Ch-4 Random Variables & processes

Probability :-

think about
↓
The concept of probability occures when we contemplate the possible outcomes of an experiment whose outcome is not always the same.

Probability of an event 'A',

$$\boxed{P(A) = \frac{\text{Number of possible favourable outcomes}}{\text{Total number of possible equally likely outcomes}}}$$

Ex :- 1) In tossing a dice, even number can occure in 3 ~~…~~ ways out of 6 equally likely ways.

Therefore $P(A) = \frac{|\{2,4,6\}|}{|\{1,2,3,4,5,6\}|} = \frac{3}{6} = \frac{1}{2}$

Properties of Probability

1) The probability of 'certain' event is unity.

$$\boxed{P(A) = 1}$$

'certain' → If an event contains all the outcomes.

2) If event is not possible, $\boxed{P(A) = 0}$

3) In rest cases, $\boxed{0 \le P(A) \le 1}$

4) If A & B are mutually exclusive events

$$P(A+B) = P(A) + P(B)$$

5) $P(\bar{A}) = 1 - P(A)$, $\bar{A}$ = complement of event A

6) If A, B are not mutually exclusive,

$$P(A+B) = P(A) + P(B) - P(AB)$$

where $P(AB)$ is called probability of events A & B both occurring simultaneously.

→ Such an event is called Joint event of A & B.

→ $P(AB)$ → is called joint probability.

→ For mutually exclusive events, $\boxed{P(AB) = 0}$

Mutually exclusive events :-

Two possible outcomes of an experiment are defined as mutually exclusive if the occurance of one outcome precludes the occurance of the other.

(Prevent something from happening)

$$P(A_1 \text{ or } A_2) = P(A_1) + P(A_2)$$

$$P(A_1 \text{ or } A_2 \text{ --- or } A_L) = \sum_{j=1}^{L} P(A_j)$$

If there are L possible events,

$$\sum_{j=1}^{L} P(A_j) = 1.$$

Joint Probability of Related or Independent events

$P(A|B) \rightarrow$ Represents Probability of event 'A' given that event 'B' has already occured.

$P(B|A) \rightarrow$ Represents Probability of event B, given that event 'A' has already occured.

∴ The probability of outcome, B_x, given that A_j is known to have occured, is called Conditional probability and is written as $P(B_x|A_j)$.

$$\rightarrow \quad P(B|A) = \frac{P(AB)}{P(A)} \quad \text{--- (1)}$$

$$P(A|B) = \frac{P(AB)}{P(B)} \quad \text{--- (2)}$$

where $P(AB)$ is the joint probability of A & B.

From eqn (1) & (2),

$$\cancel{P(AB) = P(A) \cdot P(B|A)}$$

$$P(A) \cdot P(B|A) = P(B) \cdot P(A|B)$$

$$\Rightarrow \cancel{P(A)}$$

$$\Rightarrow \boxed{P(A|B) = \frac{P(A)}{P(B)} \cdot P(B|A)}$$

This is known as Bayes Theorem.

Statistically Independent event

If A & B are 2 events in an experiment, the possibility / probability of occurrance of event B does not depend upon occurrance of event A known then these two events A & B are as statistically independent event.

$$\rightarrow \quad P(B|A) = \frac{P(AB)}{P(A)} \quad \text{—— (1)}$$

Since probability of occurrance of 'B' does not depend on occurrance of 'A', then

$$P(B|A) = P(B) \quad \text{—— (2)}$$

Putting eqn (2), in eqn (1), we have.

$$P(B) = \frac{P(AB)}{P(A)}$$

$$\Rightarrow \boxed{P(AB) = P(A).P(B)} \quad \text{—— (3)}$$

Similarly,

$$P(A|B) = \frac{P(AB)}{P(B)} \quad \text{—— (4)}$$

~~Pr statistical~~ Since probability of occurance of 'A' does n't depend on occurance of 'B'

$$P(A|B) = P(A) \quad \text{—— (5)}$$

Putting eqn (5), in eqn (4)

$$\boxed{P(AB) = P(A).P(B)}$$

In general,

$$P(A_j, B_k, C_l, ---) = P(A_j) \cdot P(B_k) \cdot P(C_l) \cdots$$

Ex-1 : Given $P(A) = 0.2$, $P(B) = 0.4$

$P(A \cup B) = 0.5$

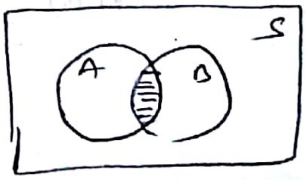
(i) Find probability of A & B jointly occuring.

(ii) Find probability that none of A or B will occure

(iii) What is the probability that A will occure if B has already occured.

(iv) ~~Very~~ Verify Bayes theorem.

Ans ÷ (i) To find $P(A \cap B)$ or $P(AB)$



$P(A+B) = P(A) + P(B) - P(AB)$

$\Rightarrow P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow 0.5 = 0.2 + 0.4 - P(A \cap B)$

$\Rightarrow P(A \cap B) = 0.6 - 0.5 = 0.1$

$\boxed{P(A \cap B) = 0.1}$ → Probability of A & B jointly occuring.

(ii) Probability that none of A or B will occure

$= S - (A \cup B)$

$= 1 - (0.5)$

$= 0.5$

(iii) $P(A|B) = \frac{P(AB)}{P(B)} = \frac{0.1}{0.4} = \frac{1}{4} = 0.25$

$P(B|A) = \frac{P(AB)}{P(A)} = \frac{0.1}{0.2} = \frac{1}{2} = 0.5$

By Bayes theorem

$$P(A|B) = \frac{P(A) \cdot P(B|A)}{P(B)}$$

L.H.S

$$P(A|B) = 0.25$$

R.H.S

$$\frac{P(A) \cdot P(B|A)}{P(B)} = \frac{0.2}{0.4} \times 0.5 = \frac{1}{2} \times 0.5 = 0.25$$

L.H.S = R.H.S (Proved)

$\therefore$ Bayes theorem is verified.

Random variable: -

A function which can take on any value from the sample space and its range is some set of real numbers is called a random variable of an experiment.

→ Random variables are denoted by upper case letters such as X, Y etc. and the values taken by them are denoted by lower case letters with subscripts such as x_1, x_2, y_1, y_2 etc.

Random Variables may be Classified as

1) Discrete ~~~~ random variable.
2) Continuous random variable.

