boxed{\frac{1}{\lambda_g^2} = \frac{1}{\lambda_0^2} - \frac{1}{\lambda_c^2}} \quad \text{--- (64)}$$

(Proved)

Imp. BPUT

Transverse Electric mode (TE mode) in Rectangular wave guide

In TE mode, the electric field is transverse (or normal) to the direction of wave propagation.

For TE mode, $E_{zs} = 0$

So other field components E_x, H_x, E_y, H_y and H_z

Can be determined using eqn (23) & eqn (37) and putting the boundary conditions.

From eqn (23),

$$H_{zs}(x,y,z) = (B_1 \cos K_x x + B_2 \sin K_x x)(B_3 \cos K_y y + B_4 \sin K_y y) \cdot e^{-\gamma z} \quad (65)$$

From eqn 37 (a)

$$E_{xs} = \frac{-\gamma}{h^2} \frac{\partial}{\partial x} E_{zs} - \frac{j\omega\mu}{h^2} \frac{\partial}{\partial y} H_{zs}$$

For TE mode, $E_{zs} = 0$,

$$\Rightarrow E_{xs} = -\frac{j\omega\mu}{h^2} \frac{\partial}{\partial y} H_{zs} \quad (66)$$

The boundary conditions are obtained from the requirement that tangential components of the electric field be continuous at the walls (perfect conductors) of the wave guides;

$$\left.\begin{array}{llll} E_{xs} = 0, & \text{at } y = 0 & (\text{bottom wall}) & (a) \\ E_{xs} = 0, & \text{at } y = b & (\text{top wall}) & (b) \\ E_{ys} = 0, & \text{at } x = 0 & (\text{Left wall}) & (c) \\ E_{ys} = 0, & \text{at } x = a & (\text{Right wall}) & (d) \end{array}\right\} (67)$$

At $y = 0$, $E_{xs} = 0$, putting this condn in eqn (66),

we have

$$\frac{\partial}{\partial y} H_{zs} = 0$$

$$\therefore \frac{\partial}{\partial y} H_{zs} = 0, \quad \text{at } y = 0 \quad (68)$$

From eqn (65), we have $\frac{\partial}{\partial y} H_{zs}$

$$= \frac{\partial}{\partial y}\left[(B_1 \cos k_x x + B_2 \sin k_x x)(B_3 \cos k_y y + B_4 \sin k_y y)\, e^{-\gamma z}\right]$$

$$= [B_1 \cos k_x x + B_2 \sin k_x x]\,[B_3 (-\sin k_y y) . k_y + B_4 (\cos k_y y) . k_y]\, e^{-\gamma z}$$

At $y = 0$, $\frac{\partial}{\partial y} H_{zs} = 0$

$$\Rightarrow [B_1 \cos k_x x + B_2 \sin k_x x][0 + B_4 k_y] . e^{-\gamma z} = 0$$

$$\Rightarrow \boxed{B_4 = 0} \quad \text{———} \quad (69)$$

[Note:- $k_y \neq 0$, otherwise H_{zs} will be always zero]

Similarly $\frac{\partial}{\partial y} H_{zs} = 0$, for $y = b$

$\Rightarrow$ At $y = b$, $\frac{\partial}{\partial y} H_{zs} = 0$

$B_4 = 0 \rightarrow$ eqn (69)

$$\Rightarrow [B_1 \cos k_x x + B_2 \sin k_x x]\,[-B_3 \sin k_y b + \cancel{B_4} (\cos k_y b)\, k_y]\, e^{-\gamma z} = 0$$

(Note:- $B_3 \neq 0$, if B_3 is zero & $B_4 = 0$ H_{zs} will be always zero)

$$\Rightarrow \sin k_y b = 0$$

$$\rightarrow k_y b = n\pi$$

$$\Rightarrow \boxed{k_y = \frac{n\pi}{b}} \qquad n = 0, 1, 2, 3 \ldots \quad \text{———} \quad (70)$$

Similarly from eqn 37 (b)

$$E_{ys} = \frac{j\omega\mu}{h^2} \frac{\partial}{\partial x} H_{zs} \quad \text{for TE mode, As } E_{zs} = 0$$

and

$E_{ys} = 0$ for $x = 0$ & a [from eqn 67]

$$\Rightarrow \left.\frac{\partial}{\partial x} H_{zs}\right]_{x = 0 \,\&\, a} = 0$$

$\therefore \frac{\partial}{\partial x}[H_{zs}]$

$$= \frac{\partial}{\partial x}\left[(B_1 \cos k_x x + B_2 \sin k_x x)(B_3 \cos k_y y + B_4 \sin k_y y)\, e^{-\gamma z}\right]$$

$$= (B_3 \cos k_y y + B_4 \sin k_y y) \cdot e^{-\gamma z} \left[-B_1 (\sin k_x x) \cdot k_x + B_2 (\cos k_x x) \cdot k_x\right]$$

At $x = 0$, $\frac{\partial}{\partial x} H_{zs} = 0$

$$\Rightarrow (B_3 \cos k_y y + B_4 \sin k_y y)(e^{-\gamma z})(B_2 k_x) = 0$$

$$\Rightarrow \boxed{B_2 = 0} \quad \text{—(71)}$$

(Note :- k_x can n't be zero, otherwise H_{zs} will be always zero)

At $x = a$, $\frac{\partial}{\partial x} H_{zs} = 0$

$$\Rightarrow \frac{\partial}{\partial x}\left[(B_1 \cos k_x x + \cancel{B_2 \sin k_x x})(B_3 \cos k_y y + B_4 \sin k_y y)e^{-\gamma z}\right] = 0$$

