Module-3: Electromagnetic Wave Propagation

(Chapter-6)

→ The existence of EM waves, predicted by Maxwell's equations, was first investigated by Heinrich Hertz. Hertz succeeded in generating and detecting radio waves.

→ In general, waves are means of transporting energy or information.

→ Typical examples of EM waves include radio waves, TV signals, radar beams, and light rays. All forms of EM energy share three fundamental characteristics

1. They travel at high velocity.
2. In traveling, they assume the properties of waves.
3. They radiate outward from a source.

In this chapter, our major goal is to solve Maxwell's equations and describe EM wave motion in the following media:

1. Free space ($\sigma = 0$, $\epsilon = \epsilon_0$, $\mu = \mu_0$) — σ: conductivity
2. Lossless dielectric ($\sigma = 0$, $\epsilon = \epsilon_r \epsilon_0$, $\mu = \mu_r \mu_0$)
3. Lossy dielectric ($\sigma \neq 0$, $\epsilon = \epsilon_r \epsilon_0$, $\mu = \mu_r \mu_0$)
4. Good Conductors ($\sigma \simeq \infty$, $\epsilon = \epsilon_0$, $\mu = \mu_r \mu_0$)

→ Case 3, for lossy dielectrics, is the most general case and will be considered first. Other cases will be derived from it as special cases by changing the values of σ, ϵ, and μ.

Wave Propagation in lossy dielectric

→ A lossy dielectric is a medium in which an EM wave, as it propagates, loses power owing to imperfect dielectric.

→ In other words, a lossy dielectric is a partially conducting medium (imperfect dielectric or imperfect conductor) with $\sigma \neq 0$, as distinct from lossless dielectric in which $\sigma = 0$.

→ Consider a linear, isotropic, homogeneous, lossy dielectric medium that is charge free ($\rho_v = 0$). Supressing the time factor $e^{j\omega t}$, from Table 5.2, the maxwell's equations we have

$$\nabla \cdot D_s = \rho_{vs} \Rightarrow \nabla \cdot E_s = 0 \quad \text{—(6.1)}$$

$$(\because \quad D_s = \epsilon E_s \ \& \ \rho_{vs} = 0)$$

$$\nabla \cdot B_s = 0 \Rightarrow \nabla \cdot H_s = 0 \quad \text{—(6.2)}$$

$$(\because \quad B_s = \mu H_s)$$

$$\nabla \times E_s = -j\omega B_s \Rightarrow \nabla \times E_s = -j\omega\mu H_s \quad \text{—(6.3)}$$

$$\nabla \times H_s = J_s + j\omega D_s \Rightarrow \nabla \times H_s = \sigma E_s + j\omega\epsilon E_s$$

$$\Rightarrow \nabla \times H_s = (\sigma + j\omega\epsilon) E_s \quad \text{—(6.4)}$$

Taking curl of both sides of eqn (6.3), we have

$$\nabla \times (\nabla \times E_s) = -j\omega\mu (\nabla \times H_s) \quad \text{— (6.5)}$$

Applying the Vector identity

$$\nabla \times (\nabla \times A) = \nabla (\nabla \cdot A) - \nabla^2 A,$$ and eqn (6.4) we have

$$\nabla (\nabla \cdot E_s) - \nabla^2 E_s = -j\omega\mu (\sigma + j\omega\epsilon) E_s \quad \text{— (6.6)}$$

From eqn (6.1), $\nabla \cdot E_s = 0$

∴ Eqn (6.6) becomes

$$\cancel{-} \nabla^2 E_s = \cancel{-} j\omega\mu (\sigma + j\omega\epsilon) E_s$$

$$\Rightarrow \nabla^2 E_s - j\omega\mu (\sigma + j\omega\epsilon) E_s = 0$$

$$\Rightarrow \boxed{\nabla^2 E_s - \gamma^2 E_s = 0} \quad \text{— (6.7)}$$

where

$$\gamma^2 = j\omega\mu (\sigma + j\omega\epsilon) \quad \text{— (6.8)}$$, is called the propagation constant of the medium. By similar procedure, it can be shown that for H field,

$$\boxed{\nabla^2 H_s - \gamma^2 H_s = 0} \quad \text{— (6.9)}$$

(2.1)

Equation (6.7) and (6.9) are known as homogeneous vector Helmholtz's equations or simply vector wave equations. In Cartesian coordinates, eqn (6.7), for example, is equivalent to three scalar wave equations, one for each component of E along a_x, a_y and a_z.

Since γ is a complex quantity, from eqn (6.8), we may let

$$\boxed{\gamma = \alpha + j\beta} \quad \text{——— (6.10)}$$

From eqn 6.8, $\gamma^2 = j\omega\mu(\sigma + j\omega\epsilon)$

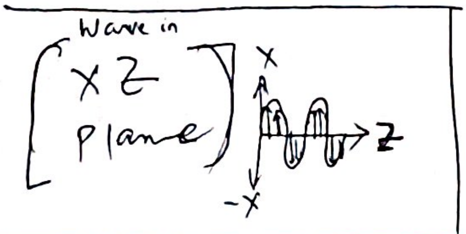
$$\Rightarrow \gamma = \sqrt{j\omega\mu(\sigma + j\omega\epsilon)} = \alpha + j\beta$$

To find the root over of a complex number, as we do in our mathematics, we obtain

$$\alpha = \omega\sqrt{\frac{\mu\epsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2} - 1\right]} \quad \text{— (6.11)}$$

$$\beta = \omega\sqrt{\frac{\mu\epsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2} + 1\right]} \quad \text{— (6.12)}$$

Assuming, the wave propagates along $+a_z$ and that E_s has only x-component, then



$$E_s = E_{xs}(z)\, a_x \quad \text{——— (6.13)}$$

Substituting this into equation (6.7)

$$\left(\nabla^2 - \gamma^2\right) E_{xs}(z) = 0 \quad \text{——— (6.14)}$$

$$\Rightarrow \frac{\partial^2 E_{xs}(z)}{\partial x^2} + \frac{\partial^2 E_{xs}(z)}{\partial y^2} + \frac{\partial^2 E_{xs}(z)}{\partial z^2} - \gamma^2 E_{xs}(z) = 0$$

$$\Rightarrow \underset{(\because f^n \text{ of } z \text{ w.r.to } x}{0} + \underset{,\ f^n \text{ of } z \text{ w.r.to } y)}{0} + \left[\frac{\partial^2}{\partial z^2} - \gamma^2\right] E_{xs}(z) = 0$$

$$\Rightarrow \left[\frac{d^2}{dz^2} - \gamma^2\right] E_{xs}(z) = 0 \quad \text{——— (6.15)}$$

This is a linear homogenous differential equation, which has the following solution

$$E_{xs}(z) = E_0 e^{-\gamma z} + E_0' e^{\gamma z} \quad \text{——— (6.16)}$$

Where E_0 and E_0' are the constants.

Since $\underline{e^{\gamma z}}$ denotes a wave traveling along $(-a_z)$ where as we assume wave $\underline{\text{propagation along}}$ $(+a_z)$, therefore $E_0' = 0$.

$e^{\gamma z} \to$ Along $-a_z$

$e^{-\gamma z} \to$ Along $+a_z$

$\therefore$ Equation (6.16) becomes

$$E_{xs}(z) = E_0 e^{-\gamma z} \quad \text{——— (6.17)}$$

Inserting time factor $\left(e^{j\omega t}\right)$ into equation 6.17, we have

$$E(z,t) = Re\left[E_{xs}(z) \cdot e^{j\omega t} a_x\right] \quad \text{——— (6.18)}$$

Putting eqn (6.17) into equation (6.18), we have

$$E(z,t) = Re\left[E_0 \, e^{-\gamma z} \, e^{j\omega t} \, a_x\right]$$

$$= Re\left[E_0 \, . \, e^{-(\alpha + j\beta)z} \, . \, e^{j\omega t} \, a_x\right]$$

$$= Re\left[E_0 \, e^{-\alpha z} \, . \, e^{j(\omega t - \beta z)} \, a_x\right]$$

$$= E_0 \, e^{-\alpha z} \, Re\left[e^{j(\omega t - \beta z)} a_x\right]$$

$$\Rightarrow \boxed{E(z,t) = E_0 \, e^{-\alpha z} \cos(\omega t - \beta z) \, a_x} \quad -(6.19)$$

A sketch of $|E|$ at times $t=0$ and $t=\Delta t$ is portrayed in fig 6.1, where it is evident that E has only x-component and it is traveling in the $+z$ direction.

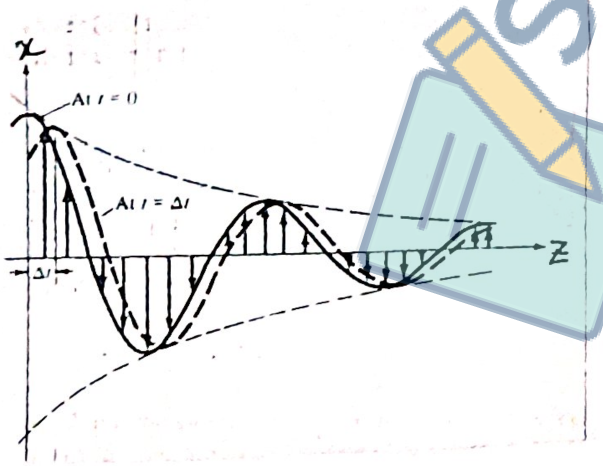


Fig 6.1: An E-field with an x-component travelling in the z-direction at times $t=0$ and $t=\Delta t$; arrows indicate instantaneous values of E

- Having obtained $E(z,t)$ we obtain $H(z,t)$ ~~either~~ by taking similar steps to solve equation (6.9). We will eventually arrive at

$$H(z,t) = Re\left[H_0 e^{-\alpha z} e^{j(\omega t-\beta z)} a_y\right] \quad -(6.20)$$

Where
$$H_0 = \frac{E_0}{\eta} \quad \text{———} \quad (6.21)$$

and η is a complex quantity known as the intrinsic impedance, in ohm, of the medium.

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}} = |\eta| \angle \theta_\eta = |\eta| e^{j\theta_\eta} \quad -(6.22)$$

(In polar form)

with
$$|\eta| = \frac{\sqrt{\mu/\epsilon}}{\left[1+\left(\frac{\sigma}{\omega\epsilon}\right)^2\right]^{\frac{1}{4}}}, \quad \tan 2\theta_\eta = \frac{\sigma}{\omega\epsilon} \quad -(6.23)$$

Substituting eqn (6.21) & (6.22) into eqn (6.20), we have

$$H(z,t) = Re\left[\frac{E_0}{|\eta| e^{j\theta_\eta}} e^{-\alpha z} e^{j(\omega t-\beta z)} a_y\right]$$

or
$$\boxed{H(z,t) = \frac{E_0}{|\eta|} e^{-\alpha z} \cdot \cos(\omega t - \beta z - \theta_\eta)\, a_y} \quad -(6.24)$$

$$\left[\because Re\, e^{j(\omega t-\beta z-\theta_\eta)} = \cos(\omega t - \beta z - \theta_\eta)\right]$$

From eqn (6.19) and (6.24), ~~it is~~ it is clear that the wave is propagating along a_z, and it decreases or attenuates in amplitude by a factor $e^{-\alpha z}$, and hence 'α' is known as the attenuation constant or attenuation coefficient of the medium. Its unit is nepers per meter (Np/m) and can be expressed in decibels per meter (dB/m)

$$1\ Np = 20 \log_{10} e = 8.686\ dB \quad -(6.28)$$

An attenuation of 1 neper denotes a reduction to (e^{-1}) of the original value.

→ From eqn (6.11), we notice that if $\sigma = 0$, as in the case of lossless medium and free space, $\alpha = 0$ and wave is not attenuated as it propagates.

→ The quantity β is a measure of the phase shift per unit length in radian per meter.

→ is called the phase constant or wave number.

→ We know $v = f\lambda$ −(6.26)

and $\omega = 2\pi f$ −(6.27)

v = velocity
f = freq
λ = wavelength

$$\Rightarrow \frac{\omega}{v} = \frac{2\pi \cancel{f}}{\cancel{f}\lambda} = \frac{2\pi}{\lambda} = \beta$$

$$\therefore \quad \boxed{\beta = \frac{2\pi}{\lambda}} \quad \text{—— (6.28)}$$

and $$\boxed{\frac{\omega}{u} = \beta} \quad \text{—— (6.29)}$$

or $$\boxed{U = \frac{\omega}{\beta}} \quad \text{—— (6.30)}$$

→ From eqs (6.19) and (6.24), it is observed that E and H are out of phase by θ_η at any instant of time due to complex intrinsic impedance of the medium. Thus at any time, E leads H by θ_η.

→ Finally, we notice that the ratio of the magnitude of the conduction current density (J_c) to that of displacement current density (J_d) in a lossy medium is

$$\frac{|J_{cs}|}{|J_{ds}|} = \frac{|\sigma E_s|}{|j\omega \epsilon E_s|} = \frac{\sigma}{|j\omega\epsilon|} = \frac{\sigma}{\omega\epsilon} = \tan\theta$$

$$\left(\because \; J = \sigma E, \; J_d = \frac{\partial D}{\partial t} = j\omega . D = j\omega . \epsilon E \right)$$

or $$\boxed{\tan\theta = \frac{\sigma}{\omega\epsilon}} \quad \text{—— (6.31)}$$

Where $\tan\theta$ is known as the loss tangent and 'θ' is the loss angle of the medium.

