Mathematical Representation of Noise

→ Noise can be defined as random, unwanted interference on transmitted signal. Noise can be classified into 2 broad groups.

(i) External noise

(ii) Internal noise.

External noise may be defined as that type of noise which is generated external to a communication system i.e whose sources are external to the communication system.

So, external noise are interference from other sources like lightening, electrical switching, automobile ignition, other communicating signal etc.

Internal noise is that type of of noise which is generated internally or within communication system or receiver. It is due to thermal motion of electrons (thermal noise) or random emissions, diffusion, recombination of carriers. (Shot noise)

Some sources of noise :-

1) Ex:- Constant agitation at the molecular level. Individual molecules vibrate about their position of equilibrium in a crystal lattice. Conduction electrons of metal wander randomly throughout the volume of metal.

These agitations of molecules are called thermal agitations because they increase with temperature.

In case of a resistor, because of random and erratic wandering of electrons, there will be statistical fluctuations away from neutrality. Thus the distribution of charges may not be uniform ~~and~~ and a voltage difference appears between resistor terminals. The random, erratic, unpredictable voltage which so appears is referred to as thermal resistor noise. This noise increases with temperature and resistance value of resistor.

2) The second type noise results from, when the charge carriers, electrons or holes, enter the junction region from one side, drift or accelerated across the junction and are collected on the other side. This noise is known as Shot noise

→ Shot noise is also encountered as a result of randomness of emission of electrons from a heated surface, generally associated with thermionic devices. [ex:- Vacuum tube devices]

3) → When a signal reaches a receiver, it is greatly attenuated. So it needs amplification. During amplification with active devices (transistors, etc), it may be corrupted by thermal & shot noise.

4) Signals are contaminated by noise in the channel.

→ If noise is added to the signal, it is called additive noise.

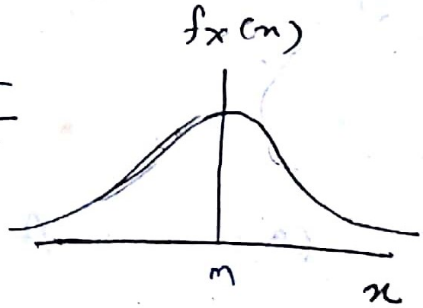
→ If noise multiply the signal, & the effect is called fading.

Note:- Q 1 - 2010 BPUT - 2 mark

1) Write the Probability density function $f(x)$ of the noise.

Ans :- It is assumed that, PDF of noise is gaussian.

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}}$$



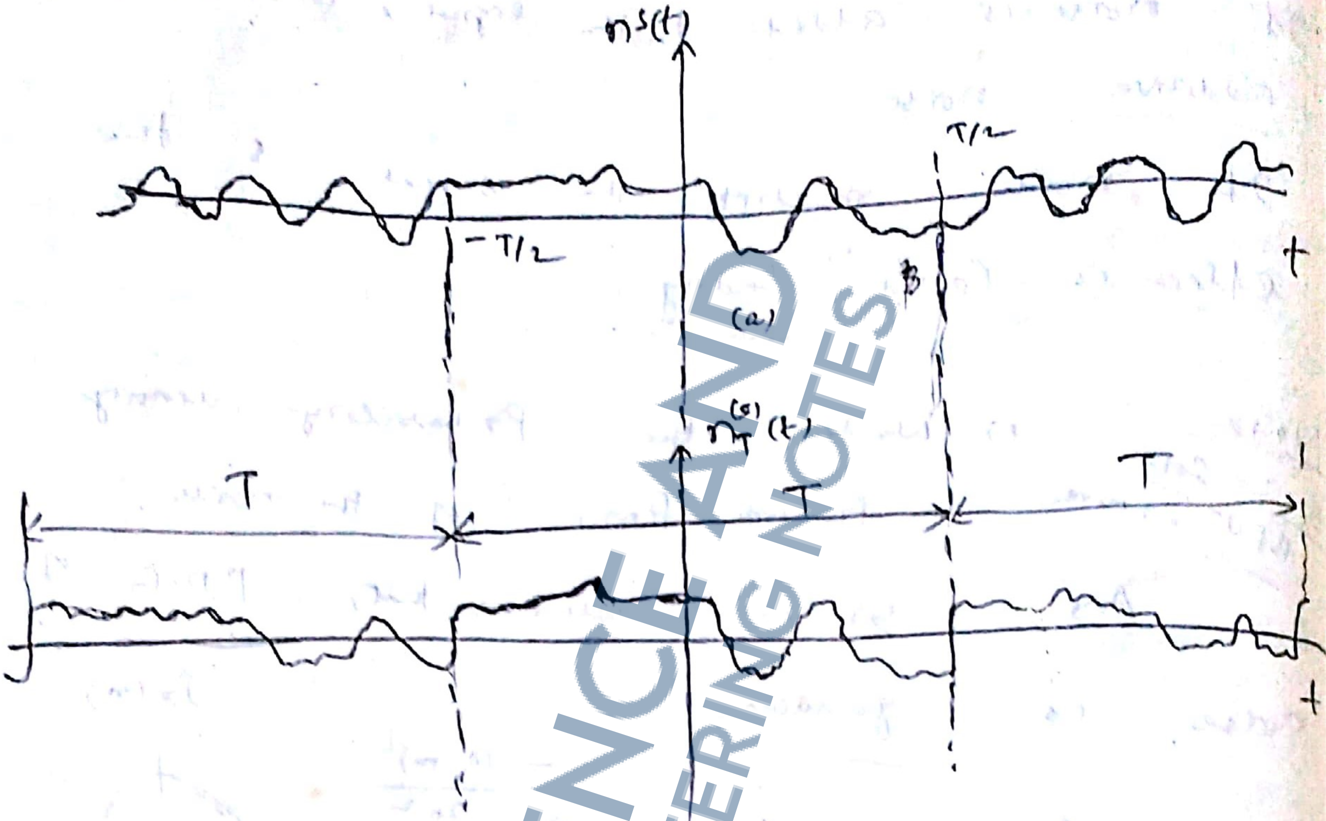
2012 - BPUT 5 marks.

2) Explain Freq - domain representation of noise.

Ans :- Let us select a particular sample function of the noise, and ~~the~~ select from the sample function an interval of duration T, extending from $t = -T/2$ to $t = \frac{T}{2}$.

Such a noise sample function $n^{(s)}(t)$ is shown in fig 1.(a).

Let us generate, as shown in fig 1 (b), a periodic waveform $n_T^{(s)}(t)$ in which the waveform in selected interval is repeated every T sec.



(a) A sample noise waveform

(b) A periodic waveform as generated by repeating the interval in (a) form $-\frac{T}{2}$ to $\frac{T}{2}$.