$B_2 = 0$

$$\Rightarrow B_1 \cdot -(\sin k_x a)\, k_x\, (B_3 \cos k_y y + B_4 \sin k_y y) \cdot e^{-\gamma z} = 0.$$

$$\Rightarrow \sin k_x a = 0$$

$$\Rightarrow k_x a = m\pi \quad , \quad m = 0, 1, 2, 3 \ldots$$

$$\Rightarrow k_x = \frac{m\pi}{a} \quad \text{——— (72)}$$

Combining eqn $\boxed{69, 70, 71, 72}$, we have

$B_4 = 0,\ B_2 = 0, \quad K_x = \frac{m\pi}{a}, \quad K_y = \frac{n\pi}{b}$

$$H_{zs} = (B_1 \cos k_x x + B_2 \sin k_x x)(B_3 \cos k_y y + B_4 \sin k_y y)\, e^{-\gamma z}$$

Putting the above conditions, we have

$$H_{zs} = \left(B_1 \cos \frac{m\pi}{a} x\right)\left(B_3 \cos\left(\frac{n\pi}{b}\right) y\right) e^{-\gamma z}$$

$$H_{zs} = B_1 B_3 \cos\left(\frac{m\pi}{a}\right)x \cos\left(\frac{n\pi}{b}\right)y\ e^{-\gamma z}$$

$$\boxed{H_{zs} = H_0 \cos\left(\frac{m\pi}{a}\right)x \cos\left(\frac{n\pi}{b}\right)y\ e^{-\gamma z}} \quad (73)$$

where $H_0 = B_1 B_3$, and m & n denote number of half cycles along x & y-axis respectively.

Other components can be found out using eqn (37)

$$E_{xs} = \frac{-j\omega\mu}{h^2} \frac{\partial H_{zs}}{\partial y} \quad \left[\begin{matrix}\text{from}\\ 37(a)\end{matrix}\right] \left[\because E_{zs} = 0 \text{ for TE mode}\right]$$

$$= \frac{-j\omega\mu}{h^2} \frac{\partial}{\partial y}\left[H_0 \cos\left(\frac{m\pi}{a}\right)x \cos\left(\frac{n\pi}{b}\right)y \cdot e^{-\gamma z}\right]$$

$$= \frac{-j\omega\mu}{h^2} H_0 \cos\left(\frac{m\pi}{a}\right)x \cdot e^{-\gamma z}\left[-\sin\left(\frac{n\pi}{b}\right)y \cdot \frac{n\pi}{b}\right]$$

$$\therefore \boxed{E_{xs} = \frac{j\omega\mu}{h^2}\left(\frac{n\pi}{b}\right) H_0 \cos\left(\frac{m\pi}{a}\right)x \sin\left(\frac{n\pi}{b}\right)y\ e^{-\gamma z}} \quad (74)$$

From 37 (b),

$$E_{ys} = \frac{j\omega\mu}{h^2} \frac{\partial H_{zs}}{\partial x} \quad \left[\because E_{zs} = 0\right]$$

$$\Rightarrow E_{ys} = \frac{j\omega\mu}{h^2} \frac{\partial}{\partial x}\left[H_0 \cos\left(\frac{m\pi}{a}\right)x \cos\left(\frac{n\pi}{b}\right)y \, e^{-\gamma z}\right]$$

$$\Rightarrow E_{ys} = \frac{j\omega\mu}{h^2} . H_0 . \cos\left(\frac{n\pi}{b}\right)y . e^{-\gamma z} . \left[- \sin\left(\frac{m\pi}{a}\right)x . \frac{m\pi}{a}\right]$$

$$\Rightarrow \boxed{E_{ys} = \frac{-j\omega\mu}{h^2} . \left(\frac{m\pi}{a}\right) . H_0 . \sin\left(\frac{m\pi}{a}\right)x \cos\left(\frac{n\pi}{b}\right)y . e^{-\gamma z}} \quad (75)$$

from eqn 37 (c)

$$H_{xs} = -\frac{\gamma}{h^2} \frac{\partial H_{zs}}{\partial x} \qquad \left[\because E_{zs} = 0\right]$$

$$= \frac{-\gamma}{h^2} \frac{\partial}{\partial x}\left[H_0 \cos\left(\frac{m\pi}{a}\right)x \cos\left(\frac{n\pi}{b}\right)y \, e^{-\gamma z}\right]$$

$$= \frac{-\gamma}{h^2} . H_0 . \cos\left(\frac{n\pi}{b}\right)y . e^{-\gamma z} . \frac{\partial}{\partial x}\left[\cos\left(\frac{m\pi}{a}\right)x\right]$$

$$H_{xs} = -\frac{\gamma}{h^2} H_0 . \cos\left(\frac{n\pi}{b}\right)y . e^{-\gamma z} \left[- \sin\left(\frac{m\pi}{a}\right)x . \frac{m\pi}{a}\right]$$

$$\Rightarrow \boxed{H_{xs} = \frac{\gamma}{h^2}\left(\frac{m\pi}{a}\right) . H_0 \sin\left(\frac{m\pi}{a}\right)x \cos\left(\frac{n\pi}{b}\right)y \, e^{-\gamma z}} \quad (76)$$

from eqn 37 (d),

$$H_{ys} = -\frac{\gamma}{h^2} \frac{\partial}{\partial y} H_{zs} \qquad \left[\because E_{zs} = 0\right]$$

$$\Rightarrow H_{ys} = -\frac{\gamma}{h^2} \frac{\partial}{\partial y}\left[H_0 \cos\left(\frac{m\pi}{a}\right)x \cos\left(\frac{n\pi}{b}\right)y \, e^{-\gamma z}\right]$$

$$\Rightarrow \boxed{H_{ys} = \frac{\gamma}{h^2}\left(\frac{n\pi}{b}\right) . H_0 \; \cos\left(\frac{m\pi}{a}\right)x . \; \sin\left(\frac{n\pi}{b}\right)y \; e^{-\gamma z}} \quad (77)$$

where $h^2 = K_x^2 + K_y^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$

and $\gamma = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - K^2}$

[As defined for TM mode]

The value of f_c, λ_c, β is same as that for TM mode, i.e.