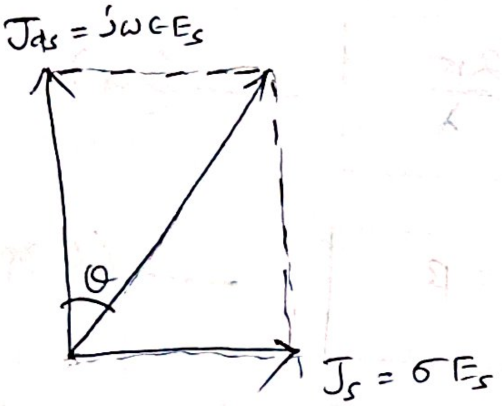


Fig 6.2: Loss angle of a lossy medium

→ A medium is said to be a good (lossless) dielectric if $\tan\theta$ is very small ($\sigma << \omega\epsilon$) or a good conductor if $\tan\theta$ is very large ($\sigma >> \omega\epsilon$)

→ The characteristic behavior of a medium depends not only on its constitutive parameters σ, ϵ, and μ but also on the frequency of operation. A medium that is regarded as a good conductor at low frequency may be a good dielectric at high frequency.

→ Note from eqn (6.23) and (6.31)

$$\theta = 2\theta_\eta \qquad (6.32)$$

From equation (6.4)

$$\nabla \times H_s = (\sigma + j\omega\epsilon) E_s = j\omega\epsilon\left(1 - j\frac{\sigma}{\omega\epsilon}\right) E_s$$

$$\Rightarrow \boxed{\nabla \times H_s = j\omega\epsilon_c E_s} \qquad (6.33)$$

Where

$$\epsilon_c = \epsilon\left(1 - j\frac{\sigma}{\omega\epsilon}\right) \quad \text{——— (6.34)}$$

or

$$\epsilon_c = \epsilon' - j\epsilon''$$

with $\epsilon' = \epsilon$, $\epsilon'' = \dfrac{\sigma}{\omega}$; ϵ_c is called complex permittivity of the medium. We observe that ratio of ϵ'' to ϵ is the loss tangent of the medm.

$$\boxed{\tan\theta = \frac{\epsilon''}{\epsilon} = \frac{\sigma}{\omega\epsilon}} \quad \text{——— (6.35)}$$

We will consider wave propagation in media of other types that may be regarded as special cases of what we have considered here.

Plane waves in lossless dielectrics

In a lossless dielectric, $\boxed{\sigma << \omega\epsilon}$. It is a special case of wave propagation in lossy dielectric except that

$$\boxed{\sigma \simeq 0, \quad \epsilon = \epsilon_0\epsilon_r, \quad \mu = \mu_0\mu_r} \quad \text{—(6.36)}$$

Substituting these into eqn (6.11) and (6.12), gives

$$\left.\begin{aligned} \alpha &= 0 \\ \beta &= \omega\sqrt{\mu\epsilon} \end{aligned}\right\} \quad \text{——— (6.37a)}$$

$$\left. u = \frac{\omega}{\beta} = \frac{\omega}{\omega\sqrt{\mu\epsilon}} = \frac{1}{\sqrt{\mu\epsilon}}, \quad \lambda = \frac{2\pi}{\beta} \right\} \text{—(6.37b)}$$

From eqn (6.22) & (6.23)

$$\eta = \sqrt{\frac{\mu}{\epsilon}}\angle 0^\circ \quad \text{——— (6.38)}$$

(Since, $\tan 2\theta_\eta = \frac{\sigma}{\omega\epsilon} = \frac{0}{\omega\epsilon} = 0^\circ$

$\theta_\eta = 0^\circ$)

Since $\theta_\eta = 0$, From eqn (6.19) and (6.24),

E and H are in time phase with each other.

Plane waves in free space

Plane waves in free space comprise a special case of wave propagation in lossy dielectric / lossless dielectric.

In this case

$$\boxed{\sigma = 0, \quad \epsilon = \epsilon_0, \quad \mu = \mu_0} \quad \text{——} (6.39)$$

Thus replacing ϵ by ϵ_0, μ by μ_0 in eqn (6.37), we have

$$\alpha = 0, \quad \beta = \omega\sqrt{\mu_0\epsilon_0} = \frac{\omega}{c} \quad \text{——} (6.40a)$$

$$u = \frac{1}{\sqrt{\mu_0\epsilon_0}} = c, \quad \lambda = \frac{2\pi}{\beta} \quad \text{——} (6.40b)$$

$\because$ $c = \frac{1}{\sqrt{\mu_0\epsilon_0}}$

$\frac{1}{c} = \sqrt{\mu_0\epsilon_0}$

Where

$c \simeq 3\times10^8 \frac{m}{sec}$, the speed of light in a vacuum.

The fact that EM waves travel in free space at the speed of light is significant. It provides some evidence that light is the manifestation of an EM wave. In other words, light is characteristically electromagnetic.

→ Eqn (6.38) becomes,

$$\eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} \quad \text{———} \quad (6.41)$$

where η_0 is called the intrinsic impedance of free space is given by

$$\eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = 120\pi \simeq 377\,\Omega \quad \text{———} \quad (6.42)$$

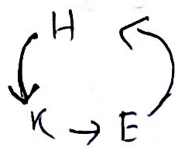
→ From eqn (6.19), and (6.24)

$$E = E_0 \cos(\omega t - \beta z)\, a_x \quad \text{———} \quad (6.43a)$$

and
$$\left.\begin{aligned} H &= \frac{E_0}{\eta_0} \cos(\omega t - \beta z)\, a_y \\ H &= H_0 \cos(\omega t - \beta z)\, a_y \end{aligned}\right\} \quad \text{———} \quad (6.43b)$$

$$\frac{|E|}{|H|} = \frac{E_0}{H_0} = \frac{E_0}{\frac{E_0}{\eta_0}} = \eta_0 \quad \text{———} \quad (6.44)$$

The plot of E and H are shown in Figure 6.3(a). In general, if a_E, a_H, and a_K are unit vectors along E-field, the H-field, and the direction of wave propagation it can be shown that



$$a_E \times a_H = a_K \quad \text{———} \quad (6.45)$$

Both E and H fields (EM waves) are everywhere normal to the direction of wave propagation, a_K.

That means that the fields lie on a plane that is transverse or orthogonal to the direction of wave propagation. They form an EM wave that has no electric or magnetic field components along the direction of propagation; such a wave is called transverse electromagnetic (TEM) wave.

A combination of E and H is called a uniform plane wave because E (or H) has the same magnitude throughout any transverse plane, defined by $z =$ constant. The direction in which the electric field points is the polarization of a TEM wave. The wave in eqn (6.49), for example, is polarized in the x- direction.

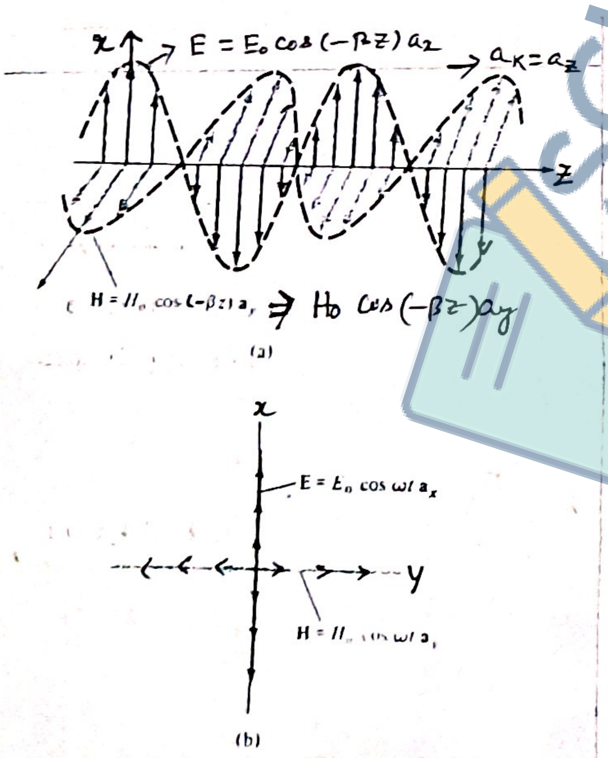


Fig 6.3: Plots of E and H (a) as function of z at t=0; (b) at z=0. The arrow indicates instantaneous values.

Plane waves in good Conductors

Plane waves in good conductors comprise another special cases of that considered for lossy dielectric.

A perfect, or good conductor, is one in which $\sigma >> \omega\epsilon$ so that $\frac{\sigma}{\omega\epsilon} >> 1$; that is,

$$\boxed{\sigma \simeq \infty, \quad \epsilon = \epsilon_0, \quad \mu = \mu_0\mu_r} \quad \text{——} (6.46)$$

Hence eq^ns (6.12) and (6.13) becomes

$$\alpha = \omega\sqrt{\frac{\mu\cancel{\epsilon}}{2} \times \frac{\sigma}{\omega\cancel{\epsilon}}}$$

$$= \frac{\sqrt{\omega}}{\cancel{\omega}} \times \sqrt{\frac{\mu\sigma}{2}} \times \frac{1}{\cancel{\sqrt{\omega}}}$$

From (6.11), if $\frac{\sigma}{\omega\epsilon} >> 1$

$$\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2} - 1 \simeq \frac{\sigma}{\omega\epsilon} - 1 \simeq \frac{\sigma}{\omega\epsilon}$$

$$\alpha = \sqrt{\frac{\omega\mu\sigma}{2}}$$

Also,

$$\beta = \sqrt{\frac{\omega\mu\sigma}{2}} \quad \text{——} (6.47)$$

Putting $\omega = 2\pi f$

$$\alpha = \beta = \sqrt{\pi f \mu\sigma} \quad \text{——} (6.48)$$

$$u = \frac{\omega}{\beta} = \frac{\omega}{\sqrt{\frac{\omega\mu\sigma}{2}}} = \frac{\sqrt{2} \times \cancel{\sqrt{\omega}}\sqrt{\omega}}{\cancel{\sqrt{\omega}}\sqrt{\mu\sigma}} = \sqrt{\frac{2\omega}{\mu\sigma}}$$

$$u = \sqrt{\frac{2\omega}{\mu\sigma}} \quad \text{——} 6.49(a)$$

$$\lambda = \frac{2\pi}{\beta} \quad \text{——} 6.49(b)$$

From eqⁿ (6.22)

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}} = \sqrt{\frac{j\omega\mu}{\sigma}} \quad -6.50(a)$$

$\because \sigma >> \omega\epsilon$

$$\eta = \sqrt{\frac{\omega\mu}{\sigma}} \angle 45^\circ \quad -6.50(b)$$

$\because \sqrt{Z} = \sqrt{r} \angle(\phi/2)$

Here

$Z = \sqrt{\frac{j\omega\mu}{\sigma}} = \sqrt{x+jy}$

$x = 0, \; y = \frac{\omega\mu}{\sigma}$

$r = \sqrt{x^2+y^2} = \frac{\omega\mu}{\sigma}$

$\phi = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}(\infty) = 90^\circ$

$\frac{\phi}{2} = 45^\circ$

$\therefore \theta_\eta = 45^\circ$

Thus, E field leads H field by 45°. If

$$E = E_0 e^{-\alpha z} \cos(\omega t - \beta z) a_x$$

then

$$H = \frac{E_0}{\sqrt{\frac{\mu\omega}{\sigma}}} e^{-\alpha z} \cos(\omega t - \beta z - 45^\circ) a_y \quad -(6.51)$$

(From eqⁿ (6.19) and (6.24))

$\because \eta = \sqrt{\frac{\omega\mu}{\sigma}}$

→ From eqⁿ (6.51), it is observed that, as E (or H) travels in conducting medium, its amplitude is attenuated by the factor $e^{-\alpha z}$.

The distance δ, shown in figure 6.4, through which the wave amplitude decreases to a factor e^{-1} (about 37% of the original value) is called Skin depth or Penetration depth of the medium; that is

$$e^{-\alpha z} = e^{-1}$$

$$\Rightarrow e^{-\alpha \delta} = e^{-1}$$

$$\Rightarrow \alpha \delta = 1$$

$$\Rightarrow \boxed{\delta = \frac{1}{\alpha}} \qquad (6.52)$$

The skin depth (δ) is a measure of the depth to which EM wave can penetrate the medium.

→ Eqn (6.52) is generally valid for any material medium.

→ For good conductors,

$$\alpha = \sqrt{\pi f \mu \sigma} \qquad \text{(from eqn (6.48))}$$

Thus,

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma}} = \frac{1}{\alpha} \qquad (6.53)$$

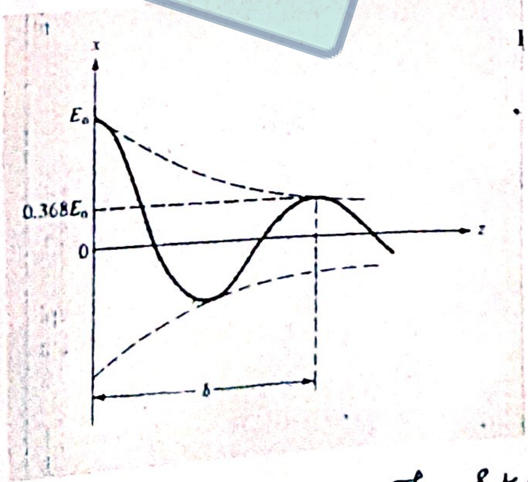


Fig 6.4: Illustration of Skin depth

Example 6.1

A lossy dielectric has an intrinsic impedance of $200 \angle 30^\circ \Omega$ at a particular radian freq. ω. If at that frequency, the plane wave propagating through the dielectric has the ~~magnitude~~ magnetic field component

$$H = 10 e^{-\alpha x} \cos(\omega t - \tfrac{1}{2}x)\, a_y \ \frac{A}{m}$$

find E and α. Determine the skin depth and wave polarization.