Discrete random variables

A discrete random variable may be defined as the random variable which can take on only finite number of values in a finite observation interval.

Ex:- An experiment of tossing 3 coins simultaneously.

→ There are 8 possible out comes.

→ Sample space (S)

$S = \{ \underset{000}{TTT}, \underset{001}{TTH}, \underset{010}{THT}, \underset{011}{THH}, \underset{100}{HTT}, \underset{101}{HTH}, \underset{110}{HHT}, \underset{111}{HHH} \}$

$X = \{ x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 \}$.

Let X = Number of heads appearing in toss.

			No. of heads	
~~ If ~~~~ O/P is	TTT,	then ~~~~	0	(x_1)
	TTH,	~~~~	1	(x_2)

If $S = \{TTT, TTH, THT, THH, HTT, HTH, HHT, HHH\}$

Then, $X = \{0, 1, 1, 2, 1, 2, 2, 3\}$

∴ X can take finite number of values i.e 8.

Therefore, it is a discrete random variable.

Continuous Random Variable:-

A random variable that takes on an infinite number of values is called a continuous random variable.

Ex:- A noise voltage generated by an electronic amplifier has continuous amplitude. This means that sample space S of the noise voltage amplitude is continuous. Therefore, in this case, the random variable X has a continuous range of values.

Probability function or Probability Distribution of a discrete random variable

Let X be a discrete random variable and also let x_1, x_2, x_3 -- be the values that X can take.

Then

$$P(X = x_j) = f(x_j)$$

where $j = 1, 2, 3$ ---, will be the probability of x_j.

The function $f(x_j)$ or simply $f(x)$ as called the probability function or probability distribution of the discrete random variable.

Ex:- In an experiment, 3 coins are tossed simultaneously. If the number of heads is random variable, find the probability function of this random variable.

Ans :

$S = \{TTT, TTH, THT, THH, HTT, HTH, HHT, HHH\}$

$X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$

$= \{0, 1, 1, 2, 1, 2, 2, 3\}$

$P(X=0) = P(x_1) = \frac{1}{8}$

$P(X=1) = P(x_2) + P(x_3) + P(x_5) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}.$

$P(X=2) = P(x_4) + P(x_6) + P(x_7) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}.$

$P(X=3) = P(x_8) = \frac{1}{8}.$

∴ Probability function for discrete random variable X is given as.

X	0	1	2	3
f(x)	1/8	3/8	3/8	1/8

Cummulative Distribution Function (CDF)

The CDF of a random variable 'x' may be defined as the probability that a random variable 'x' takes a value less than or equal to x.

i.e

$$\boxed{\text{CDF}: \quad F_X(x) = P(X \leq x)}$$

where $F_X(x)$ is called cummulative distribution function of random variable X.

Ex :- Rolling two dice

$$S = \{ (1,1), (1,2), (1,3), (1,4), (1,5), (1,6)$$
$$(2,1), (2,2), (2,3), (2,4), (2,5), (2,6)$$
$$(3,1), (3,2), (3,3), (3,4), (3,5), (3,6)$$
$$(4,1), (4,2), (4,3), (4,4), (4,5), (4,6)$$
$$(5,1), (5,2), (5,3), (5,4), (5,5), (5,6)$$
$$(6,1), (6,2), (6,3), (6,4), (6,5), (6,6) \}$$

X = Sum of number appearing on the dice)

$$F_X(3) = P(X \leq 3) = P(1) + P(2) + P(3)$$

$P(1) = 0$,

$P(2) = \frac{1}{36}$ [because only one case (1,1)]

$P(3) = \frac{2}{36}$ [$\because$ (1,2), (2,1)]

$$F_X(3) = 0 + \frac{1}{36} + \frac{2}{36} = \frac{3}{36} = \frac{1}{12}.$$

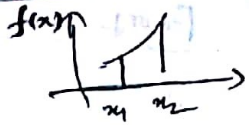
Properties of CDF:-

1) $0 \leq F_X(x) \leq 1.$

i.e the value of CDF lies betn 0 to 1.

2) $F_X(-\infty) = 0$ and $F_X(\infty) = 1.$

3) $\boxed{F_X(x_1) \leq F_X(x_2) \text{ , if } x_1 \leq x_2}$



That is CDF, $F_X(x)$ is a ~~monotone~~ non-decreasing function of x.

i.e monotone increasing.

$f(x_2) \geq f(x_1)$ if $x_2 > x_1$

Probability Density function (PDF)

The derivative of Cummulative distribution f^n with respect to some dummy variable is known as Probability Density Function (PDF).

PDF is generally denoted as $\boxed{f_X(x)}$.

Mathematically,

$$\text{PDF: } \boxed{f_X(x) = \frac{d}{dx} F_X(x)}$$

where x is a dummy variable.

PDF is the more convenient representation for continuous random variable.

Properties of PDF

1) PDF is always nonzero for all values of

i.e $f_X(x) \geq 0$, for all values of x.

2) The area under PDF curve is always equal to unity.

$$\int_{-\infty}^{\infty} f_X(x)\, dx = 1$$

<u>Proof</u> :-

$$\int_{-\infty}^{\infty} f_X(x)\, dx$$

$$= \int_{-\infty}^{\infty} \frac{d}{dx}\left[F_X(x) \right] dx$$

$$= \left[F_X(x) \right]_{-\infty}^{\infty}$$

$$= F_X(\infty) - F_X(-\infty)$$

$$= 1 - 0$$

$$= 1$$

3) The CDF may be obtained by integrating PDF, Mathematically

$$\boxed{F_X(x) = \int_{-\infty}^{x} f_X(x)\, dx}$$

Proof :-

$$f_X(x) = \frac{d}{dx} F_X(x)$$

Integrating both the sides,

$$\int_{-\infty}^{x} f_X(x) = \int_{-\infty}^{x} \left[\frac{d}{dx} F_X(x)\right] dx$$

$$= \left[F_X(x)\right]_{-\infty}^{x}$$

$$= F_X(x) - F_X(-\infty)$$

$$= F_X(x) - 0$$

$$= F_X(x) \quad \text{(Proved)}$$

4) Probability of the event $\{x_1 \leq X \leq x_2\}$ is simply given by area under the PDF curve in the range $x_1 \leq X \leq x_2$.