Imp

$$\boxed{f_c = \frac{1}{2\sqrt{\mu\epsilon}} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}} \quad (78)$$

or

$$\boxed{f_c = \frac{c'}{2} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}} \quad (79)$$

Imp

$$\boxed{\lambda_c = \frac{2}{\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}}} \quad (80)$$

Imp

$$\boxed{\beta = \beta' \sqrt{1 - \left(\frac{f_c}{f}\right)^2}} \quad (81)$$

where $\beta' = \omega\sqrt{\mu\epsilon}$

But value of intrinsic wave impedance of TE mode differs from that of TM mode

$$\eta_{TE} = \frac{E_x}{H_y} = \frac{j\omega\mu}{\gamma} = \frac{-E_y}{H_x}$$ [eqn (74) & (77) eqn (75) & (76)]

$$\eta_{TE} = \frac{j\omega\mu}{\gamma} = \frac{\cancel{j}\omega\mu}{\cancel{j}\beta}$$ [∵ $\gamma = j\beta$ for Propagation mode $\alpha = 0$]

$$\eta_{TE} = \frac{\omega\mu}{\beta} = \frac{\omega\mu}{\beta' \sqrt{1-\left(\frac{f_c}{f}\right)^2}} = \frac{\omega\mu}{\cancel{\omega}\sqrt{\mu\epsilon}\sqrt{1-\left(\frac{f_c}{f}\right)^2}}$$

$$\eta_{TE} = \sqrt{\frac{\mu}{\epsilon}} \left[\frac{1}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}}\right]$$

Imp

$$\boxed{\eta_{TE} = \eta' \left[\frac{1}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}}\right]} \quad \text{—— (82)}.$$

η_{TE} = In Presence of wave guide.

η' = In Absence of waveguide.

From eqn (58) & (82)

Imp

$$\boxed{\eta_{TE} \cdot \eta_{TM} = (\eta')^2} \quad \text{—— (83)}$$

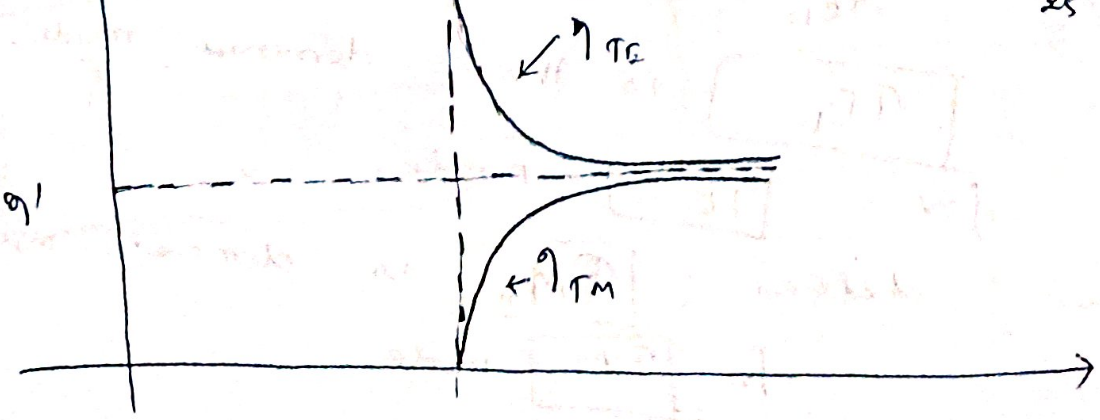


fig:- variation of wave impedance with freq for TE & TM mode.

Dominant Mode :-

The dominant mode is the mode with lowest cutoff frequency (or longest cutoff wavelength).

For TE mode, (m, n) may be (0,1) or (1,0) but not (0,0). Both m & n cannot be zero at the same time because this will force the field components to vanish.

For m=1, n=0

$$f_{c_{10}} = \frac{u'}{2}\sqrt{\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2} = \frac{u'}{2}\times\frac{1}{a} = \frac{u'}{2a}$$

For m=0, n=1

$$f_{c_{01}} = \frac{u'}{2b}$$

Generally for rectangular wave guide $a > b$,

$$\frac{u'}{2a} < \frac{u'}{2b}$$

$$\Rightarrow f_{c_{10}} < f_{c_{01}}$$

$\therefore$ $f_{c_{10}}$ is the lowest cut off freq.

$\therefore$ $\boxed{TE_{10}}$ is the dominant mode for $\boxed{TE}$ mode.

We have defined, $\boxed{TM_{11}}$ is dominant mode for $\boxed{TM}$ mode.

Overall for a rectangular waveguide

$$(f_c)_{TE_{10}} = \frac{u'}{2a}.$$

$$(f_c)_{TM_{11}} = \frac{u'}{2}\sqrt{\frac{1}{a^2}+\frac{1}{b^2}} = \frac{u'}{2}\sqrt{\frac{a^2+b^2}{a^2b^2}}$$

$$= \frac{u'}{2a}\cdot\sqrt{\frac{a^2+b^2}{b^2}}$$

$$= \frac{u'}{2a}\sqrt{1+\frac{a^2}{b^2}}$$

$$(f_c)_{TM_{11}} > (f_c)_{TE_{10}}$$

$\therefore$ $(f_c)_{TE_{10}}$ is lowest ;

$\therefore$ For a rectangular waveguide $\boxed{TE_{10}}$ is the dominant mode.