Ans:- Given, $\eta = 200 \angle 30^\circ = |\eta| \angle \theta_\eta$

$$\therefore \quad |\eta| = 200, \quad \theta_\eta = 30^\circ \quad \text{——— (1)}$$

The wave travels along a_x so that

$$a_k = a_x, \quad a_H = a_y \ (\text{Given})$$

(Refer eqn (6.45), i.e

$z \to y \to x$

$$a_E \times a_H = a_x$$

$$\Rightarrow a_E \times a_y = a_x$$

$$\Rightarrow -(a_y \times a_E) = a_x$$

$$\Rightarrow a_y \times a_E = -a_x$$

So if $a_E = -a_z$, then the eqn is satisfied. $\left[\text{because } a_y \times (-a_z) = -a_y \times a_z = -a_x\right]$

$\therefore \quad a_E = -a_z$ —— (2)

$H_0 = 10$, <u>We know</u>

$$\eta = \frac{E_0}{H_0} = \frac{E_0}{10}$$

$$\Rightarrow 200\angle 30^\circ = \frac{E_0}{10}$$

$$\Rightarrow E_0 = 2000\angle 30^\circ = 2000\, e^{j\frac{\pi}{6}}$$

Except for the amplitude and phase difference, E and H have the same form.

<u>Referring eqn (6.17) and (6.18)</u>

$$E = Re\left[E_0\, e^{-\gamma z}\, e^{j\omega t}\, a_x\right]$$

<u>Here</u>, direction of propagation is along 'x' instead of z.

Direction of Electric field along $(-a_z)$

So E becomes

$$E = Re\left[E_0\, e^{-\gamma x}\, e^{j\omega t}\,(-a_z)\right]$$

$$= -Re\left[2000\, e^{j(\pi/6)} \cdot e^{-\gamma x}\, e^{j\omega t}\, a_z\right]$$

$$= -Re\left[\underline{2000}\; \underline{\underline{e^{j\frac{\pi}{6}}}} \cdot \underline{e^{-\alpha x}} \cdot \underline{\underline{e^{-j\beta x}}} \cdot \underline{\underline{e^{j\omega t}}}\, a_z\right]$$

$$E = -2000\, e^{-\alpha x} \cos\left(\omega t - \beta x + \frac{\pi}{6}\right) a_z$$

From eqn of H, $\beta = \frac{1}{2}$

$$\therefore \boxed{E = -2 e^{-\alpha x} \cos\left(\omega t - \frac{x}{2} + \frac{\pi}{6}\right) a_z \ \frac{kV}{m}}.$$

$$\alpha = \omega \sqrt{\frac{\mu \epsilon}{2}\left[\sqrt{1 + \left(\frac{\sigma}{\omega \epsilon}\right)^2} - 1\right]}$$

$$\beta = \omega \sqrt{\frac{\mu \epsilon}{2}\left[\sqrt{1 + \left(\frac{\sigma}{\omega \epsilon}\right)^2} + 1\right]}$$

But

$$\frac{\sigma}{\omega \epsilon} = \tan 2\theta_\eta \qquad \left(\text{Eqn } 6.23\right)$$

But $\theta_\eta = 30°$

$$\frac{\sigma}{\omega \epsilon} = \tan 60° = \sqrt{3}$$

$$\frac{\alpha}{\beta} = \frac{\sqrt{\sqrt{1+3} - 1}}{\sqrt{\sqrt{1+3} + 1}} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \alpha = \frac{\beta}{\sqrt{3}} = \frac{\left(\frac{1}{2}\right)}{\sqrt{3}} = \frac{1}{2\sqrt{3}} = 0.2887$$

$$\Rightarrow \boxed{\alpha = 0.2887 \ \frac{Np}{m}}$$

and <u>Skin depth</u> $\delta = \frac{1}{\alpha} = \frac{1}{\left(\frac{1}{2\sqrt{3}}\right)}$

$\Rightarrow \boxed{\delta = 2\sqrt{3} = 3.464 \text{ meter}}$

$\therefore$ Since the wave has only E_z component; hence it is polarized along the z-direction.

<u>Example 6.2</u>

In a <u>lossless dielectric</u> for which $\underline{\eta = 60\pi}$, $\underline{\mu_r = 1}$, and $H = -0.1 \cos(\omega t - z) a_x + 0.5 \sin(\omega t - z) a_y \ \frac{A}{m}$, Calculate <u>$\epsilon_r$, ω and E.</u>

<u>Ans</u> :- For <u>lossless dielectric</u>

$\boxed{\sigma = 0, \ \alpha = 0,}$ and $\boxed{\beta = 1}$ (From expression of H)

$\cos(\omega t - \beta z) \rightarrow \cos(\omega t - z)$

$\Rightarrow \beta = 1$

$\therefore$

$\eta = \sqrt{\frac{\mu}{\epsilon}}$ (From eqn 6.38)

$= \sqrt{\frac{\mu_0}{\epsilon_0}} \sqrt{\frac{\mu_r}{\epsilon_r}} = 120\pi \times \sqrt{\frac{1}{\epsilon_r}}$

$\Rightarrow \cancel{60\pi} = \cancel{120\pi}^{\,2} \times \frac{1}{\sqrt{\epsilon_r}}$

$\Rightarrow \sqrt{\epsilon_r} = 2 \Rightarrow \boxed{\epsilon_r = 4}$

$$\Rightarrow \quad v = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\epsilon}}$$

$$\Rightarrow \quad \omega = \frac{\beta}{\sqrt{\mu\epsilon}} = \frac{1}{\sqrt{\mu_0\epsilon_0}\sqrt{\mu_r\epsilon_r}}$$

$$\Rightarrow \quad \omega = \frac{3\times10^8}{\sqrt{1\times4}}$$

$\frac{1}{\sqrt{\mu_0\epsilon_0}} = c$ = Velocity of light in free space

$$\Rightarrow \quad \boxed{\omega = 1.5\times10^8 \ \frac{rad}{sec}}$$

To find E

From Maxwell's eqn $\nabla \times H = J + \frac{\partial D}{\partial t}$

$$\nabla \times H = \sigma E + \frac{\partial}{\partial t}(\epsilon E)$$

$\sigma \to 0$, $\quad \nabla \times H = \epsilon \frac{\partial E}{\partial t}$

$$\Rightarrow \quad E = \frac{1}{\epsilon}\int (\nabla \times H)\, dt$$

$$\nabla \times H = \begin{vmatrix} a_x & a_y & a_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & 0 \end{vmatrix} = a_x\left(-\frac{\partial H_y}{\partial z}\right) - a_y\left(-\frac{\partial H_x}{\partial z}\right) + a_z\left(\cancelto{0}{\frac{\partial}{\partial x} H_y} - \cancelto{0}{\frac{\partial}{\partial y} H_x}\right)$$

$(\because \; H_y = 0.5 \sin(\omega t - z)$, $H_x = -0.1 \cos(\omega t - z))$

$$\Rightarrow \nabla \times H = -\frac{\partial H_y}{\partial z} a_x + \frac{\partial}{\partial z} H_x a_y$$

$$\Rightarrow \nabla \times H = \frac{-\partial}{\partial z}\left[0.5 \sin(\omega t - z)\right] a_x + \frac{\partial}{\partial z}\left[-0.1 \cos(\omega t - z)\right] a_y$$

$$\Rightarrow \nabla \times H = -0.5 \cos(\omega t - z) \times (-1)\, a_x + (-0.1)(-\sin(\omega t - z)) \times (-1)\, a_y$$

$$\Rightarrow \nabla \times H = 0.5 \cos(\omega t - z) a_x - 0.1 \sin(\omega t - z) a_y .$$

$$\therefore \quad E = \frac{1}{\epsilon} \int (\nabla \times H)\, dt$$

$$= \frac{1}{\epsilon} \int \left[0.5 \cos(\omega t - z) a_x - 0.1 \sin(\omega t - z) a_y\right] dt$$

$$= \frac{1}{\epsilon_0 \epsilon_r}\left[\frac{0.5}{\omega} \sin(\omega t - z) a_x + \frac{0.1}{\omega} \cos(\omega t - z) a_y\right]$$

$$= \frac{1}{8.854 \times 10^{-12} \times 4}\left[\frac{0.5}{1.5 \times 10^8} \sin(\omega t - z) a_x + \frac{0.1}{1.5 \times 10^8} \cos(\omega t - z) a_y\right]$$

$$E = 94.11 \sin(\omega t - z) a_x + 18.82 \cos(\omega t - z) a_y \ \frac{V}{m} .$$

Example 6.3 :- A Uniform plane wave propagating in a medium has

$$E = 2 e^{-\alpha z} \sin(10^8 t - \beta z) a_y \ \frac{V}{m}$$

If the medium is characterized by $\epsilon_r = 1$, $\mu_r = 20$, and $\sigma = 3 \frac{S}{m}$. find α, β, and H.

Ans :- We need to determine the loss tangent to be able to tell whether the medium is a lossy dielectric or a good conductor.

$$\frac{\sigma}{\omega \epsilon} = \frac{3}{10^8 \times \epsilon_0 \epsilon_r}$$

$\because \sin(10^8 t - \beta z)$

$\omega = 10^8$

$$= \frac{3}{10^8 \times \frac{10^{-9}}{36\pi} \times 1}$$

$$= 108\pi \times 10$$

$$= 1080\pi$$

$$\frac{\sigma}{\omega \epsilon} = 3393 >> 1$$

Since $\frac{\sigma}{\omega \epsilon} >> 1$, showing that the medium may be regarded as a good conductor at the frequency of operation. Hence, for a good conductor,

$$\alpha = \beta = \sqrt{\frac{\mu \omega \sigma}{2}}$$

From eqn (6.47)

$$= \sqrt{\frac{\mu_0 \mu_r \omega \sigma}{2}}$$

$$= \sqrt{\frac{4\pi \times 10^{-7} \times 20 \times 10^8 \times 3}{2}} = 61.4$$

$\therefore \quad \alpha = \beta = 61.4$

$$\alpha = 61.4 \ \frac{Np}{m}, \qquad \beta = 61.4 \ \frac{rad}{m}$$

$$|\eta| = \sqrt{\frac{\mu\omega}{\sigma}} = \sqrt{\frac{\mu_0 \mu_r \omega}{\sigma}} \qquad [\text{Eq}^n\ (6.50)$$

$$= \sqrt{\frac{4\pi \times 10^{-7} \times 20 \times 10^{8}}{3}}$$

$$\Rightarrow \quad |\eta| = \sqrt{\frac{800\pi}{3}} = 28.94$$

From eq^n 6.50 (b), $\theta_\eta = 45^\circ$ [For good conductor]

or

$$\tan 2\theta_\eta = \frac{\sigma}{\omega\epsilon} = 3393$$

$$\Rightarrow \quad 2\theta_\eta = \tan^{-1}(3393) \rightarrow \text{radian}$$

$$\Rightarrow \quad \theta_\eta = \frac{89.99}{2} \simeq 45^\circ$$

$$\boxed{H = H_0 \, e^{-\alpha z} \cos(\omega t - \beta z - 45^\circ) \, a_H}$$

$$a_H = a_k \times a_E$$

$$= a_z \times a_y$$

$$\boxed{a_H = -a_x}$$

And $H_0 = \dfrac{E_0}{|\eta|} = \dfrac{2}{28.94} = 69.1 \times 10^{-3}$

Thus,

$$H = -69.1 \, e^{-61.4z} \sin\left(10^8 t - 61.42 z - \frac{\pi}{4}\right) a_x \ \frac{mA}{m}$$

Example 6.4 A Plane wave

$E = E_0 \cos(\omega t - \beta z) a_x$ is incident on a good conductor at $z \geq 0$. Find the current density on the conductor.

Ans:- Since the current density

$J = \sigma E$, we expect J to satisfy the wave eqn in eqn (6.7)

i.e $\nabla^2 E_s - \gamma^2 E_s = 0$

$\Rightarrow \cancel{\nabla^2 \sigma E_s - \gamma^2}$

$\Rightarrow \nabla^2 \left(\frac{J_s}{\sigma}\right) - \gamma^2 \left(\frac{J_s}{\sigma}\right) = 0$

$\Rightarrow \nabla^2 J_s - \gamma^2 J_s = 0$

Also the incident E has only an x-component and varies with z. Hence $J = J_x(z,t) a_x$

and

$$\frac{d^2}{dz^2} J_{sx} - \gamma^2 J_{sx} = 0$$

which is a ordinary differential equation with solution

$$J_{sx} = A e^{-\gamma z} + B e^{+\gamma z}$$

The constant B must be zero because J_{sx} is finite as $z \to \infty$.

$$J_{sx} = A e^{-\gamma z}$$

Also in a good conductor,

$\sigma >> \omega\epsilon$, $\underline{\alpha = \beta}$

$\therefore \gamma = \alpha + j\beta = \alpha + j\alpha = \alpha(1+j)$

$$\gamma = \frac{1+j}{\delta} \qquad \left(\because \alpha = \frac{1}{\delta}\right)$$

$$\therefore J_{sx} = A e^{-\left(\frac{1+j}{\delta}\right)z}$$

or $$\boxed{J_{sx} = J_{sx}(0)\, e^{-(1+j)z/\delta}}$$

Where $J_{sx}(0)$ is the current density on the conductor surface.