Mathematically,

$$\boxed{P(x_1 \leq X \leq x_2) = \int_{x_1}^{x_2} f_X(x)\, dx}$$

Ex :- 3) Consider the PDF

$$f_X(x) = a\, e^{-b|x|}$$, where 'X' is a random variable whose allowable value range from $x = -\infty$ to $x = \infty$.

Find

(a) Cummulative distribution function $F(x)$

(b) The relationship between a & b

(c) The probability that outcome X lies bet^n 1 & 2.

Ans :- (a) $CDF = P(X \le x) = F_X(x)$

$\therefore$ $F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx$

$= \int_{-\infty}^{x} a\,e^{-b|x|}\,dx$

$\boxed{\text{Note:- } |x| = \begin{cases} x, & x \ge 0 \\ -x, & x < 0 \end{cases}}$

$= \cancel{\begin{cases} \int_{-\infty}^{x} a\,e^{-b(-x)}\,dx, & x \le 0 \\ \int_{-\infty}^{x} a\,e^{-bx}\,dx, & x \ge 0 \end{cases}}$

~~For~~ ~~Now~~, We shall break the above expression for $x < 0$, $x = 0$, $x > 0$

$F_X(x) = \int_{-\infty}^{x} a\,\cancel{e^{-b(-x)}}\,dx + \int_{x=0} a\,e^{-bx}\,\cancel{dx} + \int_{0^+}^{x} a\,e^{-bx}\,dx$

Case - I

For $x < 0$ (x is $-ve$), $|x| = -x$

$\int_{-\infty}^{x} a\,e^{bx}\,dx = a\,\frac{e^{bx}}{b}\Big]_{-\infty}^{x}$

$= \frac{a}{b}\left[e^{bx} - 0\right]$

$= \frac{a}{b}\,e^{bx}$

Case - II

For $x \ge 0$, [x is $+ve$]

$\int_{-\infty}^{0} a\,e^{+bx}\,dx + \int_{0^+}^{x} a\,e^{-bx}\,dx$ $\left[\int_{-\infty}^{0} \rightarrow |x| = -x,\ \int_{0}^{x} |x| = x\right]$

$= a\,\frac{e^{+bx}}{b}\Big]_{-\infty}^{0} + a\,\frac{e^{-bx}}{-b}\Big]_{0^+}^{x}$

$$= \frac{a}{b}\left[1-0\right] - \frac{a}{b}\left[e^{-bx}-1\right]$$

$$= \frac{a}{b} + \frac{a}{b} - \frac{a}{b}\cdot e^{-bx}$$

$$= \frac{a}{b}\left[2 - e^{-bx}\right].$$

$$\therefore \quad F_X(x) = \begin{cases} \frac{a}{b}\cdot e^{bx} & , x < 0 \\ \frac{a}{b}\left[2 - e^{-bx}\right] & , x \geq 0 \end{cases}$$

(b) <u>Rel^n between a & b</u>

We know $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

$$\int_{-\infty}^{\infty} a\cdot e^{-b|x|}\,dx = 1$$

$$\Rightarrow \int_{-\infty}^{0} a\cdot e^{+bx}\,dx + \int_{0}^{\infty} a\cdot e^{-bx}\,dx = 1.$$

$$\Rightarrow a\cdot \frac{e^{bx}}{b}\Big]_{-\infty}^{0} + a\cdot \frac{e^{-bx}}{-b}\Big]_{0}^{\infty} = 1$$

$$\Rightarrow \frac{a}{b}\left[1-0\right] - \frac{a}{b}\left[0-1\right] = 1$$

$$\Rightarrow \frac{a}{b} + \frac{a}{b} = 1$$

$$\Rightarrow \boxed{\frac{2a}{b} = 1}$$

or $\boxed{b = 2a}$

(c) Probability that outcome lies betn x_1 & x_2 is given by the expression,

$$P(x_1 \leq X \leq x_2) = \int_{x_1}^{x_2} f_X(x)\, dx$$

Here $x_1 = 1$, $x_2 = 2$

$$P(1 \leq X \leq 2) = \int_1^2 a e^{-b|x|}\, dx$$

$$= \int_1^2 a e^{-bx}\, dx$$

$$= a \cdot \left. \frac{e^{-bx}}{-b} \right]_1^2$$

$$= -\frac{a}{b} \left[e^{-2b} - e^{-b} \right]$$

$$= \frac{a}{b} \left[e^{-b} - e^{-2b} \right]$$

From last question, $b = 2a \Rightarrow a = \frac{b}{2}$

$$P(1 \leq X \leq 2) = \frac{\cancel{b}/2}{\cancel{b}} \left[e^{-b} - e^{-2b} \right]$$

$$\boxed{P(1 \leq X \leq 2) = \frac{1}{2} \left[e^{-b} - e^{-2b} \right]}$$

Joint Cummulative Distribution function

The Joint Distribution Function or Joint CDF $F_{XY}(x, y)$ of 2 random variable X and Y is defined as the probability that the random variable X is less than or equal to a specific value x and the random variable Y is less than or equal to a specified value y.

$$F_{XY}(x, y) = P(X \leq x, Y \leq y)$$

$$= \int_{-\infty}^{y} \int_{-\infty}^{x} f_{XY}(x, y)\, dx\, dy.$$

where, f_{XY} = Joint PDF $= \dfrac{\partial^2}{\partial x \partial y} F_{XY}(x, y)$

→ Similarly,

$$P(X \leq x, -\infty \leq Y \leq \infty) = \int_{-\infty}^{\infty} \int_{-\infty}^{x} f_{XY}(x, y)\, dx\, dy.$$

→ $P(x_1 < X < x_2, y_1 < Y < y_2)$

$$= \int_{y_1}^{y_2} \int_{x_1}^{x_2} f_{XY}(x, y)\, dx\, dy. \quad (1)$$

Imp

→ If the 2 random variables X and Y are statistically independent, then Joint PDF of these two random variables becomes a product of two separate PDFs.

$$\boxed{f_{XY}(x, y) = f_X(x) \cdot f_Y(y)} \quad (2)$$

$\therefore$ Putting eq^n ②, in eq^n ①

~~$P(x_1 < X < X_2,$~~

$P(x_1 < X < x_2,\ y_1 < Y < y_2)$

$$= \int_{y_1}^{y_2} \int_{x_1}^{x_2} f_X(x) \cdot f_Y(y) \cdot dx\, dy.$$

$$= \left[\int_{y_1}^{y_2} f_Y(y)\, dy\right] \cdot \left[\int_{x_1}^{x_2} f_X(x)\, dx\right]$$

(For statistically independent random variable X and Y.