→ The periodic waveform $n_T^{(s)}(t)$ can be expanded in a Fourier series. The fundamental frequency of the expansion $\Delta f = \frac{1}{T}$ and assuming no d.c component ($a_0 = 0$), we have

$$n_T^{(s)}(t) = \sum_{k=1}^{\infty} \left(a_k \cos 2\pi k \Delta f t + b_k \sin 2\pi k \Delta f t \right)$$

or alternately

$$n_T^{(s)}(t) = \sum_{k=1}^{\infty} C_k \cos (2\pi k \Delta f t + \theta_k)$$

in which a_k, b_k, and c_k are constant coefficients of spectral terms and θ_k is a phase angle.

$$\therefore \; c_k^2 = a_k^2 + b_k^2 \quad \text{——— (1)}$$

$$\theta_k = -\tan^{-1}\left(\frac{b_k}{a_k}\right)$$

→ The power spectrum of expression is shown in fig (c)

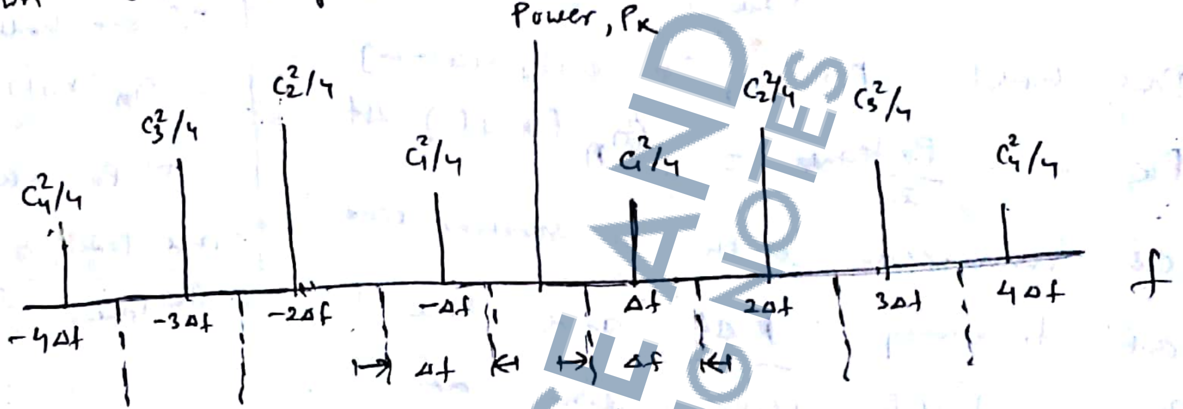


fig c :- The power spectrum of the waveform $n_T^{(s)}$

→ The power associated with each spectral term is $\frac{c_k^2}{2} = \frac{a_k^2 + b_k^2}{2}$.

→ Since a two-sided spectrum shown, each power spectral-line is of height $\frac{c_k^2}{4}$.

→ The frequency axis has been marked off in the intervals Δf and a power spectral line is located at the center of each interval.

→ Power spectral density at frequency $(k\Delta f)$

'n' stands for noise.

$$G_n(k\Delta f) = G_n(-k\Delta f) = \frac{c_k^2}{4\Delta f} = \frac{a_k^2 + b_k^2}{4\Delta f} \quad \text{——— (2)}$$

[$\because$ $PSD = \dfrac{Power}{BW}$) $= \dfrac{\left(\frac{C_k^2}{4}\right)}{\Delta f} = \dfrac{C_k^2}{4\Delta f}$)

$\therefore$ $\boxed{\text{Total Power}}$ P_k associated with frequency interval Δf at frequency $K\Delta f$ is

$$(P_k)_{Total} = 2\, G_n(K\Delta f)\, \Delta f$$

One half Power, [one side spectrum]

$$P_K = \frac{(P_K)_{Total}}{2} = G_n(K\Delta f) \cdot \Delta f$$

is associated with a spectral line at frequency $K\Delta f$. and the other half with a line at freq $-K\Delta f$.

~~Total power~~ ~~$\therefore G_K(K\Delta f) = \frac{P_K}{2\Delta f}$~~

$\because$ one half PSD,

$G_n(K\Delta f) = \dfrac{P_K}{\Delta f}$

$\Rightarrow P_K = G_n(K\Delta f) \cdot \Delta f$

Total Power, $(P_K)_{Total} = 2\, G_n(K\Delta f) \cdot \Delta f$

$\therefore$ $G_n(K\Delta f) = G_n(K\Delta f) =$ Mean power spectral density within each interval.

→ Since periodic noise is random in nature, we consider that a_K, b_K, C_K are not fixed numbers but are instead random variables.

→ Assume $T \to \infty$ ($\Delta f \to 0$), so that the periodic sample functions of noise revert to the actual noise sample function. Then the noise $n(t)$ can be presented as the

$$n(t) = \lim_{\Delta f \to 0} \sum_{K=1}^{\infty} (a_K \cos 2\pi K \Delta f t + b_K \sin 2\pi K \Delta f t)$$

or

$$n(t) = \lim_{\Delta f \to 0} \sum_{K=1}^{\infty} C_K \cos(2\pi K \Delta f t + \theta_K)$$

→ Now we replace C_K^2 by $\overline{C_K^2}$, i.e expected value of the square of the random variable C_K.

→ Further as $\Delta f \to 0$, spectral line get closer and closer finally forming continuous spectrum. So in eqn (2), we replace $K\Delta f$ by the continuous frequency variable f.

and from eqn ①, we have,

$$\overline{C_K^2} = \overline{a_K^2} + \overline{b_K^2}$$

Finally

∴ eqn ② becomes,

$$G_n(f) = \lim_{\Delta f \to 0} \frac{\overline{C_K^2}}{4\Delta f} = \lim_{\Delta f \to 0} \frac{\overline{a_K^2} + \overline{b_K^2}}{4\Delta f}$$

→ Power in frequency range from f_1 to f_2,

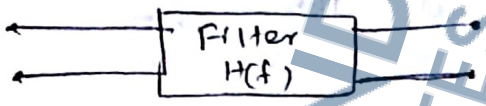
$$P(f_1 \to f_2) = \int_{-f_2}^{-f_1} G_n(f)\,df + \int_{f_1}^{f_2} G_n(f)\,df$$

$$= 2\int_{f_1}^{f_2} G_n(f)\,df$$

Total Power $P_T = \int_{-\infty}^{\infty} G_n(f)\,df = 2\int_{0}^{\infty} G_n(f)\,df$

Effect of filtering on probability density of Gaussian Noise

If gaussian noise $n_i(t)$ is applied to the i/p of a filter, the o/p noise $n_o(t)$ is also gaussian.

$n_i(t)$  $n_o(t)$

(Gaussian), white (Gaussian) , Nonwhite

fig (a):- Gaussian noise $n_i(t)$ applied to a linear filter whose o/p is $n_o(t)$

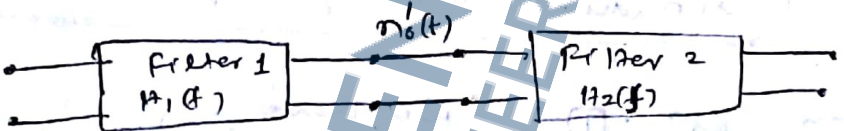
$n_i(t)$  $n_o(t)$

fig:-(b) The filter in fig (a) is split into two parts.

From fig(a),

$$n_o(t) = \int_{-\infty}^{\infty} n_i(\tau)\, h(t-\tau)\, d\tau \qquad \text{—— (1)}$$

$[\because n_o(t) = n_i(t) * h(t)]$

In general, the upper limit may actually be set of $\tau = t$, since variable of integration is τ.

and $h(t-\tau) = 0$, for $\tau > t$.