Skin Effect

Referring to plane waves in good conductor section,

$$\eta = \sqrt{\frac{\omega\mu}{\sigma}} \angle 45^\circ \quad \text{--- (1)} \quad [\text{Equation } 6.50(b)]$$

But, $$\alpha = \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\frac{\omega\mu}{\sigma} \times \frac{\sigma^2}{2}} = \sqrt{\frac{\omega\mu}{\sigma}} \times \frac{\sigma}{\sqrt{2}}$$

$$\Rightarrow \sqrt{\frac{\omega\mu}{\sigma}} = \frac{\sqrt{2}\,\alpha}{\sigma} \quad \text{--- (6.54)}$$

$$\therefore \eta = \frac{\sqrt{2}\,\alpha}{\sigma} e^{j\frac{\pi}{4}} \qquad \left(\text{Putting (6.54) in eqn (1)}\right)$$

$$= \frac{\sqrt{2}}{\sigma\delta} e^{j\frac{\pi}{4}} \qquad \left[\because \alpha = \frac{1}{\delta}\right]$$

$$\therefore \quad \eta = \frac{\sqrt{2} \times \left(\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right)}{\sigma \delta}$$

$$e^{j\frac{\pi}{4}} = \cos\frac{\pi}{4} + j\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$$

$$\therefore \quad \boxed{\eta = \frac{1+j}{\sigma \delta}} \quad \text{——— (6.55)}$$

Noting that for good conductor we have $\alpha = \beta = \frac{1}{\delta}$, eqn (6.51) can be written as

$$E = E_0 e^{-\alpha z} \cos(\omega t - \beta z) a_x$$

$$\Rightarrow E = E_0 e^{-\frac{z}{\delta}} \cos\left(\omega t - \frac{z}{\delta}\right) a_x \quad \text{——— (6.56)}$$

Eqn (6.56) is showing that 'δ' measures the exponential damping of the waves as it travels through the conductor.

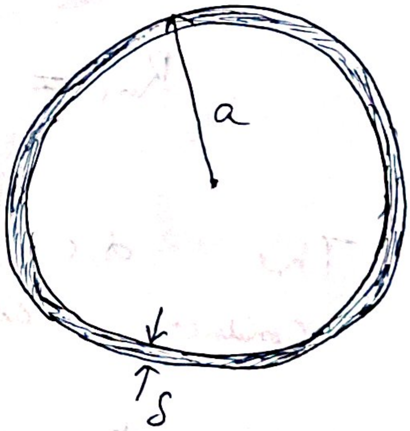
<u>From</u> eqn (6.53)

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma}} = \frac{1}{\alpha}$$

So the skin depth decreases with increase in frequency. So for high freq (microwaves) waves good conductors provides attenuation $[f\uparrow, \delta\downarrow, \alpha\uparrow]$. Therefore E and H can hardly propagate through good conductor.

Thus transmission lines provides attenuation to high frequency waves, hence we use wave guides for microwave signals.

The <u>phenomenon</u> whereby field intensity in a conductor rapidly decreases is known as <u>skin effect.</u> It is a tendency of charges to migrate from the bulk of the conducting material to the surface, resulting in higher resistance. The fields and associated currents are confined to a very thin layer (skin) of the conducting surface. For a wire of radius 'a', for example, it is a good approximation at high frequencies to assume that all of the current flows in the circular ring of thickness 'δ' as shown in figure 6.5.



<u>Fig</u>. 6.5: Skin depth at high frequencies, $\delta << a$.

→ Although skin effect causes attenuation, it is used to advantage in many applications. For example, because the skin depth in <u>silver</u> is very small, the difference in performance between <u>pure silver component</u> and a <u>silver-plated brass</u> component is negligible, so silver plating is often used to reduce the material cost of waveguide components.

→ Skin depth is useful in calculating the ac resistance due to skin effect.

We define surface or skin resistance (R_s) as real part of 'η' for good conductor. Thus from eqn (6.55)

$$R_s = \frac{1}{\sigma \delta} = \frac{\sqrt{\pi f \mu \sigma}}{\sigma} = \sqrt{\frac{\pi f \mu}{\sigma}} \quad \text{---(6.57)}$$

$$\left(\because \delta = \frac{1}{\sqrt{\pi f \mu \sigma}} \right)$$

The ac resistance of the conductor is given by

$$R_{ac} = \frac{l}{\sigma \delta w} = \frac{R_s l}{w} \quad \text{---(6.58)}$$

$\because R_s = \frac{1}{\sigma \delta}$ and $w = 2\pi a$

The d.c resistance of conductor is given by

$$R_{dc} = \frac{l}{\sigma S} \quad \text{---(6.59)}$$

$\because$ We know

$R = \frac{\rho L}{A}$, Here

$R = \frac{1 l}{\sigma S}$ | $\rho \to \frac{1}{\sigma}$, $L \to l$, $A \to S$

$$\therefore \frac{R_{ac}}{R_{dc}} = \frac{\cancel{l}}{\cancel{\sigma} \delta w} \times \frac{\cancel{\sigma} S}{\cancel{l}}$$

$$\frac{R_{ac}}{R_{dc}} = \frac{\cancel{\pi} \cancel{a^2} a}{\delta \times 2\cancel{\pi} \cancel{a^2}} = \frac{a}{2\delta} \quad \text{— (6.60)}$$

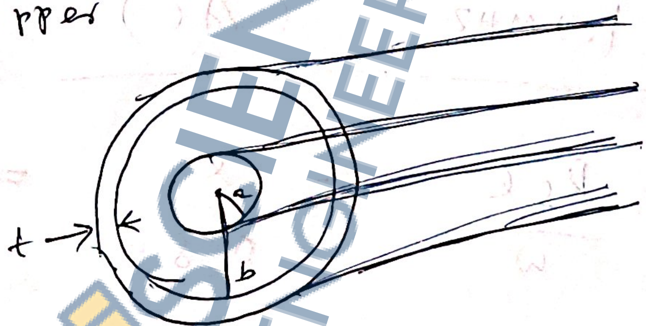
$$\Rightarrow \frac{R_{ac}}{R_{dc}} = \frac{a}{2}\sqrt{\pi f \mu \sigma} \quad \text{— (6.61)}$$

Since $\delta << a$ at high frequencies, (Refer eqn (6.60))

this show R_{ac} is far greater than R_{dc}.

Example 6.5

For a copper coaxial cable, let



$a = 2mm$

$b = 6\,mm$

$t = 1\,mm$

$\sigma = 5.8 \times 10^7 \frac{S}{m}$

Calculate the resistance of a 2m length of the cable at dc and at 100 MHz.

Ans: Let $R = R_o + R_i$

Where R_o and R_i are the resistance of the inner and outer conductors.

At dc, $R_i = \frac{l}{\sigma S} = \frac{l}{\sigma \pi a^2} = \frac{2}{5.8\times10^7 \times \pi \times (2\times10^{-3})^2}$

$$\Rightarrow \boxed{R_i = 2.744\, m\Omega}$$

$$R_o = \frac{l}{\sigma s} = \frac{l}{\sigma\left[\pi (b+t)^2 - \pi b^2\right]}$$

$$= \frac{2}{5.8\times10^7\, \pi \left[7^2 - 6^2\right]\times10^{-6}} \quad \Bigg| \quad \begin{array}{l} b = 6mm \\ t = 1mm \end{array}$$

$$\Rightarrow R_o = 0.844\, m\Omega$$

$$\therefore R_{dc} = R_i + R_o = 2.744 + 0.844 = 3.588\, m\Omega$$

At 100 MHz (AC)

$$R_i = \frac{R_s l}{w} = \frac{l}{\sigma \delta w} = \frac{l}{\sigma\, \delta\, 2\pi a}$$

$$\Rightarrow R_i = \frac{l\sqrt{\pi f \mu \sigma}}{\sigma \times 2\pi a} \quad \Bigg| \quad \because \delta = \frac{1}{\sqrt{\pi f \mu \sigma}}$$

$$= \frac{\cancel{2} \times \sqrt{\cancel{\pi}\times 100\times10^6 \times 4\cancel{\pi}\times\cancel{10^{-7}} \times 5.8\times\cancel{10^7}}}{5.8\times10^7 \times \cancel{2}\cancel{\pi} \times 2\times10^{-3}}$$

$$= \frac{\cancel{10^4}\times \cancel{2}\sqrt{5.8}}{5.8\times\cancel{2}\times\cancel{10^4}} = \frac{1}{\sqrt{5.8}} = 0.41\,\Omega$$

$$\boxed{R_i = 0.41\,\Omega}$$

$$R_o = \frac{R_s l}{w} = \frac{l}{\sigma \delta w} = \frac{l}{\sigma \delta 2\pi b}$$

$\because$
$b = 6mm$
$a = 2mm$
$\Rightarrow \frac{b}{a} = 3$
$\Rightarrow b = 3a$

$$R_o = \frac{l}{\sigma \delta\, 2\pi (3a)}$$

$$= \frac{l}{3 \times \sigma \delta 2\pi a}$$

$$R_o = \frac{R_i}{3}$$

$$\Rightarrow \boxed{R_o = 0.1384\,\Omega}$$

Hence,

$$R_{ac} = 0.41 + 0.1384 = 0.5484\,\Omega$$

Which is about 150 times greater than R_{dc}.

Thus, for same effective current, (i), the ohmic loss (i^2R) of the cable at 100 MHZ is greater than the dc Power loss by a factor of 150.

Power and Poynting Vector

As mentioned earlier, energy can be transported from one point (where a transmitter is located) to another point (with a receiver) by means of EM waves. The rate of such energy transportation can be obtained from Maxwell's equations

$$\nabla \times E = -\mu \frac{\partial H}{\partial t} \quad \text{---} (6.62)$$

$$\nabla \times H = \sigma E + \epsilon \frac{\partial E}{\partial t} \quad \text{---} (6.63)$$

$\because$ $\nabla \times E = -\frac{\partial B}{\partial t}$

$= -\frac{\partial \mu H}{\partial t}$

$= -\mu \frac{\partial H}{\partial t}$

$\nabla \times H = \sigma E + \frac{\partial D}{\partial t}$

$= \sigma E + \epsilon \frac{\partial E}{\partial t}$

Dotting both sides of eqn (6.62) with E gives

$$E \cdot (\nabla \times H) = \sigma E^2 + E \cdot \epsilon \frac{\partial E}{\partial t} \quad \text{---} (6.64)$$

$(\because E \cdot E = E_x E_x \cos 0° = E^2)$

From vector Identity

$$\nabla \cdot (A \times B) = B \cdot (\nabla \times A) - A \cdot (\nabla \times B)$$

$$\Rightarrow B \cdot (\nabla \times A) = A \cdot (\nabla \times B) + \nabla \cdot (A \times B) \quad \text{---} (6.65)$$

{comparing eqn (6.64) & (6.65)}

$\langle B = E, \; A = H \rangle$, using (6.65) vector identity, the

$\therefore$ Eqn (6.64) becomes

$$H \cdot (\nabla \times E) + \nabla \cdot (H \times E) = \sigma E^2 + E \cdot \epsilon \frac{\partial E}{\partial t}$$

$$\Rightarrow H\cdot(\nabla\times E) + \nabla\cdot(H\times E) = \sigma E^2 + \frac{1}{2}\epsilon\frac{\partial E^2}{\partial t} \quad —(6.66)$$

Dotting both sides of eqn (6.62) with H, we have

$$H\cdot(\nabla\times E) = H\cdot\left(-\mu\frac{\partial H}{\partial t}\right)$$

$$= -\frac{\mu}{2}\frac{\partial}{\partial t}(H\cdot H)$$

$$H\cdot(\nabla\times E) = -\frac{\mu}{2}\frac{\partial}{\partial t}H^2 \quad —(6.67)$$

$\because \frac{\partial}{\partial t}E^2 = 2E\frac{\partial E}{\partial t}$

$\therefore \frac{1}{2}\epsilon\frac{\partial E^2}{\partial t} = \frac{1}{\cancel{2}}\epsilon\cdot\cancel{2}E\frac{\partial E}{\partial t} = E\cdot\epsilon\frac{\partial E}{\partial t}$

$(\because A\cdot B = B\cdot A)$

Thus eqn (6.66) becomes

$$\Rightarrow -\frac{\mu}{2}\frac{\partial H^2}{\partial t} + \nabla\cdot(H\times E) = \sigma E^2 + \frac{1}{2}\epsilon\frac{\partial E^2}{\partial t}$$

$$\Rightarrow -\nabla\cdot(E\times H) = \frac{\mu}{2}\left[\frac{\partial H^2}{\partial t}\right] + \frac{\epsilon}{2}\left[\frac{\partial E^2}{\partial t}\right] + \sigma E^2$$

$(\because H\times E = -(E\times H))$

$$\Rightarrow \nabla\cdot(E\times H) = -\frac{\partial}{\partial t}\left[\left(\frac{1}{2}\epsilon E^2 + \frac{1}{2}\mu H^2\right)\right] - \sigma E^2$$

Taking volume integral both the sides

$$\int_V \nabla\cdot(E\times H) = -\frac{\partial}{\partial t}\int_V\left[\frac{1}{2}\epsilon E^2 + \frac{1}{2}\mu H^2\right]dv - \int_V \sigma E^2 dv \quad —(6.68)$$

Applying divergence thm $\left(\int_V \nabla\cdot A\, dv = \oint_S A\cdot ds\right)$

$$\oint_S (E\times H)\cdot ds = -\frac{\partial}{\partial t}\int_V\left(\frac{1}{2}\epsilon E^2 + \frac{1}{2}\mu H^2\right)dv - \int_V \sigma E^2 dv \quad (6.69)$$

$$\Rightarrow \begin{pmatrix} \text{Total} \\ \text{power} \\ \text{leaving} \\ \text{the} \\ \text{volume} \end{pmatrix} = \begin{pmatrix} \text{Rate of} \\ \text{decrease} \\ \text{in energy} \\ \text{stored in} \\ \text{electric} \\ \text{and} \\ \text{magnetic fields} \end{pmatrix} - \begin{pmatrix} \text{Ohmic} \\ \text{power} \\ \text{dissipated} \end{pmatrix} \quad (6.70)$$

Equation (6.69) is referred to as Poynting's theorem. The various terms in the equations are identified using energy-conservation arguments for EM fields.