Ex:- The joint probability density of the random variables X and Y is

$$f_{XY}(x, y) = \frac{1}{4} \cdot e^{-|x| - |y|}, \quad \begin{matrix} -\infty < x < \infty \\ -\infty < y < \infty \end{matrix}$$

① (a) Are X and Y are statistically independent random variables

(b) Calculate the probability that $X \le 1$ and $Y \le 0$.

Ans. $f_{XY}(x, y) = \frac{1}{4} \cdot e^{-|x| - |y|}$

$$= \left(\frac{1}{2} \cdot e^{-|x|}\right) \cdot \left(\frac{1}{2} \cdot e^{-|y|}\right)$$

$f_{XY}(x,y) = f_X(x) \cdot f_Y(y)$.

$\therefore$ X and Y are Statistically Independent

(b) $P(X \le 1, Y \le 0)$

$$= \int_{-\infty}^{1} f_X(x)\,dx \int_{-\infty}^{0} f_Y(y)\,dy$$

$$= \int_{-\infty}^{1} \frac{1}{2} \cdot e^{-|x|}\,dx \cdot \int_{-\infty}^{0} \frac{1}{2} \cdot e^{-|y|}\,dy.$$

$$= \frac{1}{2}\left\{\left[\int_{-\infty}^{0} e^{x}\,dx\right] + \left[\int_{0}^{1} e^{-x}\,dx\right]\right\} \frac{1}{2}\left\{\left[\int_{-\infty}^{0} e^{y}\,dy\right]\right\}$$

$\left(\because \begin{matrix}|x| = x, x \ge 0 \\ |x| = -x \text{ when } x < 0\end{matrix}\right)$

$$= \frac{1}{2}\left\{\left[e^{x}\right]_{-\infty}^{0} - \left[e^{-x}\right]_{0}^{1}\right\}\left\{\frac{1}{2} \cdot \left[e^{y}\right]_{-\infty}^{0}\right\}$$

$$= \frac{1}{4}\left[(1-0) - (e^{-1} - 1)\right]\left[1 - 0\right]$$

$$= \frac{1}{4}\left[1 - e^{-1} + 1\right]$$

$$= \boxed{\frac{1}{4}\left[2 - e^{-1}\right]}$$

Average / Mean of a random variable

Mean or Average

The mean or average of any random variable is expressed by summation of the values of random variables X weighted by their probabilities.

→ Denoted by m_x

→ Mean value is also known as expected value of random variable X, $E[x]$

∴ Mean is denoted by $\boxed{m_x}$ or $\boxed{E[x]}$.

Mean value of discrete random variable

Let the discrete random variable X be take the following values

$$X = \{x_1, x_2, x_3 \ldots x_n\}$$

The mean or average m_x, is expressed as

$$m_x = x_1 P(x_1) + x_2 P(x_2) \ldots + x_n P(x_n)$$

$$\boxed{m_x = E[x] = \bar{X} = \sum_{i=1}^{n} x_i P(x_i)}$$

Here, $\bar{X}$, is also notation for mean value.

Mean Value of Continuous Random Variable

For continuous random variable X, the mean or average value is expressed as,

$$\boxed{m_x = \int_{-\infty}^{\infty} x\, f_X(x)\, dx}$$

where $f_X(x)$ = Probability Density Function.

In general,

The average value or expectation, of a function $g(X)$ of a random variable X is

$$\overline{g(X)} = E[g(X)] = \int_{-\infty}^{\infty} g(x)\, f_X(x)\, dx$$

Note: If $g(X) = X^n$, the avg value $E[X^n]$ is referred to as nth moment of the random variable.

→ So, avg value $\bar{X}$ is also called first moment of X.

→ If a random variable Z is a function of 2 two random variables X and Y say $Z = w(X, Y)$, then

$$\boxed{\bar{Z} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w(x,y)\, f_{XY}(x,y)\, dx\, dy}$$

In particular, if $Z = XY$, then

$$\bar{Z} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{XY}(x,y)\, dx\, dy$$

If X and Y are independent random variables, then

$$\bar{Z} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_X(x)\, f_Y(y)\, dx\, dy$$

$$\bar{Z} = \int_{-\infty}^{\infty} x\, f_X(x)\, dx \int_{-\infty}^{\infty} y\, f_Y(y)\, dy = \bar{X}.\bar{Y}$$

$$= m_x . m_y$$

Moments & Variance of a random variable:-

→ The nth moment of any random variable X may be defined as the mean value of X^n

i.e $g(X) = X^n$

Mean of X^n $= \overline{g(X)} = \int_{-\infty}^{\infty} x^n f_X(x)\, dx$

$$\Rightarrow \boxed{E[X^n] = \overline{X^n} = \int_{-\infty}^{\infty} x^n f_X(x)\, dx}$$

For, $\boxed{n=1,\ E[X] = \bar{X} = \int_{-\infty}^{\infty} x\, f_X(x)\, dx}$

$\therefore$ mean or avg value is called first moment of random variable X.

$n=2$, $\overline{X^2} = E[x^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx$

$\overline{X^2}$ is known as mean square value of random variable or second moment of random variable X.

→ Central Moments are the moments of the difference between random variable X and its mean 'm_x'.

→ n th Central Moment may be given as

$$\boxed{E[(X-m_x)^n] = \int_{-\infty}^{\infty} (x-m_x)^n f_X(x)\, dx}$$

The second central moment for $n=2$, is known as variance of random variable X

$$\text{Variance}[X] = E[(X-m_x)^2] = \sigma_x^2$$

$$\boxed{\sigma_x^2 = \int_{-\infty}^{\infty} (x-m_x)^2 f_X(x)\, dx}$$

$$\sigma_x^2 = \int_{-\infty}^{\infty} (x^2+m_x^2 - 2x m_x)\, f_X(x)\, dx$$

$$= \int_{-\infty}^{\infty} x^2 f_X(x)\, dx + \int_{-\infty}^{\infty} m_x^2 f_X(x)\, dx - 2\int_{-\infty}^{\infty} x\, m_x f_X(x)\, dx$$

$$= E[x^2] + m_x^2 \int_{-\infty}^{\infty} f_X(x)\, dx - 2 m_x \int_{-\infty}^{\infty} x f_X(x)\, dx$$