So eqn (1) becomes,

$$n_o(t) = \int_{-\infty}^{t} n_i(\tau)\, h(t-\tau)\, d\tau \qquad \text{—— (2)}$$

~~Suppose~~ If $n_i(t)$ is Gaussian & white noise

O/P $n_o(t)$ is Gaussian but non white.

For fig (b).

Suppose the filter of transfer function $H(f)$ [fig a] is split into 2 parts, as shown in fig (b), with $H(f) = H_1(f) \cdot H_2(f)$

→ If $n_i(t)$ is white and gaussian, $n_o'(t)$ & $n_o(t)$ are non white gaussian noise.

⇒ Conclusion is If i/p is gaussian noise ~~o/p~~, then o/p of the filter is always gaussian.

White Noise:-

White noise is noise whose power spectral density is uniform over the entire frequency range of interest. The term white is used in analogy ~~the~~ with white light, which is superposition of all visible spectral components. Similarly, white noise consists of all frequency components in equal amount. The power density spectrum of white noise is independent of frequency. It is constant over the entire spectrum (including

Positive and -ve frequencies.

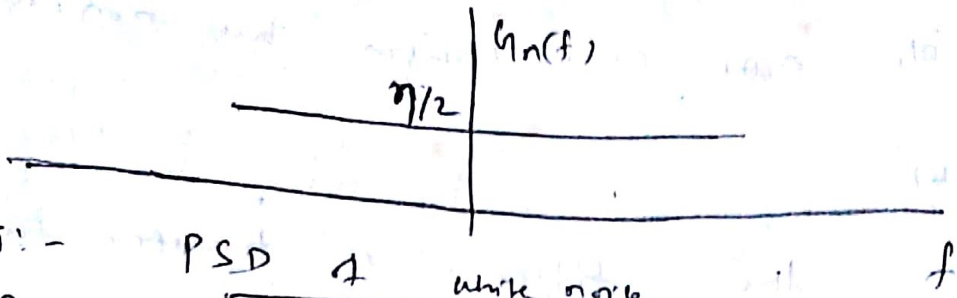


fig:- PSD of white noise.

PSD $\rightarrow$ $\boxed{G_n(f) = \eta/2}$ PSD $\rightarrow$ Power Spectral Density

Where η is a constant.

$\rightarrow$ If the probability of occurrence of a white noise is specified by a Gaussian distribution function, it is called white gaussian noise.

$\rightarrow$ White noise is ideal noise.

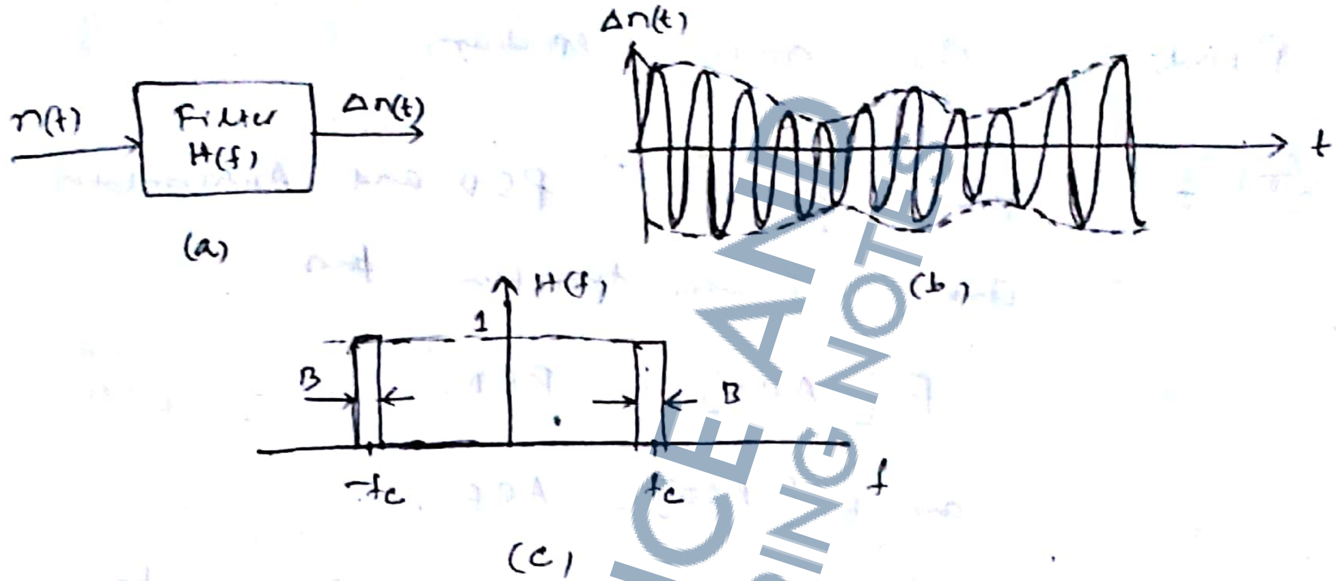
$\rightarrow$ PSD of Thermal noise & Shot noise is independent of operating frequencies, therefore shot noise & thermal noise can be treated as white gaussian noise for all practical purposes.

$\rightarrow$ Thermal noise has PSD which is quite uniform up to frequencies of the order 10^{13} Hz.

$\rightarrow$ Shot noise has PSD which is reasonably constant up to frequencies which are of the order of the reciprocal of transit time of charge carriers across the junction.

Response of a Narrowband filter to Noise :-

If the representation of noise as superposition of spectral components is reasonable one, then when noise is passed through a narrowband filter the o/p of filter look like sinusoid.



As shown in fig (b), the o/p waveform looks like a sinusoid except that, as expected, the amplitude varies randomly.

The spectral range of the envelope of the filter output encompasses the spectral range from $-B/2$ to $B/2$, where B is the filter BW. The avg. frequency of the waveform is the center frequency 'f_c' of the filter. If $B << f_c$, the average amplitude remains constant, and the waveform becomes more and more sinusoidal.

[$B << f_c$, narrowband, allow single freq, sinusoidal shape contains single freq]

Ex:- The autocorrelation function of a noise signal is triangular and defined as

$$R_n(\tau) = \begin{cases} (1-|\tau|), & \text{for } |\tau| < 1 \\ 0, & |\tau| > 1 \end{cases}$$

Find its noise spectrum.

Ans :- We know PSD and Autocorrelation f^n are Fourier transform pair.

$$F[ACF] = PSD$$

and $F^{-1}[PSD] = ACF$.

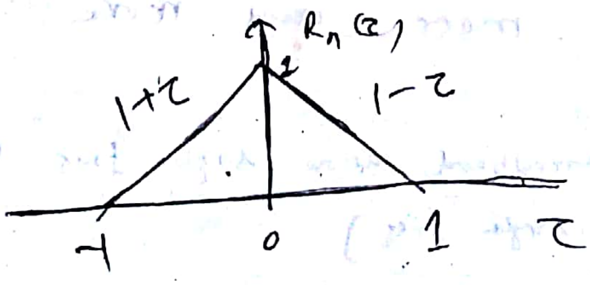
-> Noise power spectrum can be obtained by Fourier transform of $R_n(\tau)$.

$$R_n(\tau) = \begin{cases} 1-(-\tau), & \text{for } -1 < \tau < 0 \\ 1-\tau, & \text{for } 0 \leq \tau < 1 \\ 0, & \text{elsewhere} \end{cases}$$

because, $|\tau| < 1$ means $-1 < \tau < 1$.

and $|\tau| = -\tau$, for $\tau < 0$

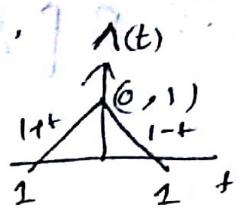
$|\tau| = \tau$, for $\tau \geq 0$.