The first term on the right-hand side of eqn (6.69) is interpreted as the rate of decrease in energy stored in the electric and magnetic fields. The second term is the power dissipated because the medium is conducting ($\sigma \neq 0$). The quantity $E \times H$ on the left-hand side of eqn (6.69) is known as the Poynting vector ($\mathcal{P}$), measured in watts per square meter ($\frac{W}{m^2}$) i.e.

$$\mathcal{P} = E \times H \quad (6.71)$$

It represents the instantaneous power density vector associated with the EM field at a given point. The integration of the Poynting vector over any closed surface gives the net power flowing out of the surface.

Thus, Poynting's theorem states that the net power flowing out of a given volume v, is equal to the time rate of decrease in the energy stored within v minus the ohmic losses. The thⁿ is illustrated in fig (6.6)

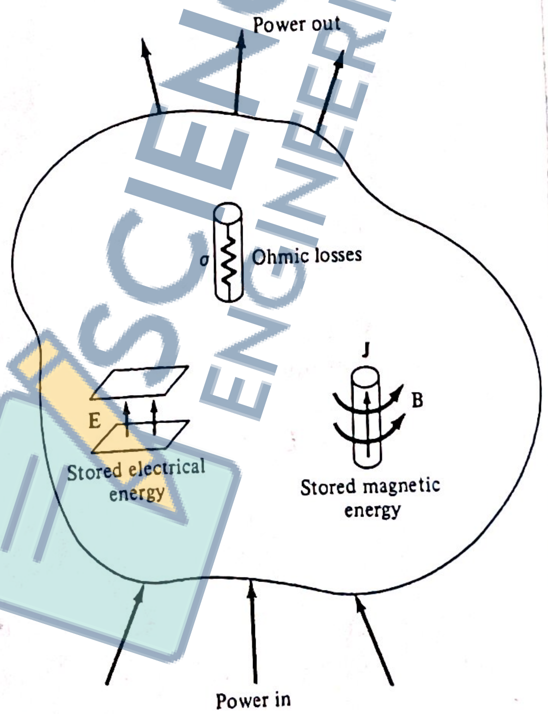


Fig 6.6: Illustration of power balance for EM fields

- It should be noted that $\mathcal{P}$ is normal to both E and H and is therefore along the direction of wave propagation a_k for

Uniform plane waves. Thus

$$a_k = a_E \times a_H \qquad (6.72)$$

If we assume

$$E(z,t) = E_0 e^{-\alpha z} \cos(\omega t - \beta z)\, a_x$$

then $$H(z,t) = \frac{E_0}{|\eta|} e^{-\alpha z} \cos(\omega t - \beta z - \theta_\eta)\, a_y$$

$$\mathcal{P}(z,t) = E \times H$$

$$= \frac{E_0^2}{(2)|\eta|} e^{-2\alpha z} (2\times)\left[\cos(\omega t - \beta z) \cdot \cos(\omega t - \beta z - \theta_\eta)\right] a_z$$

multiplying & dividing '2'

$$\mathcal{P}(z,t) = \frac{E_0^2}{2|\eta|} e^{-2\alpha z} \left[\cos(2\omega t - 2\beta z - \theta_\eta) + \cos(\theta_\eta)\right] \qquad (6.73)$$

$$\left(\because \; 2\cos(A)\cos(B) = \cancel{2}\left[\cos(A+B) + \cos(A-B)\right] \right)$$

To determine the time-avg Poynting vector $\mathcal{P}_{avg}(z)$, which is of more practical value than the instantaneous Poynting vector $\mathcal{P}(z,t)$, we integrate eqn (6.73) over the time period $T = \frac{2\pi}{\omega}$; i.e

$$\mathcal{P}_{avg}(z) = \frac{1}{T} \int_0^T \mathcal{P}(z,t)\, dt \qquad (6.74)$$

It can be shown that

Assignment prove that

$$\mathcal{P}_{avg}(z) = \frac{1}{2} \operatorname{Re}\left(E_s \times H_s^*\right) \quad -(6.75)$$

Putting $E_s = E_0 e^{-\gamma z} = E_0 e^{-(\alpha + j\beta)z} = E_0 e^{-\alpha z} \cdot e^{-j\beta z} a_x$

$$H_s = H_0 e^{-\gamma z} = \frac{E_0}{\eta} e^{-\gamma z} = \frac{E_0}{|\eta| e^{j\theta_\eta}} e^{-(\alpha + j\beta)z}$$

$$H_s = \frac{E_0}{|\eta|} e^{-\alpha z} e^{-j(\beta z + \theta_\eta)}$$

$$H_s^* = \frac{E_0}{|\eta|} e^{-\alpha z} e^{+j(\beta z + \theta_\eta)} a_y$$

$$\mathcal{P}_{avg}(z) = \frac{1}{2} \operatorname{Re}\left(E_s \times H_s^*\right)$$

$$= \frac{1}{2} \operatorname{Re}\left[E_0 \cdot e^{-\alpha z} \cdot e^{-j\beta z} \cdot \frac{E_0}{|\eta|} e^{-\alpha z} \cdot e^{+j(\beta z + \theta_\eta)}\right] a_z$$

$$= \frac{1}{2} \frac{E_0^2}{|\eta|} e^{-2\alpha z} \operatorname{Re}\left[e^{j\theta_\eta}\right] a_z$$

$$\Rightarrow \boxed{\mathcal{P}_{avg}(z) = \frac{E_0^2}{2|\eta|} e^{-2\alpha z} \cos(\theta_\eta) \, a_z} \quad -(6.76)$$

The total time-avg power crossing a given surface S is given by

$$P_{avg} = \int_S \mathcal{P}_{avg} \cdot ds \quad -(6.77)$$

We should note the difference between $\mathcal{P}$, $\mathcal{P}_{avg}$ and P_{avg}; where $\mathcal{P}(x, y, z, t)$ is

<Italic> <Normal>

the Poynting vector in watts per square meter and is time varying; $\mathcal{P}_{avg}(x, y, z)$, also watts per square meter, is the time average of the Poynting vector $\mathcal{P}$; it is ~~the~~ a vector but is time invariant. Finally, P_{avg} is a total time-average power through a surface in watts; it is a scalar.

Example 6.6

In a nonmagnetic medium

$$E = 4 \sin(2\pi \times 10^7 t - 0.8x) a_z \ \frac{V}{m}$$

Find

(a) ϵ_r, η

(b) The time-avg power carried by the wave

(c) The total power crossing 100 cm^2 of plane $2x+y=5$

Ans:-

$$E = E_0 e^{-\alpha z} \sin(\omega t - \beta x) a_z \ \frac{V}{m}$$

Here,

$$E = 4 e^{-0.z} \sin(2\pi \times 10^7 t - 0.8x) a_z \ \frac{V}{m}$$

$\therefore \quad \underline{\alpha = 0} \quad , \beta = 0.8 \ \frac{rad}{m}$

Let's check <for free space>

$\beta = \frac{\omega}{u} = \frac{\omega}{c} = \frac{2\pi \times \cancel{10^7}}{3 \times \cancel{10^8}/10} = \frac{2\pi}{30} = \frac{\pi}{15} = 0.209$

Since $\beta = \underline{0.8}$, the medium is not free space.

But since $\alpha = 0$, it a lossless medium.

As medium is nonmagnetic so given $(\mu = \mu_0)$

$\therefore \quad \alpha = 0, \quad \beta = 0.8 \frac{rad}{m}, \ \epsilon = \epsilon_0 \epsilon_r, \quad \mu = \mu_0, \ \underline{\omega = 2\pi \times 10^7}$

$\therefore \quad \beta = \omega\sqrt{\mu\epsilon}$ | For lossless medium

$\Rightarrow \beta = \omega \sqrt{\mu_0 \epsilon_0 \epsilon_r}$

$\Rightarrow \beta = \omega \sqrt{\epsilon_r} \times \frac{1}{c}$

$\therefore \ c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$

$\Rightarrow \frac{1}{c} = \sqrt{\mu_0 \epsilon_0}$

$\Rightarrow \sqrt{\epsilon_r} = \frac{\beta c}{\omega}$

$\Rightarrow \sqrt{\epsilon_r} = \frac{0.8 \times 3 \times \cancel{10^8}/10}{2\pi \times \cancel{10^7}} = \frac{24}{2\pi} = \frac{12}{\pi}$

$\Rightarrow \epsilon_r = \left(\frac{12}{\pi}\right)^2 = 14.59$

$\eta = \sqrt{\frac{\mu}{\epsilon}} = \sqrt{\frac{\mu_0}{\epsilon_0 \epsilon_r}} = \frac{\sqrt{\frac{\mu_0}{\epsilon_0}}}{\sqrt{\epsilon_r}} = \frac{120\pi}{\left(\frac{12}{\pi}\right)} = 10\pi^2 = 98.69\Omega$

(b) Poynting vector

$\mathcal{P} = E \times H$

$\Rightarrow \mathcal{P} = E \times H$ z, x, y

$$\Rightarrow \mathcal{P} = 4 \sin(2\pi \times 10^7 t - 0.8x) a_z \times \frac{4}{\eta} \sin(2\pi \times 10^7 t - 0.8x) a_y$$

$$\Rightarrow \mathcal{P} = \frac{4^2}{\eta} \sin^2(2\pi \times 10^7 t - 0.8x) a_x$$

From eqn (6.74)

$$\mathcal{P}_{avg} = \frac{1}{T}\int_0^T \mathcal{P}(t)\,dt \quad \left[\text{From eqn } (6.74)\right]$$

$$= \frac{1}{T}\int_0^T \frac{4^2}{\eta} \sin^2(2\pi \times 10^7 t - 0.8x)\, a_x \, dt$$

$$= \frac{4^2}{\eta} \times \frac{1}{T}\int_0^T \frac{1 - \cos 2(2\pi \times 10^7 t - 0.8x)}{2} dt \quad \Bigg| \quad \sin^2\theta = \frac{1 - \cos 2\theta}{2}$$

$$= \frac{4^2}{2\eta} \times \frac{1}{T}\left[t - \frac{\sin 2(2\pi \times 10^7 t - 0.8x)}{2 \times 2\pi \times 10^7}\right]_0^T$$

$$= \frac{4^2}{2\eta} \times \frac{1}{T}\left[\left(T - \frac{\sin 2(2\pi \times 10^7 T - 0.8x)}{2 \times 2\pi \times 10^7}\right) - \left(0 - \frac{\sin(-2 \times 0.8x)}{2 \times 2\pi \times 10^7}\right)\right]$$

$$= \frac{4^2}{2\eta} \times \frac{1}{T}\left[T - \frac{\sin 2(\cancel{2\pi} \times \cancel{10^7} \times \frac{2\pi}{\cancel{2\pi} \times \cancel{10^7}} - 0.8x)}{2 \times 2\pi \times 10^7} - \frac{\sin 1.6x}{2 \times 2\pi \times 10^7}\right]$$

$$\Rightarrow P_{avg} = \frac{4^2}{2\eta} \times \frac{1}{T}\left[T - \frac{\sin(4\pi - 1.6x)}{2\times 2\pi\times 10^7} - \frac{\sin 1.6x}{2\times 2\pi\times 10^7}\right]$$

$$= \frac{16}{2\times 10\pi^2} \times \cancel{\frac{1}{T}}\left[\cancel{T} + \cancel{\frac{\sin(1.6x)}{2\times 2\pi\times 10^7}} - \cancel{\frac{\sin(1.6)x}{2\times 2\pi\times 10^7}}\right]$$

$\sin(4\pi - x)$
$= -\sin x$

$$P_{avg} = \frac{\cancel{16}\,8}{\cancel{2}\times 10\pi^2} = 81\, a_x \frac{mW}{m^2}$$

(c) On plane $2x + y = 5$

$a_n = \frac{\nabla V}{|\nabla V|}$ (gradient)

$V = 2x + y - 5$

$$a_n = \frac{2a_x + a_y}{\sqrt{2^2 + 1^2}} = \frac{2a_x + a_y}{\sqrt{5}}$$

Hence, the time power.

$$P_{avg} = \int P_{avg} \cdot ds$$

$$= P_{avg} \cdot S\, a_n$$

100 cm² surface.

$$= 81\times 10^{-3}\, a_x \cdot (100\times 10^{-4}) \cdot \left(\frac{2a_x + a_y}{\sqrt{5}}\right)$$

$$= 81\times 10^{-3} \times 100\times 10^{-4} \times \frac{2}{\sqrt{5}}$$

$$= \frac{162}{\sqrt{5}} \times 10^{-5} = 72.44\times 10^{-5}$$

$$\boxed{P_{avg} = 724.4\ \mu Watt}$$

(Ans)

Reflection of a Plane Wave at normal incidence

So far, we have considered uniform plane waves travelling on unbounded, homogeneous, isotropic media. When a plane wave from one medium meets a different medium, it is partly reflected and partly transmitted.