$$= E[x^2] + m_x^2 - 2 m_x E[x]$$

$$\left(\because \int_{-\infty}^{\infty} f_X(x)\, dx = 1\right)$$

$$\sigma_x^2 = E[x^2] + m_x^2 - 2 m_x \cdot m_x \qquad (\because E[x] = m_x)$$

$$\sigma_x^2 = E[x^2] - m_x^2$$

$$\therefore \boxed{\sigma_x^2 = E[x^2] - m^2}$$

(or)

$$E[(X-m)^2]$$

$$= E[x^2 + m^2 - 2mX]$$

$$= E[x^2] + m^2 - 2m\,E[x]$$

$$= E[x^2] + m^2 - 2m \cdot m$$

$$= E[x^2] - m^2$$

$$E[(X-m)^2] = E[x^2] - m^2$$

$$\boxed{\sigma_x^2 = E[x^2] - m^2}$$

Variance = Mean Square value − Square of the mean.

→ $\sigma_x^2 = \overline{x^2} - m^2$

→ Square root of variance is known as standard deviation of random variable X.

$$S.D = \sqrt{\text{Variance}}$$

$$\sigma_X = \sqrt{E[X^2] - m^2} = \sqrt{\overline{X^2} - m^2}$$

(Standard Deviation)

Ex :- Find if the $\boxed{\text{Value of } a}$ is fixed where PDF of X is defined as $f_X(x)$

$= a \cdot e^{-0.2x}$, for $x \geq 0$, and zero else elsewhere.

Ans : We know, $\int_{-\infty}^{\infty} f_X(x)\, dx = 1$

$$\Rightarrow \int_0^{\infty} a \cdot e^{-0.2x}\, dx = 1$$

$$\Rightarrow a \cdot \left[\frac{e^{-0.2x}}{-0.2}\right]_0^{\infty} = 1$$

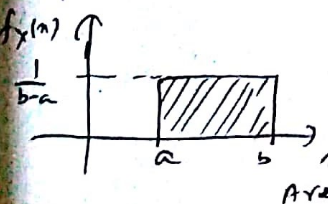
$$\Rightarrow \frac{-a}{0.2}[0-1] = 1$$

$$\Rightarrow \frac{a}{0.2} = 1$$

$$\Rightarrow \boxed{a = 0.2}$$

Ex :- Find mean & variance of random variable X which is uniformly distributed betn a & b and $a < b$.

Ans : Since X is uniformly distributed,



$$f_X(x) = \begin{cases} \frac{1}{b-a}, & \text{for } a \leq x \leq b \\ 0, & \text{else where.} \end{cases}$$

Area $= (b-a) \times \frac{1}{(b-a)} = 1$

From definition,

$$\text{mean} = m = \int_{-\infty}^{\infty} x \, f_X(x) \, dx$$

$$= \int_{a}^{b} x \cdot \left(\frac{1}{b-a}\right) \cdot dx$$

$$= \frac{1}{b-a} \int_{a}^{b} x \, dx$$

$$= \frac{1}{b-a} \cdot \left.\frac{x^2}{2}\right]_a^b$$

$$= \frac{1}{2(b-a)} \cdot \left[b^2 - a^2\right]$$

$$= \frac{1}{2\cancel{(b-a)}} (b+a)\cancel{(b-a)}$$

$$\boxed{\text{mean} = \frac{a+b}{2}}$$

Variance

$$\sigma^2 = \int_{-\infty}^{\infty} (x-m)^2 f_X(x) \, dx = E[X^2] - m^2$$

$$= \int_{a}^{b} (x^2) f_X(x) \, dx - m^2$$

$$= \int_{a}^{b} x^2 \times \frac{1}{b-a} \, dx - \left(\frac{a+b}{2}\right)^2$$

$$= \frac{1}{b-a} \cdot \left[\frac{x^3}{3}\right]_a^b - \left(\frac{a+b}{2}\right)^2$$

$$\sigma^2 = \frac{1}{3(b-a)} \times (b^3 - a^3) - \left(\frac{a+b}{2}\right)^2$$

$$= \frac{\cancel{(b-a)}(b^2 + a^2 + ab)}{3\cancel{(b-a)}} - \frac{(a+b)^2}{4}$$

$$= \frac{4b^2 + 4a^2 + 4ab - 3a^2 - 3b^2 - 6ab}{12}$$

$$= \frac{b^2 + a^2 - 2ab}{12}$$

$$\boxed{\sigma^2 = \frac{(b-a)^2}{12}}$$

Ex :- (BPUT 2003-04)

A random variable X has the uniform distribution given by

$$f_X(x) = \begin{cases} \frac{1}{2\pi}, & 0 \le x \le 2\pi \\ 0, & \text{elsewhere} \end{cases}$$

Determine, m, $\overline{X^2}$, σ_X.

Ans : $m = \int_0^{2\pi} x \cdot \frac{1}{2\pi} \cdot dx = \frac{1}{2\pi} \cdot \left.\frac{x^2}{2}\right]_0^{2\pi} = \frac{1}{2\pi} \times \frac{4\pi^2}{2} = \frac{2\pi}{2} = \pi$

$$\overline{X^2} = \int_0^{2\pi} x^2 \times \frac{1}{2\pi} dx = \frac{1}{2\pi} \times \left.\frac{x^3}{3}\right]_0^{2\pi}$$

$$= \frac{1}{3 \cdot \cancel{6\pi}} \cdot \cancel{8\pi^3} = \frac{4}{3}\pi^2$$

$$\sigma_x^2 = E[x^2] - m^2$$

$$= \frac{4}{3}\pi^2 - \pi^2$$

$$\boxed{\sigma_x^2 = \frac{1}{3}\pi^2}$$

$$\boxed{\sigma_x = \frac{\pi}{\sqrt{3}}}$$

Useful Probability Density Functions (PDFs)

1) Gaussian or Normal Distribution

→ Gaussian Distribution is also known as Normal Distribution.