(Triangular f^n)

1st method :-

Lets find the F.T of triangular f^n,



$$\Lambda(t) = \begin{cases} 1+t, & -1<t<0 \\ 1-t, & 0<t<1 \\ 0, & \text{elsewhere} \end{cases}$$

$$F[\Lambda(t)] = \int_{-\infty}^{\infty} \Lambda(t) \cdot e^{-j2\pi ft}\, dt$$

$$= \int_{-1}^{0} (t+1) \cdot e^{-j2\pi ft}\, dt + \int_{0}^{1} (-t+1)\, e^{-j2\pi ft}\, dt$$

Consider

$$\int t \cdot e^{-j2\pi ft}\, dt$$

$$= t \cdot \frac{e^{-j2\pi ft}}{-j2\pi f} - 1 \cdot \left(\frac{-1}{j2\pi f}\right) \cdot \left(\frac{e^{-j2\pi ft}}{-j2\pi f}\right)$$

$$\left(\because \int u \cdot v = u \cdot v_1 - u' v_2 + u'' v_3 - u''' v_4 + \cdots \right)$$

$$= \frac{-t \cdot e^{-j2\pi ft}}{j2\pi f} + \frac{e^{-j2\pi ft}}{(2\pi f)^2}$$

Now

$$\Lambda(t) = \int_{-1}^{0} t \cdot e^{-j2\pi ft}\, dt + \int_{-1}^{0} e^{-j2\pi ft}\, dt - \int_{0}^{1} t\, e^{-j2\pi ft}\, dt + \int_{0}^{1} e^{-j2\pi ft}\, dt$$

$$= \left[\int_{-1}^{0} t \cdot e^{-j2\pi ft}\, dt - \int_{0}^{1} t \cdot e^{-j2\pi ft}\, dt \right] + \int_{-1}^{1} e^{-j2\pi ft}\, dt$$

$$\therefore F[\Lambda(t)] = \left[\frac{e^{-j2\pi ft}}{(2\pi f)^2} - t\cdot\frac{e^{-j2\pi ft}}{j2\pi f}\right]_{-1}^{0} -$$

$$\left[\frac{e^{-j2\pi ft}}{(2\pi f)^2} - t\frac{e^{-j2\pi ft}}{j2\pi f}\right]_{0}^{1} + \left[\frac{e^{-j2\pi ft}}{-j2\pi f}\right]_{-1}^{1}$$

[∵ Putting the value of $\int t\, e^{-j2\pi ft}\, dt$]

$$\Rightarrow F[\Lambda(t)] = \left[\frac{1}{(2\pi f)^2} - 0 - \frac{e^{+j2\pi f}}{(2\pi f)^2} - \frac{e^{+j2\pi f}}{j2\pi f}\right] -$$

$$\left[\frac{e^{-j2\pi f}}{(2\pi f)^2} - \frac{e^{-j2\pi f}}{j2\pi f} - \frac{1}{(2\pi f)^2} + 0\right] +$$

$$\frac{\left[e^{-j2\pi f} - e^{+j2\pi f}\right]}{-j2\pi f}$$

$$= \frac{2}{(2\pi f)^2} - \frac{e^{j2\pi f}}{(2\pi f)^2} - \frac{e^{-j2\pi f}}{(2\pi f)^2} - \cancel{\frac{e^{+j2\pi f}}{j2\pi f}} + \cancel{\frac{e^{-j2\pi f}}{j2\pi f}}$$

$$- \cancel{\frac{e^{-j2\pi f}}{j2\pi f}} + \cancel{\frac{e^{+j2\pi f}}{j2\pi f}}$$

$$= -\left[\left(\frac{e^{j\pi f}}{2\pi f}\right)^2 + \left(\frac{e^{-j\pi f}}{2\pi f}\right)^2 - 2\cdot\left(\frac{e^{j\pi f}}{2\pi f}\right)\cdot\left(\frac{e^{-j\pi f}}{2\pi f}\right)\right]$$

$$= -\left(\frac{e^{j\pi f}}{2\pi f} - \frac{e^{-j\pi f}}{2\pi f}\right)^2 = \frac{1}{j^2}\times\left(\frac{e^{j\pi f} - e^{-j\pi f}}{2\pi f}\right)^2$$

$$\Rightarrow \quad F[\Lambda(t)] = \left(\frac{e^{j\pi f} - e^{-j\pi f}}{2\pi j f}\right)^2$$

$$= \left(\frac{\frac{e^{j\pi f} - e^{-j\pi f}}{2j}}{\pi f}\right)^2$$

$$= \left(\frac{\sin \pi f}{\pi f}\right)^2$$

$$= (\text{sinc } f)^2$$

$$\Rightarrow \quad F[\Lambda(t)] = \text{sinc}^2 f$$

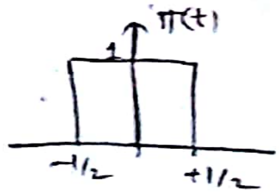
Thus $F[R_n(\tau)] = F[\Lambda(\tau)] = \text{sinc}^2 f$.

$\therefore$ PSD of noise $G_n(f) = \text{sinc}^2 f$ (Ans)

<u>2nd method :-</u>

~~$F[\Lambda(t)] =$~~

Triangular f^n can be obtained by convolution of rectangular pulse.



$$\Lambda(t) = \Pi(t) * \Pi(t)$$

$\Pi(t) =$ Rectangular Pulse.

$$F[\Lambda(t)] = F[\Pi(t) * \Pi(t)]$$

$$= F[\Pi(t)] \cdot F[\Pi(t)]$$

($\because$ F[Convolution in time domain] = Multiplication in freq. domain)

$$F[\Lambda(t)] = \{F[\Pi(t)]\}^2$$

Lets find

$$F[\pi(t)] = \int_{-\infty}^{\infty} \pi(t) \cdot e^{-j2\pi f t}\, dt$$

$$= \int_{-1/2}^{1/2} 1 \cdot e^{-j2\pi f t}\, dt$$

$$= \left[\frac{e^{-j2\pi f t}}{-j2\pi f}\right]_{-1/2}^{1/2}$$

$$= -\frac{1}{j2\pi f} \cdot \left[e^{-j2\pi f \frac{1}{2}} - e^{+j2\pi f \frac{1}{2}}\right]$$

$$= \frac{e^{j\pi f} - e^{-j\pi f}}{2j\pi f}$$

$$= \frac{\left(\dfrac{e^{j\pi f} - e^{-j\pi f}}{2j}\right)}{\pi f}$$

$$= \frac{\sin \pi f}{\pi f}$$

$\therefore$ $F[\pi(t)] = \text{sinc}(f)$

$$F[\Lambda(t)] = \{F[\pi(t)]\}^2 = \text{sinc}^2 f.$$

$\therefore$ PSD of noise $G_n(f) = \text{sinc}^2 f$.