The portion of the incident wave that is reflected or transmitted depends on the constitutive parameters (ϵ, μ, σ) of the two media involved. Here, we will assume that the incident wave is normal to the boundary between the media.

Suppose that a plane wave propagating along the $+z$ direction is incident normally on the boundary $z=0$ between medium 1 $(z<0)$ characterized by σ_1, ϵ_1, ~~and~~, μ_1 and medium 2 $(z>0)$ characterized by σ_2, ϵ_2, μ_2 as shown in figure 6.7. In the figure, subscripts i, r and t denote incident, reflected and transmitted waves, respectively. The incident, reflected, and transmitted waves shown in figure 6.7 are obtained as follows.

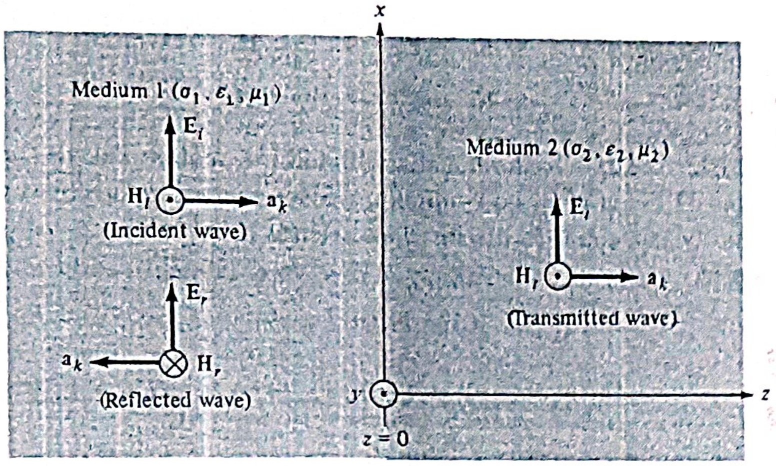


Figure 6.7 A plane wave incident normally on an interface between two different media.

Incident Wave

(E_i, H_i) is travelling along $+a_z$ in medium 1.

If we supress the time factor $e^{j\omega t}$ and assume that

$$E_{is}(z) = E_{io}\, e^{-\gamma_1 z}\, a_x \quad — (6.78)$$

then

$$H_{is}(z) = H_{io}\, e^{-\gamma_1 z}\, a_y = \frac{E_{io}}{\eta_1}\, e^{-\gamma_1 z}\, a_y \quad — (6.79)$$

Reflected Wave

(E_r, H_r) is travelling along $(-a_z)$ in medium 1.

If

$$E_{rs}(z) = E_{ro}\, e^{\gamma_1 z}\, a_x \quad — (6.80)$$

$+\gamma z \rightarrow$ Indicate along $-a_z$ direction and E-field has x-component

$a_E \times a_H = a_k \Rightarrow a_x \times a_H = -a_z \Rightarrow a_H = -a_y$

then

$$H_{rs} = H_{ro}\, e^{\gamma_1 z}\,(-a_y) = -\frac{E_{ro}}{\eta_1}\, e^{\gamma_1 z}\, a_y \quad — (6.81)$$

Where E_{rs} has been assumed to be along a_x. To satisfy the boundary conditions at the interface, we will consistently assume that

Normal incidence E_i, E_r and E_t have the same polarization. (a_x)

Transmitted Wave

(E_t, H_t) is travelling along $+a_z$ in medium 2. If

$$E_{ts}(z) = E_{to}\, e^{-\gamma_2 z}\, a_x \quad -(6.82)$$

$-\gamma z \rightarrow$ Indicates along $+a_z$ direction

$a_E \times a_H = a_z$

$\Rightarrow a_x \times a_H = a_z$

$\Rightarrow a_H = a_y$

then

$$H_{ts}(z) = H_{to}\, e^{-\gamma_2 z}\, a_y = \frac{E_{to}}{\eta_2}\, e^{-\gamma_2 z}\, a_y \quad -(6.83)$$

In eqn (6.78) to (6.83), E_{io}, E_{ro}, and E_{to} are respectively, the magnitudes of incident, reflected, and transmitted electric fields at $z=0$.

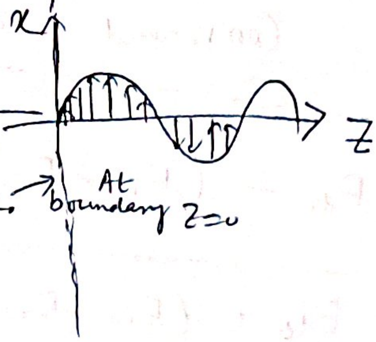
Notice from fig 6.7 that the total field in medium 1 comprises both the incident and reflected fields, whereas medium 2 has only the transmitted field, that is

$$\left.\begin{aligned} E_1 &= E_i + E_r, \quad H_1 = H_i + H_r \\ E_2 &= E_t, \quad H_2 = H_t \end{aligned}\right\} \quad -(6.84)$$

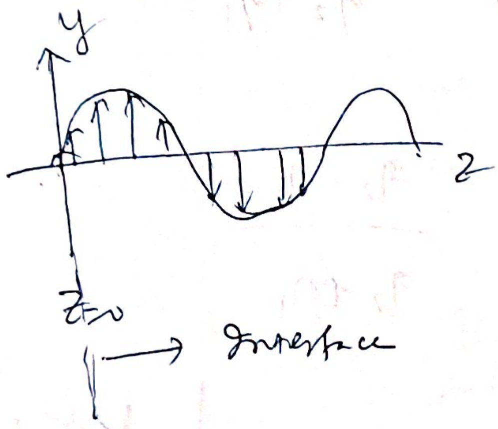
At the interface $z=0$, the boundary conditions require that the tangential components of E and H fields must be continuous. Since the waves are transverse, E and H fields are entirely tangential to the interface. Hence at $z=0$,

$$\left.\begin{aligned} E_{1tan} &= E_{2tan} \\ \text{and} \quad H_{1tan} &= H_{2tan} \end{aligned}\right\} \quad -(6.85)$$

[∴ At the boundary, E-field is tangent to the interface. At boundary $Z=0$



E-field on XZ Plane.



→ Magnetic field in YZ Plane.
At the boundary, that is $Z=0$, H-field is tangential to the interface.

→ Since the tangential component of E-field in medium 1 is continuous, the tangential component of E-field in medium 2.
= tangential component

→ Similarly for H-field]

Thus,

$$\underbrace{E_i(0) + E_r(0)}_{\text{Medium 1}} = \underbrace{E_t(0)}_{\text{medium 2}}$$

$$\Rightarrow E_{i0} + E_{r0} = E_{t0} \quad \text{--- (6.86)}$$

From eqn (6.84)

$E_{1\tan} = E_i(0) + E_r(0)$

'0' → indicates at interface, ($Z=0$)

Similarly,

$$H_i(0) + H_r(0) = H_t(0)$$

$$\Rightarrow H_{i0} + H_{r0} = H_{t0} \quad \text{--- (6.87)}$$

We denote

$E_i(0) \to E_{i0}$
$E_r(0) \to E_{r0}$
$E_t(0) \to E_{t0}$

$$\Rightarrow \frac{E_{i0}}{\eta_1} - \frac{E_{r0}}{\eta_1} = \frac{E_{t0}}{\eta_2} \quad \text{--- (6.87a)}$$

→ From eqn (6.79), (6.81) and (6.83)

$$\Rightarrow \frac{E_{i0} - E_{r0}}{E_{t0}} = \frac{\eta_1}{\eta_2}$$

$$\Rightarrow \frac{E_{t0}}{E_{i0} - E_{r0}} = \frac{\eta_2}{\eta_1}$$

Putting Componendo & dividendo

$$\frac{E_{t0} - (E_{i0} - E_{r0})}{E_{t0} + (E_{i0} - E_{r0})} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$$

$$\Rightarrow \frac{(E_{t0} - E_{i0}) + E_{r0}}{(E_{t0} - E_{r0}) + E_{i0}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$$

$$\Rightarrow \frac{E_{r0} + E_{r0}}{E_{i0} + E_{i0}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$$

Using eqn (6.86), i.e

$E_{t0} - E_{i0} = E_{r0}$

$E_{t0} - E_{r0} = E_{i0}$

$$\Rightarrow \frac{\cancel{2} E_{r0}}{\cancel{2} E_{i0}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$$

$$\Rightarrow \boxed{E_{r0} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \cdot E_{i0}} \quad \text{--- (6.88)}$$

From eqn (6.86)

$$E_{t0} = E_{i0} + E_{r0}$$

$$= E_{i0} + \left(\frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}\right) E_{i0}$$

Using (6.88)

$E_{r0} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \cdot E_{i0}$

$$= \left(1 + \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}\right) E_{i0}$$

$$= \left(\frac{\eta_2 + \cancel{\eta_1} + \eta_2 - \cancel{\eta_1}}{\eta_2 + \eta_1}\right) E_{i0} = \left(\frac{2\eta_2}{\eta_2 + \eta_1}\right) E_{i0}$$

$$\therefore \boxed{E_{to} = \left(\frac{2\eta_2}{\eta_2+\eta_1}\right) E_{io}} \quad \text{———} (6.89)$$

We now define the reflection coefficient Γ (Gamma) and transmission coefficient τ (Tow) from eqn (6.88) & (6.89), ~~[illegible]~~ respectively.

From (6.88):

$$\Gamma = \frac{\text{reflected wave}}{\text{Incident wave}} = \frac{E_{ro}}{E_{io}} = \frac{\eta_2-\eta_1}{\eta_2+\eta_1}$$

$$\boxed{\Gamma = \frac{\eta_2-\eta_1}{\eta_2+\eta_1}} \quad \text{———} (6.90)$$

or

$$\boxed{E_{ro} = \Gamma E_{io}} \quad \text{———} (6.91)$$

From (6.89):

$$\tau = \frac{\text{transmitted wave}}{\text{Incident wave}} = \frac{E_{to}}{E_{io}} = \frac{2\eta_2}{\eta_2+\eta_1}$$

$$\Rightarrow \boxed{\tau = \frac{2\eta_2}{\eta_2+\eta_1}} \quad \text{———} (6.92)$$

or

$$\boxed{E_{to} = \tau E_{io}} \quad \text{———} (6.93)$$

Note that

1. $1+\Gamma = \tau$
2. Both Γ and τ are dimensionless and may be complex.

$\because$

$$1+\Gamma = 1+\frac{\eta_2-\eta_1}{\eta_2+\eta_1}$$

$$= \frac{\eta_2+\cancel{\eta_1}+\eta_2-\cancel{\eta_1}}{\eta_2+\eta_1} = \frac{2\eta_2}{\eta_2+\eta_1} = \tau$$

3. $0 \leq |\Gamma| \leq 1$

The case just considered is the general case. Lets now consider the following special case:

1) Medium 1 is a perfect dielectric (lossless, $\sigma_1 = 0$) & Medium 2 is a perfect conductor ($\sigma_2 = \infty$)

For medium 2, $\boxed{\sigma_2 \to \infty}$, $\eta = \sqrt{\dfrac{j\omega\mu}{\sigma + j\omega\epsilon}}$, $\Rightarrow \boxed{\eta_2 \to 0}$

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{0 - \eta_1}{0 + \eta_1} = -1$$

$$\tau = 1 + \Gamma = 1 - 1 = 0$$

or $$\tau = \frac{2\eta_2}{\eta_2 + \eta_1} = \frac{0}{\eta_2 + \eta_1} = 0$$

Since $\left\{ \begin{array}{l} \text{transmission coefficient} = 0 \\ \text{reflection coefficient} = -1 \end{array} \right\}$, it shows that the wave is totally reflected. This should be expected because fields on a perfect conductor must vanish, so there can be no transmitted wave ($E_2 = 0$). The totally reflected wave combines with the incident wave to form standing wave.

A standing wave "stands" and does not travel; it consists of two travelling waves (E_i and E_r) of equal magnitude but in opposite directions.

Combining equations (6.78) and (6.80), gives the Standing wave in medium1 as

$$E_{1s} = E_{is} + E_{rs}$$

$$E_{1s} = E_{io}\, e^{-\gamma_1 z}\, a_x + E_{ro}\, e^{+\gamma_1 z}\, a_x \quad \text{——— (6.94)}$$

But $\Gamma = \frac{E_{ro}}{E_{io}} = -1$, $\sigma = 0$, $\alpha_1 = 0$, $\because \gamma_1 = j\beta_1$

Hence,

$$E_{1s} = \left(E_{io}\, e^{-j\beta_1 z} - E_{io}\, e^{j\beta_1 z}\right) a_x$$

$$\Rightarrow E_{1s} = -E_{io}\left[e^{j\beta_1 z} - e^{-j\beta_1 z}\right] a_x$$

$$\Rightarrow E_{1s} = -E_{io}\left[2j \sin(\beta_1 z)\right] a_x \quad \Big| \because \frac{e^{j\theta} - e^{-j\theta}}{2j} = \sin\theta$$

$$\Rightarrow \underline{E_{1s}} = -2j\, E_{io} \sin(\beta_1 z)\, a_x$$

<Phasor>

Thus,

$$\underline{E_1} = Re\left[E_{1s}\, e^{j\omega t}\right] a_x$$

$$= Re\left[-2j\, E_{io} \sin(\beta_1 z) .\, e^{j\omega t}\right) a_x$$

$$= Re\left[-2j\, E_{io} \sin(\beta_1 z) .\, (\cos\omega t + j\sin\omega t)\right] a_x$$

Eqn of standing wave →

$$\boxed{E_1 = 2\, E_{io} \sin(\beta_1 z) . \sin\omega t \;\; a_x} \quad \text{— (6.95)}$$

By taking similar steps, it can be shown that the magnetic field component of the wave is

$$\boxed{H_1 = \frac{2E_{io}}{\eta} \cos(\beta_1 z)\, \cos\omega t\;\; a_y} \quad \text{——— (6.96)}$$

A sketch of the standing wave in equation (6.95) is presented in figure (6.8) for $t = 0, \frac{T}{8}, \frac{T}{4}, \frac{3T}{8}, \frac{T}{2}$ and so on, where $T = \frac{2\pi}{\omega}$. From the figure, we notice that the wave does not travel but oscillates.