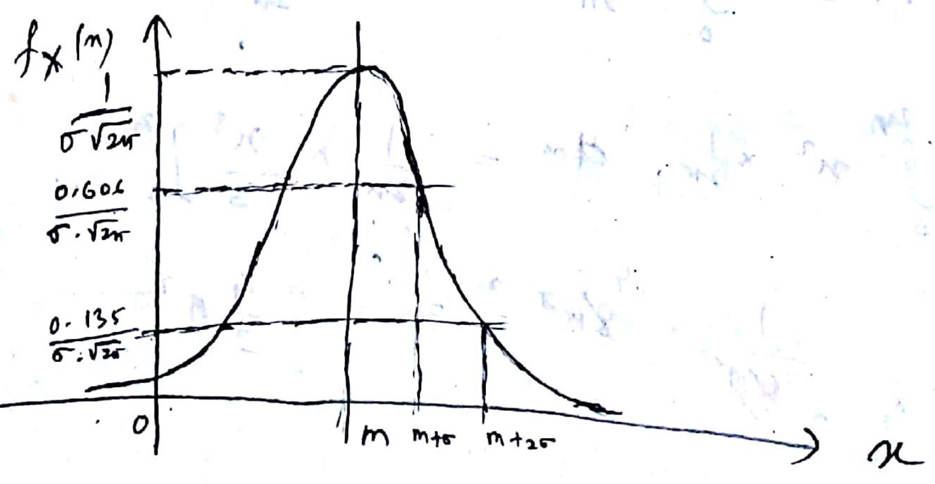
→ It is defined for a continuous random variable

→ The PDF for a Gaussian random variable is expressed as,

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}}$$

where m = Mean value of random variable

σ^2 = Variance of the random variable.



At $x = m$, $f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \times e^0 = \frac{1}{\sigma\sqrt{2\pi}}$.

At $x - m = \sigma$, $f_X(x) = \frac{1}{\sigma \cdot \sqrt{2\pi}} \cdot e^{-\frac{1}{2}} = \frac{0.606}{\sigma\sqrt{2\pi}}$.

$\Rightarrow x = m + \sigma$

At, $x - m = 2\sigma$, $f_X(x) = \frac{1}{\sigma \cdot \sqrt{2\pi}} \cdot e^{-2} = \frac{0.135}{\sigma \cdot \sqrt{2\pi}}$

$\Rightarrow x = m + 2\sigma$

Properties of Gaussian PDF

1) The peak value occures at $x = m$ i.e at mean value

Mathematically, $f_X(x) = \frac{1}{\sigma\sqrt{2\pi}}$ at $x = m$ (mean value)

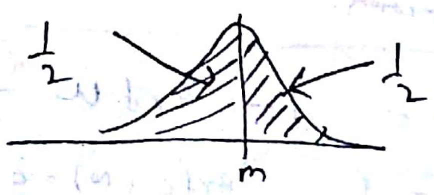
2) The plot of Gaussian PDF exhibit even symmetry around mean value.

Mathematically,

$$f_X(m - \sigma) = f_X(m + \sigma)$$

3) Area under the PDF curve is $\frac{1}{2}$ for all values of x below mean value and $\frac{1}{2}$ for all values of x above mean value.

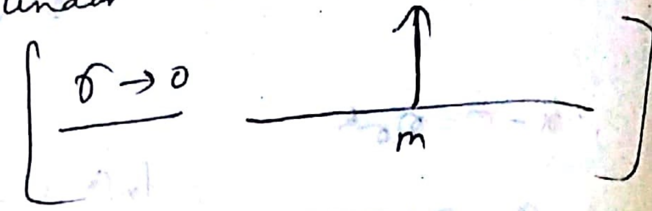
Mathematically, $P(X \leq m) = P(X > m) = \frac{1}{2}$.



Total Area $= \frac{1}{2} + \frac{1}{2} = 1$

4) As $\sigma \to 0$, Gaussian PDF approaches δ (impulse function located at $x = m$). This is because area under PDF is always unity.

Also area under impulse function is always unity.



Use/Appln of Gaussian PDF

a) The random motion of the thermally agitated electrons produce thermal noise. This thermal noise has Gaussian distribution.

b) The random errors on the experimental measurements creates the measured values to have Gaussian distribution about the true value.

c) Gaussian distribution is very important in the communication & statistical system.

2) Cummulative Gaussian Probability or Error Function

Error function →
$$\boxed{\operatorname{erf} u = \frac{2}{\sqrt{\pi}} \int_0^u e^{-u^2}\, du}$$

$$\operatorname{erf}(0) = 0, \quad \operatorname{erf}(\infty) = 1.$$

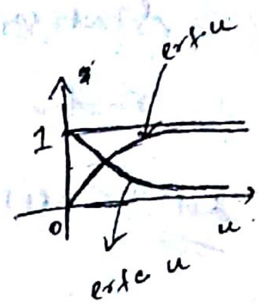
Complementary error function

$$\operatorname{erfc} u = 1 - \operatorname{erf} u = \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-u^2}\, du$$

$$\operatorname{erfc}(0) = 1, \quad \operatorname{erfc}(\infty) = 0$$

Bpur :- 2 mark: Reln betn error f^n & complementary error function

Ans:- $\boxed{erfc\ u = 1 - erf\ u}$



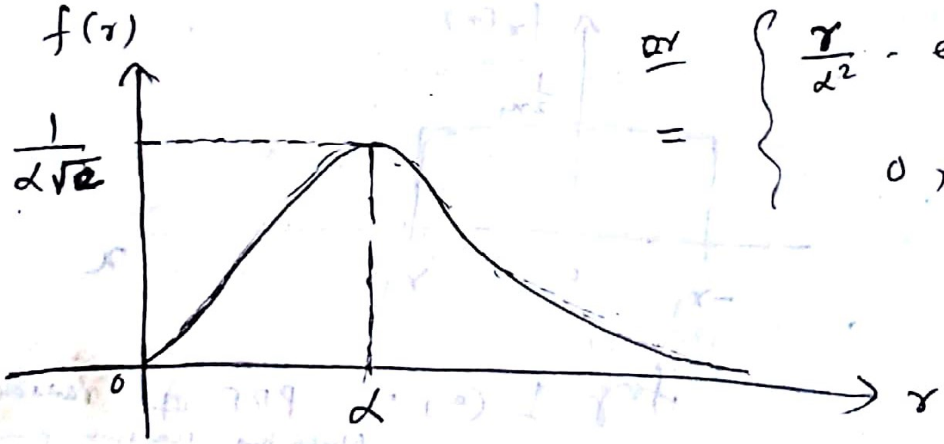
3) The Rayleigh Probability Density:-

→ Used for Continuous Random variable.