(Ans)

Superposition of noises:-

The concept of power spectrum is useful because it allows us to resolve a deterministic waveform or a random process, $f(t)$ into a sum

$$f(t) = f_1(t) + f_2(t) + \dots$$ in such a manner that superposition of power applies, i.e the power of $f(t)$ is sum of the power $f_1(t), f_2(t), \dots$.

Noise waveform was represented as a superposition of spectral components, all of which are harmonics of some fundamental frequency Δf which, in limit, approaches zero.

$$n(t) = \lim_{\Delta f \to 0} \sum_{k=1}^{\infty} C_k \cos(2\pi k \Delta f\, t + \theta_k)$$

→ Suppose we have 2 noise processes $n_1(t)$ & $n_2(t)$ whose spectral ranges overlap in part or in their entirety. Then the power P_{12} of sum $n_1(t)+n_2(t)$ would be,

$$P_{12} = E\left\{[n_1(t) + n_2(t)]^2\right\}$$

$$= E[n_1^2(t)] + E[n_2^2(t)] + 2E[n_1(t).\, n_2(t)]$$

$$P_{12} = P_1 + P_2 + 2E[n_1(t).\, n_2(t)]$$

where P_1 & P_2 are the powers, respectively, of the noise process $n_1(t)$ & $n_2(t)$ and $E[n_1(t)\, n_2(t)]$, which is

The expected value of the product, is the cross correlation of the processes. Thus superposition of power $P = P_1 + P_2$; continues to apply, provided the processes are uncorrelated.

Note :- $\rho = 0$, i.e.

$\Rightarrow$ Correlation coefficient $= 0$

$$\rho = \frac{E[XY]}{\sigma_X \sigma_Y} = 0$$

$$\Rightarrow E[XY] = 0$$

If $n_1(t)$ & $n_2(t)$ are uncorrelated

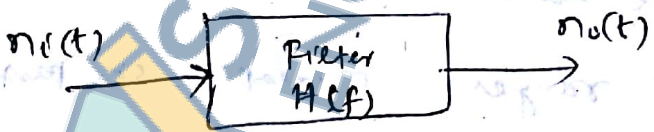
$$E[n_1(t)\, n_2(t)] = 0$$

$$\boxed{P = P_1 + P_2}$$

Ex :- If $n_1(t)$ & $n_2(t)$ were thermal noises of 2 different resistors even on same freq. bands

Then $P_{12} = P_1 + P_2$.

Reln betn o/p PSD & i/p PSD



$$\boxed{G_{n_o}(f) = |H(f)|^2\, G_{n_i}(f)}$$

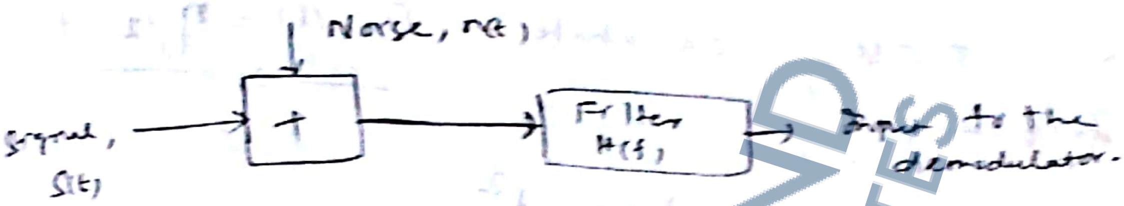
where $G_{n_o}(f)$ = PSD of o/p of filter

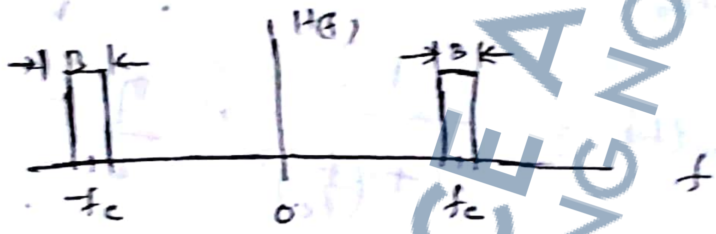
$G_{n_i}(f)$ = PSD of i/p of filter.

$H(f)$ = Transfer function of the filter.

Linear filtering of Noise :- [BPUT - 2012]

In order to minimize the noise power that is presented to the demodulator of the receiving system, filter is introduced before demodulator. [Shown in fig below]





→ The Bandwidth 'B' of the filter is made as narrow as possible so as to avoid transmitting any unnecessary noise to the demodulator.

→ Ex:- In AM system in which the baseband extends to a frequency f_m, the $BW = 2f_m$.

→ In wideband FM, $BW = 2(\Delta f + f_m)$

1) **RC low-pass filter :-**

A RC low-pass filter with 3 dB freq 'f_c' has transfer function

$$H(f) = \frac{1}{1 + j(f/f_c)} \quad -(1)$$

[Refer to - Bogletad for derivation]

If the i/p noise to this filter has PSD $G_{ni}(f)$ and PSD of output noise is $G_{no}(f)$, then

$$G_{no}(f) = |H(f)|^2 \, G_{ni}(f) \quad \text{—} \quad ②$$

If the noise is white $G_{ni}(f) = \eta/2$, for all frequencies,

$$\cancel{G_{no}(f) = |H(f)|^2}$$

$$G_{no}(f) = \frac{1}{1+\left(\frac{f}{f_c}\right)^2} \cdot \frac{\eta}{2} \quad \left[\text{using eqn } ① \,\&\, 2\right]$$

The noise power at the filter output N_o, is

$$N_o = \int_{-\infty}^{\infty} G_{no}(f)\,df = \frac{\eta}{2} \int_{-\infty}^{\infty} \frac{df}{1+\left(\frac{f}{f_c}\right)^2}$$

Put $\frac{f}{f_c} = x$, $df = f_c\,dx$

$f \to \infty$, $x \to \infty$

$f \to -\infty$, $x \to -\infty$

$$= \frac{\eta}{2} \int_{-\infty}^{\infty} \frac{f_c\,dx}{1+x^2}$$

$$= \frac{\eta}{2} \cdot f_c \int_{-\infty}^{\infty} \frac{dx}{1+x^2}$$

$$= \frac{\eta}{2} \cdot f_c \cdot \tan^{-1}[x] \Big|_{-\infty}^{\infty}$$

$$= \frac{\eta}{2} \cdot f_c \times \left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = \frac{\pi}{2} \cdot \eta \cdot f_c$$

$$\boxed{N_0 = \frac{\pi}{2} \eta f_c}$$

2) <u>Rectangular (Ideal) Low-pass filter</u>

A rectangular low pass filter has the transfer function,

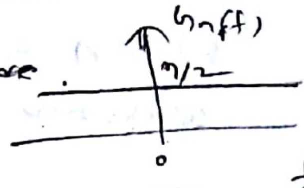
$$H(f) = \begin{cases} 1, & |f| \leq B \\ 0, & \text{elsewhere} \end{cases}$$