Note :- For TE mode Phase velocity & wavelength is also same as that of TM mode.

Imp

$$\boxed{U_p = \frac{u'}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}}} \quad (84)$$

where $u' = \frac{1}{\sqrt{\mu\epsilon}}$

Imp

$$\boxed{\lambda = \frac{\lambda'}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}}} \quad (85)$$

$\lambda' = \frac{2\pi}{\beta'}$ = wave length, in absence of waveguide.

Cutoff freq & wavelength at dominant mode

At TE_{10} mode

$m = 1, n = 0.$

$$\boxed{f_c = \frac{u'}{2a}} \quad \text{——— (86)}$$

$$\lambda_c = \frac{2}{\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}} = \frac{2}{\sqrt{\frac{1}{a^2}}} = 2a$$

$$\boxed{\lambda_c = 2a} \quad \text{——— (87)}$$

Phase velocity Vs Group velocity :-

Phase velocity (U_p) is the velocity at which loci of constant phase are propagated down the guide.

$$u_p = \frac{u'}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}} \quad \text{[As derived earlier]}$$

Group velocity (U_g) is the velocity with which the resultant repeated reflected waves are travelling down the guide. It is the energy propagation velocity in the guide and is given by

$$u_g = \frac{\partial \omega}{\partial \beta}$$

$$\beta = \beta' \sqrt{1-\left(\frac{f_c}{f}\right)^2} = \omega\sqrt{\mu\epsilon}\sqrt{1-\left(\frac{f_c}{f}\right)^2}$$

$$\therefore \beta = \omega\sqrt{\mu\epsilon}\sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2} = \cancel{\omega}\sqrt{\mu\epsilon}\frac{\sqrt{\omega^2-\omega_c^2}}{\cancel{\omega}}$$

$$\Rightarrow \frac{\partial \omega}{\partial \beta} = \frac{1}{\sqrt{\mu \epsilon}} \cdot \frac{1}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}}$$

$$\Rightarrow \frac{\partial \omega}{\partial \beta} = \frac{u'}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}} \quad \left(\because u' = \frac{1}{\sqrt{\mu \epsilon}}\right)$$

$$\Rightarrow \boxed{u_g = \frac{u'}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}}} \text{ --- } (88).$$

$$u_p \cdot u_g = \frac{u'}{\left(\sqrt{1-\left(\frac{f_c}{f}\right)^2}\right)} \times \frac{u'}{}$$

$$\therefore \beta = \sqrt{\mu \epsilon} \sqrt{\omega^2 - \omega_c^2}$$

$$\because \frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$$

$$\Rightarrow \frac{\partial \beta}{\partial \omega} = \sqrt{\mu \epsilon} \frac{\cancel{2}\omega}{\cancel{2} \times \sqrt{\omega^2 - \omega_c^2}}$$

$$= \sqrt{\mu \epsilon} \frac{\cancel{\omega}}{\cancel{\omega}\sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2}}$$

$$= \sqrt{\mu \epsilon} \cdot \frac{1}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}}$$

$$\Rightarrow \frac{\partial \omega}{\partial \beta} = \frac{1}{\sqrt{\mu \epsilon}} \cdot \sqrt{1-\left(\frac{f_c}{f}\right)^2}$$

$$\Rightarrow \frac{\partial \omega}{\partial \beta} = u' \sqrt{1-\left(\frac{f_c}{f}\right)^2} \quad \left(\because u' = \frac{1}{\sqrt{\mu \epsilon}},\ u' = c \text{ for freespace}\right)$$

$$\Rightarrow \boxed{u_g = u' \sqrt{1-\left(\frac{f_c}{f}\right)^2}} \text{ --- } (88)$$

$$\therefore \boxed{\begin{aligned} U_p \cdot U_g &= (u')^2 \quad \text{—— (89)} \\ U_p \cdot U_g &= c^2 \quad \text{for free space} \end{aligned}}$$

Power transmission inside a waveguide :-

To determine power flow in the waveguide, we first find the average Poynting vector

$$P_{avg} = \frac{1}{2} \, Re \, (E_s \times H_s^*) \quad \text{—— (90)}$$

In this case, the Poynting vector is along the z-direction, so ~~that~~

$P_{avg} \big|_{z\text{-direction}}$ can be found as follows.

$$P_{avg} = \frac{1}{2} \, Re \begin{vmatrix} a_x & a_y & a_z \\ E_{xs} & E_{ys} & E_{zs} \\ H_{xs}^* & H_{ys}^* & H_{zs}^* \end{vmatrix}$$

$$P_{avg} \big|_{z\text{-direction}} = \frac{1}{2} \, Re \left[E_{xs} H_{ys}^* - E_{ys} H_{xs}^* \right] \hat{a}_z$$

$$= \frac{1}{2} \, Re \left[E_{xs} \cdot \frac{E_{xs}^*}{\eta^*} - E_{ys} \cdot \left(- \frac{E_{ys}^*}{\eta^*} \right) \right] \hat{a}_z$$

$$\left(\because \frac{E_x}{H_y} = \eta = - \frac{E_y}{H_x} \; , \; \text{or} \; \frac{E_x^*}{H_y^*} = \eta^* = - \frac{E_y^*}{H_x^*} \right)$$

$$\Rightarrow \quad P_{avg}\Big|_{z\text{-direction}} = \frac{1}{2} \operatorname{Re}\left[\frac{|E_{xs}|^2}{\eta^*} + \frac{|E_{ys}|^2}{\eta^*} \right] a_z$$

$$\Rightarrow \quad P_{avg}\Big|_{z\text{-direction}} = \frac{|E_{xs}|^2 + |E_{ys}|^2}{2\eta} a_z \quad \left(\because \eta^* = \eta = \text{real} \right) \quad (91)$$