→ The Rayleigh density is defined as,

$$f(r) = \begin{cases} \frac{r}{\alpha^2} \cdot e^{-r^2/2\alpha^2}, & 0 \le r \le \infty \\ 0, & r < 0 \end{cases}$$

$$\text{or} = \begin{cases} \frac{r}{\alpha^2} \cdot e^{-\frac{r^2}{2\alpha^2}}, & r \geq 0 \\ 0, & r < 0 \end{cases}$$



$$f(r) = \frac{r}{\alpha^2} \cdot e^{-\frac{r^2}{2\alpha^2}}$$

At, $r = \alpha$, $f(r) = \frac{\cancel{\alpha}}{\alpha^{\cancel{2}}} \cdot e^{-\frac{\cancel{\alpha^2}}{2\cancel{\alpha^2}}}$

$$= \frac{1}{\alpha} \cdot e^{-\frac{1}{2}} = \frac{1}{\alpha \cdot \sqrt{e}}$$

→ Rayleigh's distribution has always +ve Value

→ It has appln in modelling of statistics of signals transmitted through radio channels such as ~~cell~~ Cellular radio.

2011 :- BPUT - Q) Explain Central limit theorem.

Ans :- Central-limit theorem indicates that the probability density of a sum of N independent random variables tends to approach a Gaussian density as the number N increases.

→ Fig 1 (a) shows the individual random variable has a uniform (constant) probability density.

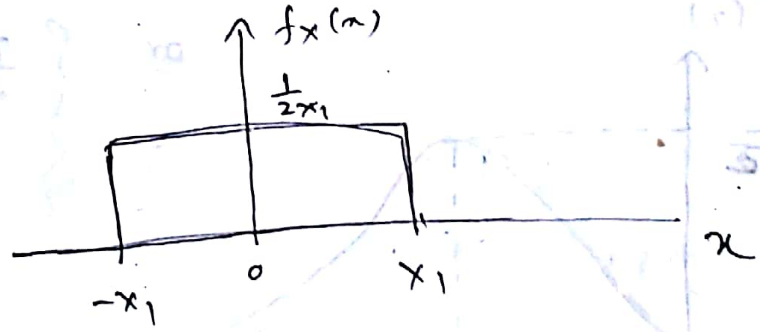


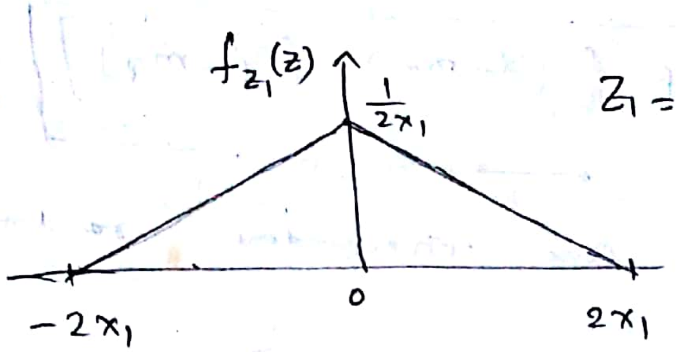
fig 1 (a) :- PDF of random variable X_1 which has uniform probability density

Area under the PDF = $\frac{1}{2x_1} \times 2x_1 = 1$.

→ Let there ~~~~ Z be the random variable which is sum of 2 random variables.

$$Z = X_1 + X_2.$$

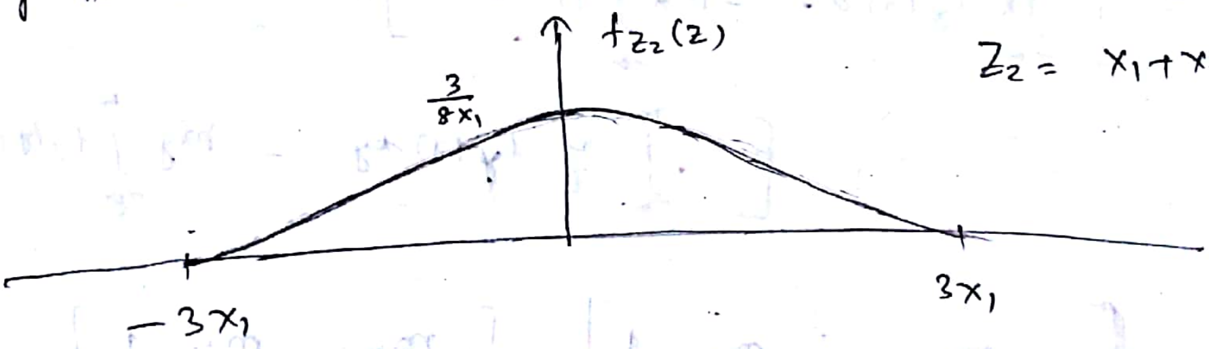
The Probability density of Z can be determined by convolution of density in fig 1 (a) with itself. The result is shown on fig 1 (b).



$Z_1 = X_1 + X_2$

fig 1 (b) :- Probability density of random variable $\frac{X_1 + X_2}{3}$

→ Similarly, the density of sum of random variables is the convolution of density in fig 1 (b) with the density on fig 1 (a). The result shown on fig 1 (c).



$Z_2 = X_1 + X_2 + X_3$

fig 1 (c) :- The density of the random variable $X_1 + X_2 + X_3$

→ Note that even for this sum of 3 terms the result suggest a gaussian density.

→ If more & more terms are added, the density indeed become Gaussian.