Note

$E = 2E_{io} \sin\beta_1 z \cdot \sin\omega t$

$\sin\omega t = \sin \frac{2\pi}{T} \cdot t$

case	t		$\sin\omega t$
0	0	→	$\sin \frac{2\pi}{T} \cdot 0 = 0$
1	$\frac{T}{8}$	→	$\sin \frac{2\pi}{T} \cdot \frac{T}{8} = \frac{1}{\sqrt{2}}$
2	$T/4$	→	$\sin \frac{2\pi}{T} \cdot \frac{T}{4} = 1$
3	$\frac{3T}{8}$	→	$\sin \frac{2\pi}{T} \cdot \frac{3T}{8} = \frac{1}{\sqrt{2}}$
4	$\frac{T}{2}$	→	$\sin \frac{2\pi}{T} \cdot \frac{T}{2} = 0$

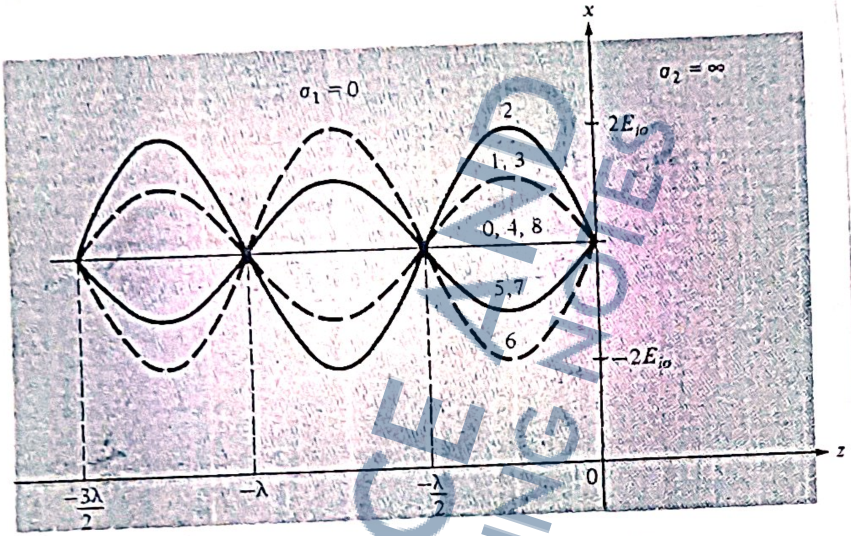


Figure 6.8 Standing waves $E = 2E_{io} \sin\beta_1 z \sin\omega t\, \mathbf{a}_x$. The curves 0, 1, 2, 3, 4, . . . , are, respectively, at times $t = 0, T/8, T/4, 3T/8, T/2, \ldots$; $\lambda = 2\pi/\beta_1$.

When media 1 and 2 are both lossless, we have another special case: $\sigma_1 = 0$, $\sigma_2 = 0$. In this case, η_1 and η_2 are real and so are Γ and τ. Lets consider two more cases:

Case 1:-

If $\eta_2 > \eta_1$, As $\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$, $\Gamma > 0$. So there is a standing wave in medium 1 but there is also a transmitted wave in medium 2. However, the incident and reflected waves have amplitude that are not equal in magnitude. It can be shown that a relative maximum of $|E_1|$ occurs at

$$-\beta_1 z_{max} = n\pi$$

or $$z_{max} = -\frac{n\pi}{\beta_1} = -\frac{n\pi}{\left(\frac{2\pi}{\lambda_1}\right)} = -\frac{n\lambda_1}{2}, \quad —(6.97)$$

where $n = 0, 1, 2, 3 \ldots$

and minimum value of $|E_1|$ occur at

$$-\beta_1 z_{min} = (2n+1)\frac{\pi}{2}$$

$$\Rightarrow z_{min} = -\frac{(2n+1)\frac{\pi}{2}}{\beta_1} = -\frac{(2n+1)\cancel{\pi} \times \lambda_1}{2 \times 2\cancel{\pi}}$$

$$\Rightarrow z_{min} = -\frac{(2n+1)}{4}\lambda_1, \quad n = 0, 1, 2, \ldots \quad —(6.98)$$

Case 2: If $n_2 < n_1$

$\Gamma = \dfrac{n_2 - n_1}{n_2 + n_1}$, $\Gamma < 0$. For this case, the locations of $|E_1|$ maximum are given by (6.98) where as those of $|E_1|$ minimum are given by (6.97). All these are illustrated in figure 6.9.

Note that

1. $|H_1|$ minimum occurs whenever there is $|E_1|$ maximum, and vice versa.

(251)

2. The transmitted wave (not shown in figure 6.8) in medium 2 is a purely travelling wave, and consequently there are no maxima or minima in this region.

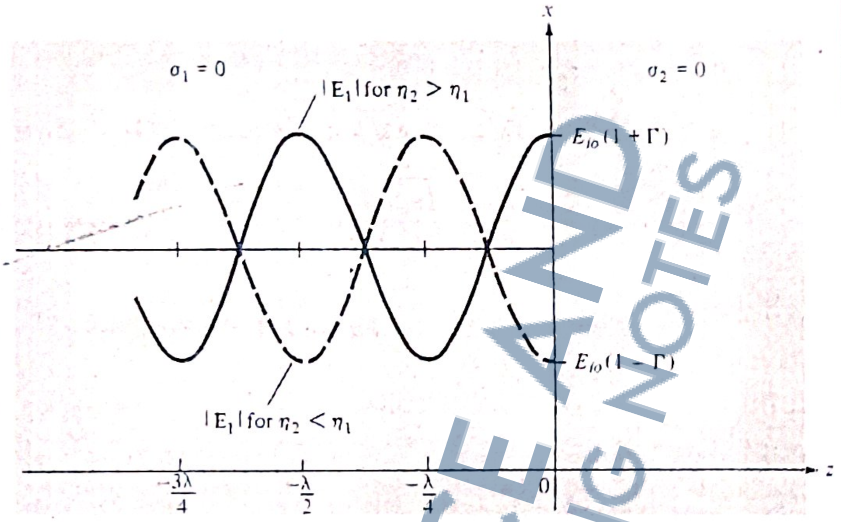


Figure 6.9 Standing waves due to reflection at an interface between two lossless media; $\lambda = 2\pi/\beta_1$.

The ratio of $|E_1|_{max}$ to $|E_1|_{min}$ (or $|H_1|_{max}$ to $|H_1|_{min}$) is called the standing wave ratio 'S', i.e.

$$S = \frac{|E_1|_{max}}{|E_1|_{min}} = \frac{|H_1|_{max}}{|H_1|_{min}} = \frac{1+|\Gamma|}{1-|\Gamma|} \quad \text{--- (6.99)}$$

$$\left(\because S = \frac{|E_{1\,max}|}{|E_{1\,min}|} = \frac{|E_i| + |E_r|}{|E_i| - |E_r|} = \frac{|E_i|\left(1 + \frac{|E_r|}{|E_i|}\right)}{|E_i|\left(1 - \frac{|E_r|}{|E_i|}\right)} = \frac{1+|\Gamma|}{1-|\Gamma|} \right)$$

$$\therefore S = \frac{1+|\Gamma|}{1-|\Gamma|}$$

Putting componendo & dividendo

$$\frac{S-1}{S+1} = \frac{\cancel{1}+|\Gamma| - \cancel{1}+|\Gamma|}{1+\cancel{|\Gamma|}+1-\cancel{|\Gamma|}} = \frac{\cancel{2}|\Gamma|}{\cancel{2}}$$

$$\Rightarrow \boxed{|\Gamma| = \frac{S-1}{S+1}} \quad \text{——— (6.100)}$$

Since $|\Gamma| \leq 1$, From eqn (6.98) i.e

$S = \frac{1+|\Gamma|}{1-|\Gamma|}$, we can have

If $|\Gamma| = 0$, $S = 1$

$|\Gamma| = 1$, $S = \infty$

$$\therefore \boxed{1 \leq S < \infty} \quad \text{——— (6.101)}$$

The standing wave ratio is dimensionless, and it is customarily expressed in dB as

$$S_{dB} = 20 \log_{10}(S) \quad \text{——— (6.102)}$$

Example 6.7

In free space ($z \leq 0$), a plane wave with

$H_i = 10 \cos(10^8 t - \beta z)\, a_x \ \frac{mA}{m}$ is

incident normally on a lossless medium ($\epsilon = 2\epsilon_0$, $\mu = 8\mu_0$) in region $z \geq 0$. Determine the reflected wave H_r, E_r and the transmitted wave H_t, E_t.

Ans :

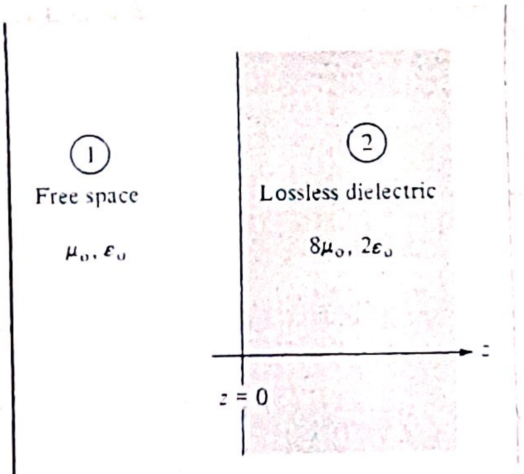


Fig 6.10

Consider the figure 6.10. For free space,

$$\beta_1 = \frac{\omega}{c} = \frac{10^8}{3\times10^8} = \frac{1}{3}$$

$\because H_i = 10\cos(10^8 t - \beta z)$, ω

$$\eta_1 = \eta_0 = 120\pi \quad (\text{for free space})$$

For lossless medium,

$$\beta_2 = \frac{\omega}{u} = \omega\sqrt{\mu\varepsilon} = \omega\sqrt{\mu_0\varepsilon_0}\sqrt{\mu_r\varepsilon_r} = \frac{\omega}{c}\sqrt{\mu_r\varepsilon_r}$$

$\because \frac{1}{c} = \sqrt{\mu_0\varepsilon_0}$

$$\Rightarrow \beta_2 = \beta_1 \times \sqrt{8\times2}$$

$$\Rightarrow \beta_2 = 4\beta_1 = \frac{4}{3}$$

$$\Rightarrow \boxed{\beta_2 = \frac{4}{3}}$$

$$\eta_2 = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_0}{\varepsilon_0}}\sqrt{\frac{\mu_r}{\varepsilon_r}} = 120\pi\times\sqrt{\frac{8}{2}} = 2\times120\pi$$

$$= 240\pi$$

or $\boxed{\eta_2 = 2\eta_0}$

Given that

$$\boxed{H_i = 10\cos(10^8 t - \beta_1 z)\,a_x \ \frac{mA}{m}}$$

We expect that, $E_i = E_{i0}\cos(10^8 t - \beta_1 z)\, a_{E_i}$ where

$$E_{i0} = \eta_1 H_{i0} = \eta_0\times10 = 10\eta_0$$

$$a_{Ei} = a_{Hi} \times a_{ki}$$

$$= a_x \times a_z$$

$$a_{Ei} = -a_y$$

$\because$ $a_E \times a_H = a_k$

$\because$ z, x, y

Hence

$E_i = E_{io} \cos(10^8 t - \beta_1 z)\, a_{Ei}$ becomes

$$\boxed{E_i = -10\eta_0 \cos\left(10^8 t - \frac{1}{3} z\right) a_y \ \frac{mV}{m}}$$

Now

$$\frac{E_{ro}}{E_{io}} = \Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{2\eta_0 - \eta_0}{2\eta_0 + \eta_0} = \frac{1}{3}$$

$$\Rightarrow \boxed{E_{ro} = \frac{1}{3} E_{io}}$$

Since reflected wave in $-z$ direction

Thus,

$$\boxed{E_r = -\frac{10}{3}\eta_0 \cos\left(10^8 t + \frac{1}{3} z\right) a_y \ \frac{mV}{m}}$$

<It is assumed E_i & E_r both vary along $-a_y$ direction>

H_r can be easily obtained as follows

$$\boxed{H_r = -\frac{10}{3} \cos\left(10^8 t + \frac{1}{3} z\right) a_x \ \frac{mA}{m}}$$

$\because$ $H_r = \frac{E_r}{\eta_1} = \frac{E_0}{\eta_0}$

$a_E \times a_H = a_k$
$\Rightarrow -a_y \times a_H = -a_z$
$\Rightarrow a_H = -a_x$

& H_r has $-a_x$ component.