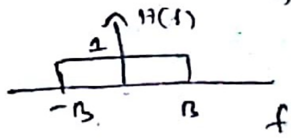
$$= \begin{cases} 1, & -B \leq f \leq B \\ 0, & \text{else where.} \end{cases}$$

Assuming that noise input to the filter is <u>white</u>,

$$G_{ni}(f) = \begin{cases} \frac{\eta}{2}, & -\infty < f < \infty \end{cases}$$

[PSD of input Noise]





$$G_{no}(f) = |H(f)|^2 \cdot G_{ni}(f)$$

$$G_{no}(f) = \begin{cases} \frac{\eta}{2}, & -B \leq f \leq B \\ 0, & \text{elsewhere} \end{cases}$$

$(\because \; H(f) = 1, \; -B \leq f \leq B)$

The o/p noise power

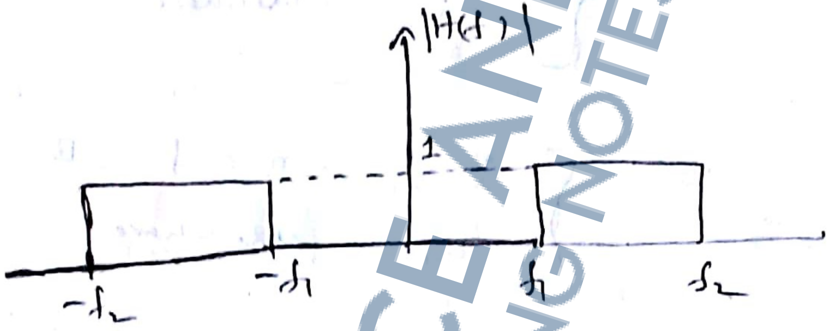
$$N_0 = \int_{-\infty}^{\infty} G_{no}(f)\, df = \int_{-B}^{B} \frac{\eta}{2} \cdot df = \frac{\eta}{2} [f]_{-B}^{B}$$

$$\therefore \quad N_0 = \frac{\eta}{\cancel{2}} \times (\cancel{2}B) = \eta B$$

$$\boxed{N_0 = \eta B}$$

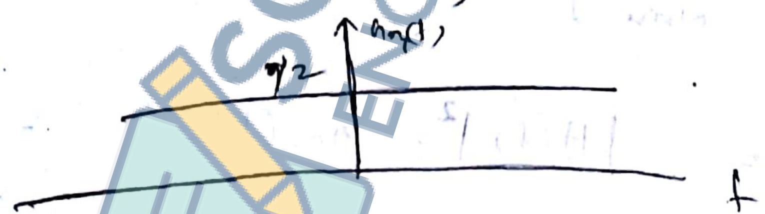
3) **A rectangular Bandpass filter**

A rectangular bandpass filter is shown below.



→ If i/p is white noise,

PSD of white noise, $G_{n_i}(f) = \left\{ \frac{\eta}{2} \right., \quad -\infty < f < \infty$



$$G_{n_0}(f) = \begin{cases} \frac{\eta}{2}, & -f_2 < f < -f_1 \\ \eta/2, & f_1 < f < f_2 \\ 0, & \text{elsewhere.} \end{cases}$$

$\left[\because G_{n_0}(f) = |H(f)|^2 . G_{n_i}(f)\right]$

O/P Noise power

$$N_0 = 2 \times \int_{f_1}^{f_2} \frac{\eta}{2} \, df$$

$$N_0 = \cancel{2} \times \frac{\eta}{\cancel{2}} \times [f]_{f_1}^{f_2}$$

$$\boxed{N_0 = \eta\,(f_2 - f_1)}$$

4) <u>A differentiating filter</u>

A differentiating filter is a n/w which yields at its o/p a waveform which is proportional to the time derivative of its i/p waveform.

Such a n/w has a transfer function

$$H(f) = j 2\pi \tau f \quad \text{— (1)}$$

where τ is a constant factor of proportionality.

If white noise with $G_{ni}(f) = \eta/2$, is passed through such a filter, then the output-noise-power spectrum density is

$$G_{no}(f) = |H(f)|^2 \cdot G_{ni}(f)$$

$$G_{no}(f) = 4\pi^2 \tau^2 f^2 \frac{\eta}{2} \quad \text{— (2)}$$

Note:- If
$F[x(t)] \rightarrow X(\omega)$
$F[\frac{d}{dt} x(t)] = j\omega X(\omega)$
$= j 2\pi f X(f)$
Here
$V_o(t) \propto \frac{d}{dt} x(t)$
$V_o(t) = K \frac{d}{dt}[x(t)]$
$V_o(f) = K . j 2\pi f \; X(f)$
$(\because V_o(f) = F[K \frac{d}{dt} x(t)]$
$= K . j 2\pi f \; X(f))$
But
$V_o(t) = H(f) . X(f)$
$\therefore H(f) = j 2\pi K f$

If the differentiator is followed by a rectangular low pass filter having a bandwidth B, the

[Refer the 2nd filter - $H(f) = \begin{cases} 1, & |f| \le B \\ 0, & \text{elsewhere} \end{cases}$]

noise power at the O/P of the low pass filter is,

$$N_0 = \int_{-B}^{B} 4\pi^2\tau^2 f^2 \frac{\eta}{2}\, df$$

$$= \frac{\cancel{4}^{2}\pi^2 \cdot \tau^2 \times \eta}{\cancel{2}} \cdot \int_{-B}^{B} f^2 df$$

$$= 2\pi^2\tau^2 \times \eta \times \left.\frac{f^3}{3}\right]_{-B}^{B}$$

$$= \frac{2}{3}\pi^2\tau^2 \times \eta \times 2B^3$$

$$\boxed{N_0 = \frac{4\pi^2\tau^2\eta B^3}{3}}$$

[5] An Integrator

Let noise $n(t)$ be applied to the i/p of an integrator at time $t=0$. We calculate the noise power at the integrator output at a time $t=T$.

A network which performs the operation of integration has a transfer function $\frac{1}{j\omega\tau}$ where

Note :- If $F[x(t)] = X(\omega)$

$$F\left[\int x(t)\right] = \frac{1}{j\omega} X(\omega)$$

τ is a constant. A delay by an interval T is represented by $e^{-j\omega T}$. Hence a n/w which performs an integration over an interval T may be represented by a n/w whose transfer function is

$$H(f) = \frac{1}{j\omega\tau} - \frac{e^{-j\omega T}}{j\omega\tau} = \frac{1-e^{-j\omega T}}{j\omega\tau}$$