Total avg power transmitted across the cross section of the waveguide ds

$$P_{avg} = \int P_{avg} \cdot ds$$

$$\boxed{P_{avg} = \int_{x=0}^{a} \int_{y=0}^{b} \left\{ \frac{|E_{xs}|^2 + |E_{ys}|^2}{2\eta} \right\} dy\, dx \quad \text{watt}}$$

For TE mode

$$\boxed{P_{avg} = \int_{x=0}^{a} \int_{y=0}^{b} \left(\frac{|E_{xs}|^2 + |E_{ys}|^2}{2\eta_{TE}} \right) dy\, dx \quad \text{watt}}$$

where $\eta_{TE} = \dfrac{\eta'}{\sqrt{1 - \left(\frac{f_c}{f}\right)^2}}$

For TM mode

$$\boxed{P_{avg} = \int_{x=0}^{a} \int_{y=0}^{b} \left(\frac{|E_{xs}|^2 + |E_{ys}|^2}{2\eta_{TM}} \right) dy\, dx \quad \text{watt}}$$

where

$$\eta_{TM} = \eta' \sqrt{1 - \left(\frac{f_c}{f}\right)^2}$$

where $\eta' = \sqrt{\dfrac{\mu}{\epsilon}}$ = Intrinsic wave impedance, in absence of wave guide, in dielectric medium

Attenuation in a lossy waveguide

Practically all waveguides are lossy in nature so that there is a loss of power along the waveguide as the wave propagates.

→ We have assumed lossless waveguides ($\sigma = 0$, $\sigma_c = \infty$) for which $\alpha = 0$, $\gamma = j\beta$.

→ When dielectric medium is lossy ($\sigma \neq 0$) and the guide walls are not perfectly conducting ($\sigma_c \neq \infty$), there is a continuous loss of power along the guide as a wave propagates.

∴ The loss occurs in the dielectric medium or in the conducting walls.

Assuming that wave propagate along z-axis.

$$E \propto e^{-\gamma z}$$

$$H \propto e^{-\gamma z}$$

$$P_{avg} \propto e^{-2\gamma z} \qquad \left[\because P_{avg} \propto E \times H\right]$$

$$\Rightarrow P_{avg} = P_0 \, e^{-2\gamma z}$$

$$\Rightarrow P_{avg} = P_0 \cdot e^{-2(\alpha + j\beta)z}$$

$$\Rightarrow P_{avg} = P_0 \cdot e^{-2\alpha z} \cdot e^{-2j\beta z}$$

$$\Rightarrow |P_{avg}| = P_0 \cdot e^{-2\alpha z} \quad \left(\because \left|e^{-2j\beta z}\right| = 1 \quad \left(\because |\cos 2\beta z - j\sin 2\beta z| = 1\right)\right)$$

In general

$$\alpha = \alpha_c + \alpha_d \quad (\cancel{\text{Total loss}})$$

where α_c ~~and~~ α_d are attenuation constants due to ohmic or conduction (α_c) losses [since $\sigma_c \neq \infty$] and dielectric (α_d) losses ($\sigma \neq 0$), respectively.

Imp

$$\boxed{\alpha_d = \frac{K^2 \tan\theta}{2\beta} \ Np/m \quad \text{for a rectangular waveguide}}$$

Note:-

Np = Neper/m

1 Np = 8.686 dB

where $K = \omega\sqrt{\mu\epsilon}$

$$\beta = \sqrt{K^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2} = \sqrt{\omega^2\mu\epsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2} = \beta' \sqrt{1-\left(\frac{f_c}{f}\right)^2}$$

$$\tan\theta = \frac{\sigma}{\omega\epsilon} = \text{Loss tangent} = \text{Ratio of conduction current to displacement current} = \frac{J_c}{J_d}$$

For lossless waveguide, $\alpha_d = 0$ because $\sigma = 0$.

Similarly, expression for α_c can be derived as

$$\alpha_c = \frac{R_s}{a^3 b \beta K \eta}\left(2b\pi^2 + a^3 K^2\right) \ Np/m$$

where $R_s = \sqrt{\frac{\omega\mu}{2\sigma}}$ = Surface resistance of the conductor

a & b are the dimension of waveguide.

K and β are same as expressed for 'α_d'.

$$\boxed{\text{Total attenuation } (\alpha) = \alpha_c + \alpha_d}$$

Imp

Important eq^n for TE & TM modes :-

TM mode

$H_{zs} = 0$

$$\eta_{TM} = \eta' \sqrt{1-\left(\frac{f_c}{f}\right)^2}$$

TE Mode

$E_{zs} = 0$

$$\eta_{TE} = \frac{\eta'}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}}$$

Imp

Common eq^n

$$f_c = \frac{\omega}{2}\sqrt{\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2}$$

where

$$\eta' = \sqrt{\frac{\mu}{\epsilon}}$$

$$\lambda_c = \frac{2}{\sqrt{\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2}}$$

$$\beta = \beta' \sqrt{1-\left(\frac{f_c}{f}\right)^2}, \quad \beta' = \omega\sqrt{\mu\epsilon} = k$$

$$u_p = \frac{u'}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}}$$ where $u' = \frac{1}{\sqrt{\mu\epsilon}}$

$$\lambda = \frac{\lambda'}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}}$$

$K = \omega\sqrt{\mu\epsilon}$, $h^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$

Problems

1) BPUT 2010 The phase constant of the TE_{10} mode of an air-filled waveguide with b = 1 cm is 102.65 rad/m. If the operating freq of the waveguide is 12 GHz, and the only mode of propagation is TE_{10}. Calculate

(a) The length 'a' of the waveguide

(b) Wave impedance.