Similarly,

$$\frac{E_{to}}{E_{io}} = \tau = 1 + \Gamma = 1 + \frac{1}{3} = \frac{4}{3}$$

$$\Rightarrow E_{to} = \frac{4}{3} E_{io}$$

Thus,

$$E_t = E_{to} \cos(10^8 t - \beta_2 z)\, a_{E_t}$$

Where

$$a_{E_t} = a_{E_i} = -a_y$$

∵ incident & transmitted wave have same direction

$$\& \; E_{to} = \frac{4}{3} E_{io} = \frac{4}{3} \times 10 \eta_0$$

Hence,

$$\boxed{E_t = -\frac{40}{3}\eta_0 \cos\left(10^8 t - \frac{4}{3} z\right) a_y \;\; \frac{mV}{m}}$$

From which we obtain,

as Same direction of E_t, +z direction

$$\boxed{H_t = \frac{20}{3} \cos\left(10^8 t - \frac{4}{3} z\right) a_x \;\; \frac{mA}{m}}$$

∵ $H_{to} = \frac{E_{to}}{\eta_2} = \frac{+\frac{40}{3}\eta_0}{2\eta_0} = +\frac{20}{3}$

Since H_i & H_t have the same component i.e a_x

To verify, our ans is correct

At boundary, $z = 0$

$$E_i(0) + E_r(0) = \left(-10\eta_0 \cos\left(10^8 t - \cancel{\tfrac{1}{3}z}^{\,0}\right) a_y\right)$$

$$+ \left(-\frac{10}{3}\eta_0 \cos\left(10^8 t + \cancel{\tfrac{1}{3}z}^{\,0}\right) a_y\right)$$

$$= -10\eta_0 \cos 10^8 t\, a_y - \frac{10}{3}\eta_0 \cos 10^8 t\, a_y$$

$$= -\frac{40}{3}\eta_0 \cos 10^8 t\, a_y = E_t(0)$$

$\therefore$ $E_i(0) + E_r(0) = E_t(0)$ is Satisfied.

and

$H_i(0) + H_r(0)$

$= 10 \cos 10^8 t \, a_x + \frac{-10}{3} \cos 10^8 t \, a_x$

$= \frac{20}{3} \cos 10^8 t \, a_x$

$= H_t(0)$

$\Rightarrow$ $H_i(0) + H_r(0) = H_t(0)$ is Satisfied.

$\therefore$ Hence our Answer is correct.

$\therefore$ These boundary conditions can always be used to Cross Check E & H

Example 6.8 Given a uniform Plane wave in air as

$$E_i = 40 \cos(\omega t - \beta z) a_x + 30 \sin(\omega t - \beta z) a_y \ \frac{V}{m}$$

(a) Find H_i

(b) If the wave encounters a perfectly conducting Plane normal to z-axis at $z=0$, find the reflected wave E_r and H_r.

(c) What are the total E and H fields for $z \le 0$?

(d) Calculate the time-avg Poynting vectors for $z \le 0$ and $z \ge 0$.

Solution :- We may treat the wave as consisting of two waves E_{i1} and E_{i2} where

$$E_{i1} = 40 \cos(\omega t - \beta z)\, a_x$$

$$E_{i2} = 30 \sin(\omega t - \beta z)\, a_y$$

At atmospheric pressure, air has $\epsilon_r = 1.006 \simeq 1$.

Thus air may be regarded as free space.

Let $H_i = H_{i1} + H_{i2}$.

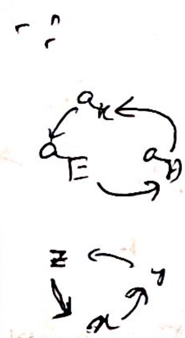
$$H_{i1} = H_{i10} \cos(\omega t - \beta z)\, a_{H1}$$

Where

$$H_{i10} = \frac{E_{i10}}{\eta_0} = \frac{40}{120\pi} = \frac{1}{3\pi}$$

$$a_{H1} = a_k \times a_E = a_z \times a_x = a_y$$

$\because$



Hence

$$H_{i1} = \frac{1}{3\pi} \cos(\omega t - \beta z)\, a_y$$

Similarly,

$$H_{i2} = H_{i20} \sin(\omega t - \beta z)\, a_{H2}$$

Where

$$H_{i20} = \frac{E_{i20}}{\eta_0} = \frac{30}{120\pi} = \frac{1}{4\pi}$$

$$a_{H2} = a_k \times a_E = a_z \times a_y = -a_x$$

Hence,

$$H_{i2} = \frac{-1}{4\pi} \sin(\omega t - \beta z)\, a_x$$

and

$$H_i = H_{i1} + H_{i2}$$

$$= \frac{1}{3\pi} \cos(\omega t - \beta z)\, a_y + \left(-\frac{1}{4\pi} \sin(\omega t - \beta z)\right) a_x$$

$$\Rightarrow \boxed{H_i = -\frac{1}{4\pi}\sin(\omega t-\beta z)\,a_x + \frac{1}{3\pi}\cos(\omega t-\beta z)\,a_y \ \frac{mA}{m}}$$

258

(b) Since medium 2 is perfectly conducting,

$$\frac{\sigma_2}{\omega \epsilon_2} >> 1$$

Since $\downarrow |\eta| = \frac{\sqrt{\mu/\epsilon}}{\left[1+\left(\frac{\sigma}{\omega\epsilon}\right)^2\right]^{1/4}} \uparrow$ we have

$$\eta_2 << \eta_1$$

$$\therefore \Gamma = \frac{\eta_2-\eta_1}{\eta_2+\eta_1} \simeq \frac{\eta_1\left(\cancel{\frac{\eta_2}{\eta_1}}^{\,0}-1\right)}{\eta_1\left(\cancel{\frac{\eta_2}{\eta_1}}^{\,0}+1\right)} \simeq -1$$

$$\tau = 1+\Gamma = 1-1 = 0$$

Since $\Gamma = -1$ or $\tau = 0$, indicates that the incident E and H fields are totally reflected.

$$\frac{E_{r0}}{E_{i0}} = \Gamma \Rightarrow \frac{E_{r0}}{E_{i0}} = -1$$

$$\Rightarrow E_{r0} = -E_{i0}$$

Indicates the reflected wave travels in -z direction.

Hence

$$E_r = -40\cos(\omega t+\beta z)\,a_x - 30\sin(\omega t+\beta z)\,a_y \ \frac{V}{m}$$

To find H_r

From eqn (6.87A), $\frac{H_{r0}}{H_{i0}} = \frac{-E_{r0}/\eta_1}{E_{i0}/\eta_1} = \frac{-E_{r0}}{E_{i0}}$

$$\Rightarrow \frac{H_{r0}}{H_{i0}} = -\left(\frac{E_{r0}}{E_{i0}}\right)$$

$$\Rightarrow \boxed{\frac{H_{r0}}{H_{i0}} = -\Gamma}$$

Here, $\Gamma = -1$, $\Rightarrow \frac{H_{r0}}{H_{i0}} = -(-1) = 1$

$$\Rightarrow H_{r0} = H_{i0}$$

$\therefore$ Here

$H_r = H_i$ → But direction of reflected wave opposite of incident.

indicates the direction of reflected wave along $(-z)$ direction.

$$H_r = -\frac{1}{4\pi} \sin(\omega t + \beta z)\, a_x + \frac{1}{3\pi} \cos(\omega t + \beta z)\, a_y \ \frac{mA}{m}$$

(c) The total fields in air

$$E_1 = E_i + E_r$$

and $H_1 = H_i + H_r$

$$\therefore E_1 = \left[40 \cos(\omega t - \beta z)\, a_x + 30 \sin(\omega t - \beta z)\, a_y\right] +$$

$$\left[-40 \cos(\omega t + \beta z)\, a_x - 30 \sin(\omega t + \beta z)\, a_y\right] \ \frac{V}{m}$$

Similarly

$$H_1 = \left[-\frac{1}{4\pi} \sin(\omega t - \beta z)\, a_x + \frac{1}{3\pi} \cos(\omega t - \beta z)\, a_y\right] +$$

$$\left[-\frac{1}{4\pi} \sin(\omega t + \beta z)\, a_x + \frac{1}{3\pi} \cos(\omega t + \beta z)\, a_y\right] \ \frac{mA}{m}$$

→ Then E_1 & H_1 are the standing waves.

→ The total field in medium 2 i.e perfect conductor

$$E_2 = E_t = 0 \quad , \quad H_2 = H_t = 0$$

(As EM wave can not be transmitted in a perfect conductor)

(d) For $z \leq 0$, from eqn (6.76)

$$\wp_{1Avg} = \frac{|E_{1s}|^2}{2\eta_1} a_k$$

$$= \frac{E_{i0}^2 a_z - E_{r0}^2 a_z}{2\eta_0}$$

$$= \frac{(40^2+30^2) a_z - (40^2+30^2) a_z}{2 \times 120\pi}$$

$$\wp_{1Avg} = 0$$

for $z \geq 0$,

$$\wp_{2avg} = \frac{|E_{2s}|^2}{2\eta_2} a_k = \frac{E_{t0}^2}{2\eta_2} a_z = 0$$

$(\because E_{t0} = 0)$

The whole incident power is reflected, so $E_{t0} = 0$, no transmitted wave, all are reflected back.

Polarization :-

→ Polarization of an antenna in a given direction is defined as " the polarization of the wave transmitted (radiated) by the antenna, Note: when the direction is not stated, the polarization is taken to be polarization in the direction of maxm gain ".

→ Polarization of a radiated wave is defined as " that property of an electromagnetic wave describing the time varying direction and relative ~~at a fixed~~ magnitude of the electric field vector; ~~location in space~~ specifically, the figure traced as a function of time by the extremity of the vector at a fixed location in space and observed along the direction of propagation.

→ Polarization then is the curve traced by the end point of the arrow representing the instantaneous electric field.

→ Polarization may be classified as linear, circular or elliptical.

→ If the vector that describes the electric field at a point in space as function of time is always directed along a line, the field is said to be linearly polarized.

→ In general, however, the figure that the electric field traces is an ellipse, and the field is said to be elliptically polarized.

→ Linear and circular polarizations are special cases of elliptical.

→ The figure of the electric field is traced in a clockwise (CW) or counterclockwise (CCW) sense.

→ Clockwise rotation of the electric field vector is designated as right-hand polarization and counterclockwise as lefthand polarization.

Mathematical Analysis:-

The instantaneous field of a plane wave, traveling in the -ve Z-direction, can be written as

$$\mathcal{E}(z;t) = \hat{a}_x \mathcal{E}(z;t) + \hat{a}_y \mathcal{E}(z;t) \quad \text{—} (1)$$

The instantaneous components are related to their complex counterparts by,

$$\mathcal{E}_x(z;t) = \text{Re}\left[E_x^- e^{j(\omega t + kz)}\right] = \text{Re}\left[E_{xo} e^{j(\omega t + kz + \phi_x)}\right]$$

$$\therefore \mathcal{E}_x(z;t) = E_{xo} \cos(\omega t + kz + \phi_x) \quad \text{———} (2)$$

Similarly

$$\mathcal{E}_y(z;t) = E_{yo} \cos(\omega t + kz + \phi_y) \quad \text{———} (3)$$

Where E_{xo} and E_{yo} are, respectively, the maximum magnitudes of the x and y Components.

A. Linear Polarization

For the wave to have linear Polarization, the time-phase difference between two Components must be

$$\Delta\phi = \phi_y - \phi_x = n\pi , \quad n = 0,1,2,3,\dots$$

B. Circular Polarization

Circular polarization can be achieved only when the magnitude of the two Components are the same and the time-phase difference between them is odd multiples of $\frac{\pi}{2}$.

i.e $\quad |E_x| = |E_y| \Rightarrow E_{xo} = E_{yo}$ ——— (4)

$$\Delta\phi = \phi_y - \phi_x = \begin{cases} +\left(\frac{1}{2} + 2n\right)\pi, & n = 0,1,2\dots \text{ for CW} \quad (5) \\ -\left(\frac{1}{2} + 2n\right)\pi, & n = 0,1,2\dots \text{ for CCW} \quad (6) \end{cases}$$

CW → Clockwise
CCW → Counter clockwise

Note :- If the direction of wave propagation is reversed (i.e $+Z$ direction), the Phases in eqn (5) & (6) for CW and CCW rotation must be interchanged.

C. Elliptical Polarization

Elliptical Polarization can be attained only when the time-phase difference between the two Components is odd multiple of $\pi/2$ and their magnitudes are not the same.

~~or~~ when the time-phase difference between two components is not equal to multiple of $\frac{\pi}{2}$ (irrespective of their magnitude)

i.e ~~when~~

$$|E_x| \neq |E_y| \Rightarrow E_{xo} \neq E_{yo} \quad -(7)$$

when $\Delta\phi = \phi_y - \phi_x = \begin{cases} +\left(\frac{1}{2} + 2n\right)\pi & \text{for CW} \\ -\left(\frac{1}{2} + 2n\right)\pi & \text{for CCW} \end{cases} \quad -(8)$

where $n = 0, 1, 2, \ldots$

or

$$\Delta\phi = \phi_y - \phi_x \neq \pm\frac{n\pi}{2} = \begin{cases} >0, & \text{for CW} \\ <0, & \text{for CCW} \end{cases} \quad -(9)$$

where $n = 0, 1, 2, 3 = \ldots$

(Irrespective of ~~...~~ their magnitude in this case)