To find $|H(f)|^2$,

$$H(f) = \frac{1-e^{-j\omega T}}{j\omega\tau} = \frac{1-(\cos(-\omega T)+j\sin(-\omega T))}{j\omega\tau}$$

$$= \frac{1-(\cos\omega T - j\sin\omega T)}{j\omega\tau}$$

$$H(f) = \frac{(1-\cos\omega T)+j\sin\omega T}{j\omega\tau}$$

$$|H(f)|^2 = \frac{(1-\cos\omega T)^2+\sin^2\omega T}{\omega^2\tau^2}$$

$$= \frac{1+\cos^2\omega T - 2\cos\omega T+\sin^2\omega T}{\omega^2\tau^2}$$

$$= \frac{1+1-2\cos\omega T}{\omega^2\tau^2} \qquad (\because \cos^2\omega T+\sin^2\omega T = 1)$$

$$|H(f)|^2 = \frac{2(1-\cos\omega T)}{\omega^2\tau^2}$$

$$= \frac{2\times 2\sin^2\frac{\omega T}{2}}{\omega^2\tau^2}$$

~~$\because \cos\theta = 1-2\sin$~~
$\cos 2\theta = 1-2\sin^2\theta$
$\cos\theta = 1-2\sin^2\frac{\theta}{2}$
$1-\cos\theta = 2\sin^2\frac{\theta}{2}$

$$|H(f)|^2 = \frac{4 \sin^2 \frac{\omega T}{2}}{\omega^2 \tau^2}$$

$$= \frac{\cancel{4} \cdot \sin^2 \frac{\cancel{2}\pi f \cdot T}{\cancel{2}}}{\cancel{4}\pi^2 f^2 \tau^2} \quad (\because \omega = 2\pi f)$$

$$= \frac{\sin^2 \pi T f}{\pi^2 f^2 \tau^2} \times \left(\frac{T^2}{T^2}\right)$$

$$= \frac{\sin^2 \pi T f}{\pi^2 f^2 T^2} \times \left(\frac{T^2}{\tau^2}\right) \quad \left(\text{Interchanging } T \text{ and } \tau\right)$$

$$|H(f)|^2 = \left(\frac{\sin \pi T f}{\pi T f}\right)^2 \times \frac{T^2}{\tau^2}$$

The noise power output of such a filter with white i/p noise of power spectral density $\eta/2$ is

$$N_o = \int_{-\infty}^{\infty} \frac{\eta}{2} |H(f)|^2 \, df$$

$$\left(\because N_o = \int_{-\infty}^{\infty} G_{N_o}(f)\, df = \int_{-\infty}^{\infty} G_{ni}(f) \cdot |H(f)|^2 df\right)$$

$$= \int_{-\infty}^{\infty} \frac{\eta}{2} \times \left(\frac{T^2}{\tau^2}\right) \times \left(\frac{\sin \pi T f}{\pi T f}\right)^2 df$$

Put $\pi T f = x$

$df = \frac{dx}{\pi T}$

$f \to -\infty, \quad x \to -\infty$

$f \to \infty, \quad x \to \infty$

$$= \frac{1}{\pi T}\left(\frac{\eta}{2} \times \frac{T^2}{\tau^2}\right) \int_{-\infty}^{\infty} \left(\frac{\sin x}{x}\right)^2 dx$$

$$\because \int_{-\infty}^{\infty} \left(\frac{\sin x}{x}\right)^2 dx = \pi$$

$$= \frac{1}{\cancel{\pi} \cancel{T}} \times \frac{\eta}{2} \times \frac{T^{\cancel{2}}}{\tau^2} \times \cancel{\pi}$$

$$\boxed{N_o = \frac{\eta T}{2\tau^2}}$$

Noise Bandwidth

Consider that white noise is present at the i/p to the receiver and a filter with transfer function $H(f)$ centered at f_0, such as indicated by solid plot.

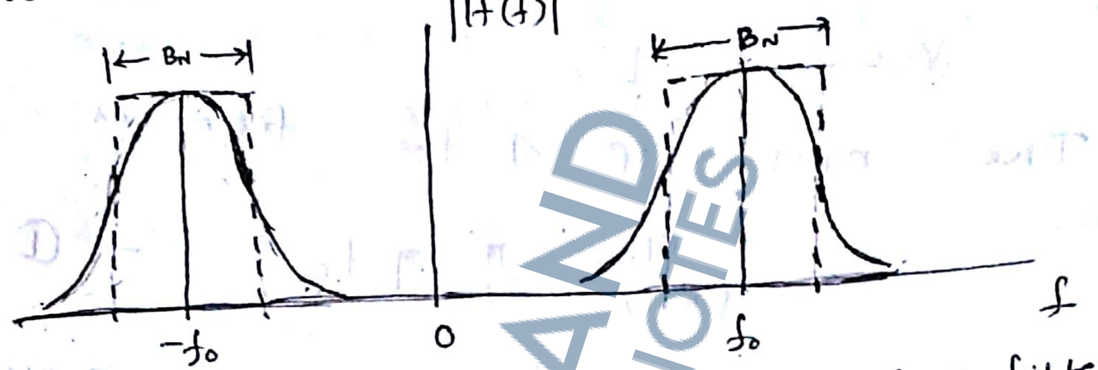


fig:- Illustration of the noise BW of a filter

→ It is being used to restrict the noise power actually passed on to the receiver.

→ The rectangular filter is shown by the dotted plot. This filter is also centered at f_0.

Let the rectangular filter bandwidth B_N be adjusted so that real filter and the rectangular filter transmit the same noise power. Then the bandwidth 'B_N' is called the noise BW of the real filter.

Thus, the noise BW is then, BW of idealized (rectangular) filter which passes the same noise power as does the real filter.

Consider RC low-pass filter

$$H(f) = \frac{1}{1+j\,f/f_c}$$

At $f=0$; $H(f)$ attains its maximum value $=1$.

The noise o/p of the filter is

$$N_{o\,(RC)} = \frac{\pi}{2}\cdot\eta f_c \quad \text{— ①} \quad [\text{Derived earlier}]$$

In the presence of such noise, a rectangular low-pass filter with $H(f)=1$, over its bandpass B_N, ie $H(f)=\begin{cases}1, & -B_N<f<B_N\\ 0, & \text{elsewhere}\end{cases}$

would yield o/p noise power

$$N_o\,(\text{Rectangular}) = \frac{\eta}{\cancel{2}}\times\cancel{2}B_N = \eta B_N \quad \text{— ②}$$

[Derived earlier]

equating eq^n ① & ② i.e setting

$N_o(RC) = N_o$ (Rectangular), we find the noise BW B_N.

$$\frac{\pi}{2}\cdot\cancel{\eta}\cdot f_c = \cancel{\eta}\,B_N$$

$$\Rightarrow \boxed{B_N = \frac{\pi}{2} f_c}$$

Thus, the noise BW of the RC filter is $\frac{\pi}{2}$ (1.57) times its 3dB bandwidth 'f_c'.

Ex:- 1) Given a white noise of magnitude $\eta = 0.001 \frac{MW}{HZ}$ is fed to following.

(a) an RC low pass filter of R = 1000 ohm and C = 0.1 μF

(b) an Ideal low pass filter of bandwidth = 1000 HZ (B)

(C) a differentiator followed by an ideal low pass filter defined in (b)

For differentiator consider proportionality const τ = 0.01 unit. Find o/p noise power in each case. How does result change in each case if low pass cut-off frequency is doubled in each case?