Ans ÷ Given

b = 1 cm, $\beta = 102.65$ rad/m. f = 12 GHz.

a = ? , $\eta_{TE} = ?$

We know

$$\beta = \beta'\sqrt{1-\left(\frac{f_c}{f}\right)^2}$$

$$\Rightarrow \beta = \omega\sqrt{\mu\epsilon}\sqrt{1-\left(\frac{f_c}{f}\right)^2}$$

$$\Rightarrow \beta = \frac{\omega}{c}\sqrt{1-\left(\frac{f_c}{f}\right)^2}$$

$$\Rightarrow \beta = \frac{2\pi f}{3\times10^8}\sqrt{1-\frac{(f_c)^2}{(12\times10^9)^2}}$$

$$c = \frac{1}{\sqrt{\mu\epsilon}}$$

$$\Rightarrow \sqrt{\mu\epsilon} = \frac{1}{c}$$

$$\Rightarrow 102.65 = \frac{2\pi \times \cancel{12}^{4} \times 10^{9}/10}{\cancel{3} \times 10^{8}} \sqrt{1 - \frac{f_c^2}{144 \times 10^{18}}}$$

$$\Rightarrow \sqrt{1 - \frac{f_c^2}{144 \times 10^{18}}} = \frac{102.65}{80\pi} = 0.408$$

$$\Rightarrow 1 - \frac{f_c^2}{144 \times 10^{18}} = 0.1668$$

$$\Rightarrow 1 - 0.1668 = \frac{f_c^2}{144 \times 10^{18}}$$

$$\Rightarrow f_c^2 = 119.97 \times 10^{18}$$

$$\Rightarrow \boxed{f_c = 10.953 \text{ GHz}}$$

For TE_{10} mode,

$$f_c = \frac{u'}{2a} = \frac{3 \times 10^8}{2a}$$

$$\Rightarrow 10.953 \times \cancel{10^{9}}/10 = \frac{3 \times \cancel{10^8}}{2a}$$

$$\Rightarrow 2a = \frac{3}{109.53}$$

$$\Rightarrow a = 0.01369 \text{ meter}$$

$$\Rightarrow \boxed{a = 1.369 \text{ cm}}$$

(a)

(b) Wave impedance $\eta_{TE} = \dfrac{\eta'}{\sqrt{1 - \left(\frac{f_c}{f}\right)^2}}$

where $\eta' = \sqrt{\frac{\mu}{\epsilon}} = 377$ ohm for freespace / air.

$$\Rightarrow \quad \eta_{TE} = \frac{377}{\sqrt{1 - \left(\frac{10.953 \times 10^9}{12 \times 10^9}\right)^2}}$$

$$\Rightarrow \quad \boxed{\eta_{TE} = 922.847 \ \Omega}$$

BPUT-2010

2/ An air-filled rectangular waveguide with dimensions $a = 3$ cm, $b = 2$ cm, it is excited with TE mode at 6 GHz. The loss tangent in air is 0.001, and $\sigma = 5.8 \times 10^7$ S/m.

Calculate

(i) Cut off freq

(ii) Phase constant

(iii) Skin depth

(iv) attenuation constant α_d & α_c.

Ans : Given

$a = 3$ cm, $b = 2$ cm, $f = 6 \times 10^9$, $\tan\delta = 0.001$

$\sigma = 5.8 \times 10^7$ S/m. For TE_{10}, mode,

(i) $f_c = \frac{u'}{2a} = \frac{\cancel{3} \times 10^8}{2 \times \cancel{3} \times 10^{-2}} = \frac{10^8 \times \cancel{10}^{5} \times 10}{\cancel{2}} = 5$ GHz.

(ii) Phase constant $(\beta) = \beta' \sqrt{1 - \left(\frac{f_c}{f}\right)^2} = \omega\sqrt{\mu\epsilon} \sqrt{1 - \left(\frac{f_c}{f}\right)^2}$

$c = \frac{1}{\sqrt{\mu\epsilon}} \Rightarrow \sqrt{\mu\epsilon} = \frac{1}{c}$ $\quad \Rightarrow \beta = \frac{\omega}{c} \sqrt{1 - \left(\frac{f_c}{f}\right)^2}$

$$\Rightarrow \beta = \frac{2\pi f}{3\times10^8}\sqrt{1-\left(\frac{f_c}{f}\right)^2}$$

$$= \frac{2\pi\times \cancel{6}^{2}\times10^{9/10}}{\cancel{3}\times10^8}\sqrt{1-\left(\frac{5}{6}\right)^2}$$

$$\Rightarrow \beta = 40\pi\sqrt{\frac{36-25}{36}} = 40\pi\times\sqrt{\frac{11}{36}}$$

$$\Rightarrow \beta = 69.46\ \frac{\text{rad}}{m}.$$

(iii) Skin depth $(\delta) = \dfrac{1}{\sqrt{\pi f \mu\sigma}}$

$$\therefore \delta = \frac{1}{\sqrt{\pi\times6\times10^9\times4\pi\times10^{-7}\times5.8\times10^7}}$$

Note
$\mu_0 = 4\pi\times10^{-7}$
$\epsilon_0 = \frac{10^{-9}}{36\pi}$

$\therefore$ $\delta = 8.53\times10^{-7}$ meter.