$(Ex \div Z = X_1 + X_2 + X_3 + X_4 + \cdots)$

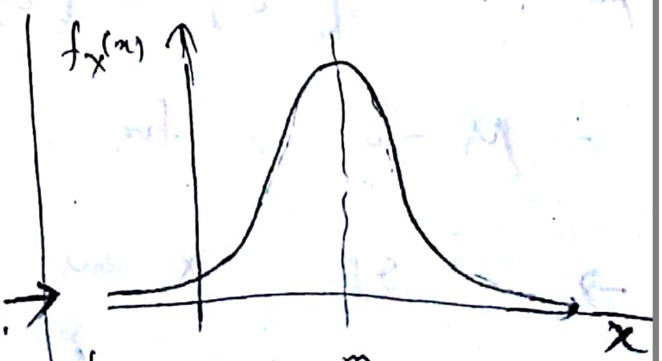


fig: 1(a)
Gaussian density function

Correlation between Random variables

→ The **Covariance** (μ) of 2 random variables X and Y is defined as

$$\boxed{\mu = E\left[(X - m_x)(Y - m_y)\right]}$$

→ If X and Y are independent random variables,

$$\mu = E\left[(X - m_x)(Y - m_y)\right]$$

$$= \left[\int_{-\infty}^{\infty} (x - m_x) f_X(x)\, dx\right]\left[\int_{-\infty}^{\infty} (y - m_y) f_Y(y)\, dy\right]$$

$$= \left[\int_{-\infty}^{\infty} x f_X(x)\, dx - m_x \int_{-\infty}^{\infty} f_X(x)\, dx\right] \times$$

$$\left[\int_{-\infty}^{\infty} y f_Y(y)\, dy - m_y \int_{-\infty}^{\infty} f_Y(y)\, dy\right]$$

$$= \left[m_x - m_x \cdot 1\right]\left[m_y - m_y \cdot 1\right]$$

$\left(\because \int_{-\infty}^{\infty} f_X(x)\, dx = \int_{-\infty}^{\infty} f_Y(y)\, dy = 1 \right.$

Area under PDF = 1)

$$\mu = [0] \times [0] = 0.$$

$\therefore \mu = 0$, for independent random variables

→ If X and Y are dependent

$$\mu = \int_{-\infty}^{\infty} (x - m_x)(y - m_y)\, f_{XY}(x, y)\, dx\, dy$$

$E\left[(X - m_x)(Y - m_y)\right]$

→ If there exist max^m dependency bet^n X and Y, such that

$X = Y$ or $X = -Y$,

Assuming $m_x = 0$, $m_y = 0$,

Note:-

$\sigma_x^2 = E[x^2] - m^2$

Since $m = 0$

$\sigma_x^2 = E[x^2]$

$$E[(X - m_x)(Y - m_y)] = E[XY]$$

If $\underline{X = Y}$, $E[XY] = E[X^2] = E[Y^2] = \sigma_x^2 = \sigma_y^2 = \sigma_x \sigma_y$

If $X = -Y$, $E[XY] = E[-X^2] = E[-Y^2] = -\sigma_x^2$

$= -\sigma_y^2 = -\sigma_x \sigma_y$.

→ Consider a quantity, ρ defined by

$$\boxed{\rho = \frac{\mu}{\sigma_x \sigma_y} = \frac{E[XY]}{\sigma_x \sigma_y}}$$

[Note:- $\mu = E[(X - m_x)(Y - m_y)]$, Since $m_x = 0$, $m_y = 0$

$\mu = E[XY]$]

ρ is called $\underline{\text{Correlation coefficient}}$ bet^n X and Y,

→ It is a $\underline{\text{measure}}$ of the extent to which X and Y are dependent.

→ $\boxed{-1 \leq \rho \leq 1}$

→ $\rho = 0$, if X and Y are independent

→ $\rho = 1$, when $X = Y$

⇒ $\rho = -1$, when $X = -Y$

→ If X and Y are neither identical nor independent, then ρ will have magnitude 0 to 1.

→ When $\rho = 0$, random variable X and Y are said to be uncorrelated.

→ When random variables are ~~correlated~~, independent, they are uncorrelated.

→ However, if they are uncorrelated does not ensure they are independent.

Ex:- Let Z be a random variable with probability density $f_Z(z) = \frac{1}{2}$ in the range $-1 \le z \le 1$. Let the random variable $X = Z$ and the random variable $Y = Z^2$. [Note:- X and Y are dependent i.e $Y = X^2$]. Show, nonetheless, (in spite of that) that X and Y are uncorrelated.

Ans :- We have

$$E[Z] = \int_{-1}^{1} z \cdot f(z)\, dz$$

$$= \int_{-1}^{1} z \cdot \left(\tfrac{1}{2}\right) dz = \frac{1}{2} \cdot \frac{z^2}{2}\Big]_{-1}^{1}$$

$$E[Z] = \frac{1}{4}[1-1] = 0.$$

$$E[Z] = 0$$

$\rightarrow$ $\boxed{E[X] = 0}$ $(\because X = Z)$

$\Rightarrow$ $\boxed{m_x = 0}$.

Similarly, $E[Y] = E[Z^2]$ $(\because Y = Z^2)$

$$= \int_{-1}^{1} z^2 f(z)\, dz$$

$$= \int_{-1}^{1} z^2 \cdot \left(\tfrac{1}{2}\right) dz$$

$$= \frac{1}{2}\left[\frac{z^3}{3}\right]_{-1}^{1}$$

$$= \frac{1}{6} \times [1+1]$$

$$= \frac{2}{6}$$

$$\boxed{E[Y] = 1/3}$$

$$M = E[(X - m_x)(Y - m_y)]$$

$$= E\left[(X-0)\left(Y - \tfrac{1}{3}\right)\right]$$

$$= E\left[Z\left(z^2 - \tfrac{1}{3}\right)\right]$$

$$= E\left[z^3 - \frac{z}{3}\right] = \int_{-1}^{1}\left(z^3 - \frac{z}{3}\right) \times \frac{1}{2}\, dz$$

$f(z)$

$$\therefore \quad \mu = \frac{1}{2}\int_{-1}^{1}\left(z^3 - \frac{z}{3}\right)dz$$

$$= \frac{1}{2}\left[\frac{z^4}{4} - \frac{1}{3}\cdot\frac{z^2}{2}\right]_{-1}^{1}$$

$$= \frac{1}{2}\left[\frac{z^4}{4} - \frac{z^2}{6}\right]_{-1}^{1}$$

$$= \left[\frac{z^4}{8} - \frac{z^2}{12}\right]_{-1}^{1}$$

$$= \frac{1}{8}\left[z^4\right]_{-1}^{1} - \frac{1}{12}\left[z^2\right]_{-1}^{1}$$

$$= \frac{1}{8}\times 0 - \frac{1}{12}\times 0$$

$$\boxed{\mu = 0}$$

$$\therefore \quad \rho = \frac{\mu}{\sigma_x \cdot \sigma_y} = 0$$

$\boxed{\rho = 0}$ i.e Variables are uncorrelated

$\therefore$ The Variables are uncorrelated, but they are dependent $(y = x^2)$.

$\Rightarrow$ Uncorrelated does not ensure they are independent

(Proved)