Ans:- (a) Cut off frequency,

$$f_c = \frac{1}{2\pi RC} = \frac{1}{2\pi \times 1000 \times 0.1 \times 10^{-6}} = \frac{5000}{\pi} HZ$$

For RC LPF,

$$N_o = \frac{\pi}{2} \cdot \eta \cdot f_c = \frac{\pi}{2} \times (0.001) \times 10^{-6} \frac{Watt}{HZ} \times \frac{5000}{\pi} HZ$$

$$\boxed{N_o = 2.5 \ MW}$$

If cutt off frequency doubles, $f_c \rightarrow 2f_c$, $(N_o)_{New} = 2 \times N_o$

∴ o/p power will be doubled = 5 MW.

(b) For Ideal LPF, [Rectangular]

$$N_0 = \eta \times B$$

$$= 0.001 \times 10^{-6} \times 1000$$

$$= 10^{-6}$$

$$\boxed{N_0 = 1 \mu W}$$

If cut off freq. doubles, $B \to 2B$.

$$(N_0)_{new} = 2 \times N_0 = 2 \times 1 \mu W = 2 \mu W$$

(c) For a differentiator, followed by Ideal LPF

$$N_0 = \frac{4\pi^2}{3} \cdot \eta \cdot \tau^2 B^3$$

$$= \frac{4 \times \pi^2}{3} \times 0.001 \times 10^{-6} \times (0.01)^2 \times (1000)^3$$

$$= \frac{4 \times \pi^2}{3} \times 10^{-3} \times 10^{-6} \times 10^{-4} \times 10^9$$

$$= 13.16 \times 10^{-4}$$

$$= \frac{13.16 \times 10^{-4} \times 10}{10}$$

$$N_0 = 1.316 \text{ mW}$$

If the cut off freq is doubled, $N_0 \propto B^3$ $\longrightarrow$ $(N_0)_{New} \propto (2B)^3$

$\therefore$ $(N_0)_{new}$ = 8 times the original value

$$= 8 \times 1.316 \text{ mW}$$

$$(N_0)_{new} = 10.528 \text{ mW}$$

Note:- When differentiator is used, noise power is increased

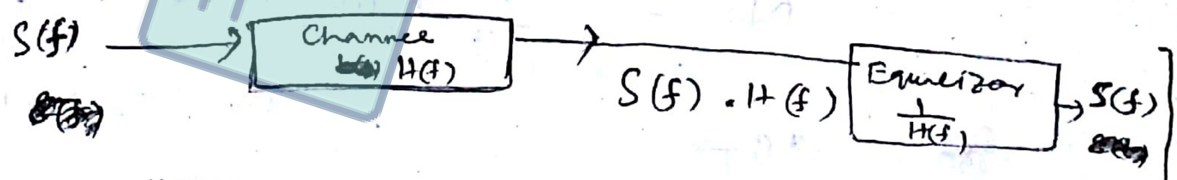
Q1 A low pass (within 4000 HZ) signal of strength $\underline{0.001 W}$ passes through a distorting channel defined as $H(f) = \frac{4000}{j4000 + f}$ and is also corrupted with a additive white Gaussian noise of magnitude $\underline{10^{-8}}$ $\frac{Watt}{HZ}$. At the receiver side there is an equalizer which exactly matches the channel within frequency of interest (up to 4000 HZ) and zero elsewhere. Find SNR at the o/p of equalizer.

<u>Ans</u>:- (Note: [Equalizer is used for <u>noise cancellation</u>]

An equalizer when exactly matches channel has a <u>transfer function</u> which is <u>inverse of channel</u> and the signal at its o/p is <u>undistorted</u> signal.

[Ex:- ~~S(f) m(t)~~

If signal is S(t), its freq domain representation S(f)



Hence signal ~~energy~~ power at equalizer o/p $\boxed{S_o = 0.001 W}$
(same as i/p)

Noise power, $N_o = \int_{-\infty}^{\infty} |H_{eq}(f)|^2 . G_{n_i}(f) \, df$

$H_{equalizer}(f) = \frac{1}{H(f)} = \frac{j4000 + f}{4000}$ $\qquad [\because G_{n_o}(f) = |H(f)|^2 G_{n_i}(f)$

$$N_0 = \int_{-4000}^{+4000} \left| \frac{j4000 + f}{4000} \right|^2 \cdot (10^{-8})\, df \cdot$$ [freq of interest given up to 4000]

$$= \int_{-4000}^{+4000} \frac{f^2 + 4000^2}{4000^2} \times 10^{-8}\, df$$

$$= 2 \times \int_0^{4000} \left(1 + \frac{f^2}{16 \times 10^6}\right) \times 10^{-8}\, df$$ [changing the limit]

$$= 2 \times 10^{-8} \left[f + \frac{f^3}{3 \times 16 \times 10^6} \right]_0^{4000}$$

$$= 2 \times 10^{-8} \left[4000 + \frac{4000^3}{3 \times 16 \times 10^6} \right]$$

$$= 2 \times 10^{-8} \left(\frac{4 \times 10^3 \times 48 \times 10^6 + 4^3 \times 10^9}{48 \times 10^6} \right)$$

$$= \frac{\cancel{2} \times 10^{-8} \times \cancel{4} \times 10^9 \left[48 + \cancel{64}^{16} \right] \rightarrow 4^2}{\cancel{48}\,6 \times 10^6}$$

$$= \frac{\cancel{16}\,4 \times 10}{6 \times 10^6}$$

$$N_0 = \frac{32}{3} \times 10^{-5} \text{ Watt}$$

~~In dB,~~

$$\cancel{N_0 = 10 \log \left(\frac{32}{3} \times 10^{-5}\right)}$$

$$SNR = \frac{S_0}{N_0} = \frac{0.001}{\frac{32}{3} \times 10^{-5}} = \frac{10^{-3} \times 3}{32 \times 10^{-5}} = \frac{300}{32}$$

In dB, $(SNR)_{dB} = 10 \log \left(\frac{300}{32}\right) = 9.7197 \text{ dB}$

(Ans)

Internal - II - Question - Soln

Q) Twenty four voice signals are sampled uniformly and then have to time division multiplexed. The highest frequency component for each voice signal is equal to 3.4 KHZ

Now

(i) If the signals are pulse amplitude modulated using sampling rate, what would be the minm Channel BW required.

(ii) If the signals are PCM modulated with 8-bit encoder, what would be the sampling rate? The bit rate of the system is given as 1.5×10^6 bits/sec.

Ans:- (i) For TDM PAM,

$$\boxed{BW = n f_m}$$

where n = no of channels multiplexed,
f_m = maxm freq in the signals.

$$BW = 24 \times 3.4 \text{ KHZ} = 81.6 \text{ KHZ}.$$

(ii)

$$\text{bit rate } (r) = 1.5 \times 10^6 \frac{\text{bits}}{\text{sec}}.$$

Since there are 24 channels, bit rate of individual channel,

$$r(\text{one channel}) = \frac{1.5 \times 10^6}{24} \frac{\text{bits}}{\text{sec}}.$$

Each sample is encoded using 8 bits, the samples per second will be

$$\text{sample/sec} = \frac{\left(\frac{\text{bits}}{\text{sec}}\right)}{(\text{bits/sample})} \quad \text{i.e } \frac{\text{bits}}{\text{sec}} \times \frac{\text{sample}}{\text{bits}} = .$$

$$f_s = \text{Sampling freq} = \text{samples/sec} = \frac{1.5 \times 10^6}{24 \times 8} = 7812.5 \text{ samples/second.}$$