(iv) Attenuation Constant

$\therefore$ $\alpha_d = \dfrac{k^2\tan\theta}{2\beta}$

$k = \omega\sqrt{\mu\epsilon} = \dfrac{\omega}{c} = \dfrac{2\pi f}{c} = \dfrac{2\pi\times\cancel{6}^{2}\times10^9}{\cancel{3}\times10^8} = 4\pi\times10 = 40\pi = 125.66$

$\tan\theta =$ loss tangent $= 0.001$ (given)

$\beta = 69.46\ \frac{\text{rad}}{m}$ (derived in (ii) bit)

$$\therefore \alpha_d = \frac{(125.66)^2\times0.001}{2\times69.46} = 0.1136\ \text{Np/m (Ans)}$$

$$= 0.9867\ \text{dB/m}$$

$$\alpha_c = \frac{R_s}{a^3 b \beta \times \eta} \left(2b\pi^2 + a^3 k^2 \right)$$

$$R_s = \sqrt{\frac{\omega\mu}{2\sigma}} = \sqrt{\frac{2\pi f \times 4\pi \times 10^{-7}}{2 \times 5.8 \times 10^7}} = \sqrt{\frac{2 \times \pi \times 6 \times 10^9 \times 4\pi \times 10^{-7}}{2 \times 5.8 \times 10^7}}$$

$$\Rightarrow \quad R_s = 0.0202\ \Omega.$$

$\eta = 377\,\Omega$ for free space.

$$\alpha_c = \frac{0.0202 \left[2 \times (2 \times 10^{-2}) \times \pi^2 + (3 \times 10^{-2})^3 (125.66)^2 \right]}{(3 \times 10^{-2})^3 (2 \times 10^{-2}) (69.46)(125.66) \times 377}$$

$$\alpha_c = \frac{(0.0202)\left[.3947 + 0.4263 \right]}{1.7769}$$

$$\alpha_c = 9.33 \times 10^{-3}\ \text{NP/m} = 0.081\ \text{dB/m}. \quad \text{(Ans)}$$

$$\left(\because 1\ \text{Np} = 8.686\ \text{dB} \right)$$

3) BPUT-2009

An air filled rectangular waveguide having dimension 4 × 8 operates in the TE_{10} mode. Find out

(i) the cut-off freq

(ii) The Phase velocity at a freq of 4 GHz.

Ans : Given $a = 8$ cm, $b = 4$ cm

$$f_c = \frac{c}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

$$TE_{10},\quad f_c = \frac{c}{2}\sqrt{\frac{1}{a^2}} = \frac{c}{2a}.$$

(i) $$f_c = \frac{3\times10^8}{2\times 8\times10^{-2}} = \frac{30}{16}\times10^9 = 1.8\ GHz$$

(ii) Phase velocity at freq 4 GHz.

$$u_p = \frac{u'}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}} = \frac{3\times10^8}{\sqrt{1-\left(\frac{1.8}{4}\right)^2}} = 3.36\times10^8\ m/s$$

Phase velocity $= 3.36\times10^8\ m/s$

4) ~~BPUT-2008~~ In an air filled square waveguide with dimensions $a = 1.2$ cms,

$$E_x = -10\sin\left(\frac{2\pi y}{a}\right)\sin(\omega t - 150 z)\ \frac{V}{m}.$$

Find (i) Mode of propagation

(ii) its cut-off wavelength

(iii) Calculate the freq of operation

(iv) wave impedance.

~~Ans:- ~~ ~~Let~~

$$\cancel{E_x = E_A \cos\left(\frac{m\pi}{a}\right)x \sin\left(\frac{n\pi}{b}\right)y}\ e^{-\gamma z}\ \frac{V}{m}$$

Comparing with given

$$\cancel{E_x = -10\cos(0)x\sin\left(\frac{2\pi}{a}\right)y}$$

Ans :- Let

$$E_x = E_A \cos\left(\frac{m\pi}{a}\right)x \sin\left(\frac{n\pi}{b}\right)y \sin(\omega t - \beta z)$$

given

$$E_x = -10 \cos(0)x \sin\left(\frac{2\pi}{a}\right)y \sin(\omega t - 150 z)$$

$(\because \cos(0)x = 1)$

Comparing

$E_A = -10$, $m = 0$, $n = 2$, $\beta = 150$

(For square waveguide $a = b$)

Two possibilities

TE_{02} or TM_{02}

If $m = 0$, TM_{mn} can n't exist. (All component vanish)

(i) : Mode of propagation is $\boxed{TE_{02}}$.

(ii) Cut off wave length

$$\lambda_c = \frac{2}{\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}} = \frac{2}{\sqrt{0 + \left(\frac{2}{b}\right)^2}} = \frac{2}{\frac{2}{b}}$$

$$\boxed{\lambda_c = b}$$

For square ~~wa~~ waveguide, $b = a = 1.2$ cm.

$$\therefore \boxed{\lambda_c = 1.2 \text{ cm}}$$

(iii) $$f_c = \frac{c}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} = \frac{c}{2}\sqrt{\left(\frac{2}{b}\right)^2} = \frac{c}{\cancel{2}} \times \frac{\cancel{2}}{b}$$

$$f_c = \frac{c}{b}.$$

$$f_c = \frac{3\times10^8}{1.2\times10^{-2}} = 2.5\times10^{10} = 25\ \text{GHz}.$$

$$\beta = \beta' \sqrt{1-\left(\frac{f_c}{f}\right)^2}$$

$$\Rightarrow\ 150 = \omega\sqrt{\mu\epsilon}\ \sqrt{1-\left(\frac{f_c}{f}\right)^2}$$

$$\Rightarrow\ 150 = \frac{\omega}{c}\ \sqrt{1-\left(\frac{f_c}{f}\right)^2}$$

$$\Rightarrow\ \frac{150c}{\omega} = \sqrt{1-\left(\frac{f_c}{f}\right)^2}$$

$$\Rightarrow\ \frac{150^2c^2}{\omega^2} = 1-\left(\frac{f_c}{f}\right)^2$$

$$\Rightarrow\ \frac{150^2c^2}{4\pi^2f^2} = 1-\frac{f_c^2}{f^2}$$

$$\Rightarrow\ 150^2c^2 = 4\pi^2f^2 - 4\pi^2\cancel{f^2}\cdot\frac{f_c^2}{\cancel{f^2}}$$

$$\Rightarrow\ 150^2c^2 = 4\pi^2f^2 - 4\pi^2f_c^2 = 4\pi^2(f^2-f_c^2)$$

$$\Rightarrow\ (150)^2\times(3\times10^8)^2 = 4\pi^2\left(f^2-(25\times10^9)^2\right)$$

$$\Rightarrow\ f^2-(25\times10^9)^2 = \frac{150^2\times9\times10^{16}}{4\pi^2} = 51.29\times10^{18}$$

$$\Rightarrow\ f^2 = 625\times10^{18} + 51.29\times10^{18} = 676.29\times10^{18}$$

$$\Rightarrow\ \boxed{f = 26.00\ \text{GHz}}$$

(iv) Wave impedance

$$\eta = \frac{\eta'}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}} = \frac{377}{\sqrt{1-\left(\frac{25}{26}\right)^2}} = 1372.55\ \Omega$$

5] An air-filled rectangular waveguide of inside dimension 7 x 3.5 cm operates in the dominant mode TE_{10} mode.

(a) Find the cutoff freq.

(b) Determine phase velocity of the wave in the guide at freq 3.5 GHz.

(c) Determine the guided wavelength at the same freq.

Ans: (a) $f_c = \frac{u'}{2a} = \frac{3\times10^8}{2\times7\times10^{-2}} = 2.14$ GHz.

(given $a = 7$ cm, $b = 3.5$ cm)

(b) $$u_p = \frac{u'}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}} = \frac{3\times10^8}{\sqrt{1-\left(\frac{2.14}{3.5}\right)^2}} = 3.79\times10^8\ \frac{m}{s}$$

(c) $$\lambda = \frac{\lambda'}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}} = \frac{\frac{c}{f}}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}} = \frac{\frac{3\times10^8}{3.5\times10^9}}{\sqrt{1-\left(\frac{2.14}{3.5}\right)^2}}$$

$$\lambda = \frac{0.857\times10^{-1}}{0.7912} = 1.08\times10^{-1}\ \text{meter}$$

$$\boxed{\lambda = 10.8\ \text{Cm}}$$

6) Design a rectangular waveguide at cutoff frequency

$f_c = 9400\,MHz$.

Ans ÷ $(f_c)_{10} = \frac{c}{2a}$

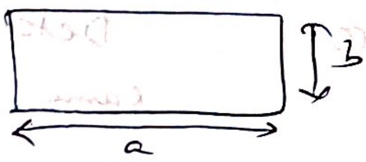
$\Rightarrow \cancel{9400} \times 10^{\cancel{6}} = \frac{3 \times 10^{\cancel{8}}}{2 \times a}$

$\Rightarrow 94 \times 2 \times a = 3$

$\Rightarrow a = \frac{3}{188} = 0.0159$ meter

$\Rightarrow a = 1.59$ cm

For ideal case $\boxed{a = 2b}$



$b = \frac{a}{2} = \frac{1.59}{2} = 0.795$ cm

$b = 0.795$ cm.

∴ Dimension of Rectangular waveguide (1.59×0.795) cm

7) A rectangular waveguide with dimension $\underline{3 \times 2 \text{ cm}}$ operates in the TM_{11} mode at 10 GHz. Determine the characteristic wave impedance.

Ans ÷ Given $a = 3$ cm, $b = 2$ cm.

$$f_c = \frac{u'}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} = \frac{u'}{2}\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}$$

$$\Rightarrow f_c = \frac{3 \times 10^8}{2}\sqrt{\frac{1}{(3 \times 10^{-2})^2} + \frac{1}{(2 \times 10^{-2})^2}} = 1.5 \times 10^8 \sqrt{\frac{10^4}{9} + \frac{10^4}{4}}$$

$$\Rightarrow f_c = 1.5 \times 10^8 \times 10^2 \times 0.60 = 9\,GHz.$$

$$\eta_{TM} = \eta' \sqrt{1-\left(\frac{f_c}{f}\right)^2}$$

$$\eta_{TM} = 377 \sqrt{1-\left(\frac{9}{10}\right)^2}$$

$$\Rightarrow \boxed{\eta_{TM} = 164.33\,\Omega}$$

8) Consider a rectangular waveguide of 8 x 4 cm. Give critical wavelength of TE_{10} = 16 cm, TM_{11} = 7.16 cm, TM_{21} = 5.6 cm for 8 x 4 cm rectangular waveguide. What mode propagates at guided wavelength of (a) 10 cm (b) 5 cm.

Ans :- $f > f_c$ [For propagation on waveguide].

or $\boxed{\lambda < \lambda_c}$

Case-I $\lambda = 10$ cm

(i) TE_{10}, $\lambda_c = 16$ cm, $\because 10 < 16$ cm, TE_{10} ✓ Propagates

(ii) TM_{11}, $\lambda_c = 7.16$ cm, $10 \not< 7.16$ cm, TM_{11} ×

(iii) TM_{21}, $\lambda_c = 5.6$ cm, $10 \not< 5.6$ cm, TM_{21} ×

Case-II $\lambda = 5$ cm

(i) TE_{10}, $\lambda_c = 16$ cm, $5 < 16$ cm, TE_{10} Propagates.

(ii) TM_{11}, $\lambda_c = 7.16$ cm, $5 < 7.16$ cm, TM_{11} ,,

(iii) TM_{21}, $\lambda_c = 5.6$ cm, $5 < 5.6$ cm, TM_{21} ,,