Chapter-4: Magnetostatic Fields

The static electric fields are characterized by E or D. On the other hand, the static magnetic fields are characterized by H or B. There are similarities and dissimilarities between electric and magnetic fields. As E and D are related according to $D = \epsilon E$, for linear, isotropic material space, H and B are related according to $B = \mu H$.

A definite link between electric and magnetic field was established by Oersted in 1820. As we have noticed, an electrostatic field is produced by static or stationary charges. If the charges are moving with constant velocity, a static magnetic (magnetostatic) field is produced.

A magnetostatic field is produced by a constant current flow (or direct current). This current flow may be due to magnetization current as in permanent magnets, electron-beam currents as in vacuum tubes, or conduction currents as in current-carrying wires.

Study of magnetostatics is an indispensable necessity. Motors, transformers, microphones, compasses, telephone bell ringers, television focusing controls, advertising displays, magnetically levitated high-speed vehicles, memory ~~tapes~~, stores, magnetic separators and so on, which play an important role in our everyday life, could not have been developed without an understanding of magnetic phenomena.

There are two major laws governing magnetostatic fields: (1) Biot-Savart's law and (2) Ampere's circuit law. Like Coulomb's law, ~~is~~ Biot-Savart's law is the general law of magnetostatics.

Just as Gauss's law is the special case of Coulomb's law, Ampere's law is a special case of Biot-Savart's law. and is easily applied in problems involving symmetrical current distribution.

The Table 1 shows the analogy between electric and magnetic field quantities.

Table 1:- Analogy between Electric and Magnetic fields

Term	Electric	Magnetic
Basic law	$F = \frac{Q_1 Q_2}{4\pi\epsilon_0 R^2} a_r$	$dB = \frac{\mu_0 I dl \times a_R}{4\pi R^2}$
	$\oint D \cdot ds = Q_{enc}$	$\oint H \cdot dl = I_{enc}$
Force law	$F = QE$	$F = Q(u \times B)$
Source element	dQ	$Qu = I\,dl$
Field intensity	$E = \frac{V}{l} \; (\frac{V}{m})$	$H = \frac{I}{l} \; (\frac{A}{m})$
Flux density	$D = \frac{\Psi}{S} \; (\frac{C}{m^2})$	$B = \frac{\Psi}{S} \; (\frac{Wb}{m^2})$
Relationship betn field and potential	$D = \epsilon E$	$B = \mu H$
	$E = -\nabla V$	$H = -\nabla V_m$
Flux	$V = \int \frac{\rho_L dl}{4\pi\epsilon_0}$	$A = \int \frac{\mu I dl}{4\pi R}$
	$\Psi = \int D \cdot ds$	$\Psi = \int B \cdot ds$
	$\Psi = Q = CV$	$\Psi = LI$
	$I = C\frac{dV}{dt}$	$V = L\frac{dI}{dt}$
Energy density	$W_E = \frac{1}{2} D \cdot E$	$W_m = \frac{1}{2} B \cdot H$
Poisson's Equation	$\nabla^2 V = -\frac{\rho_v}{\epsilon}$	$\nabla^2 A = -\mu J$

Where
J = Volume current density
= Ampere per meter squared $= \frac{A}{m^2}$

Biot-Savart's law

Biot-Savart's law states that the differential magnetic field intensity dH produced at a point P, as shown in figure 4.1, by the differential current element Idl is proportional to the product Idl and sine of the angle 'α' between the element and the line joining P to the element and is inversely proportional to the square of the distance R between P and the element.

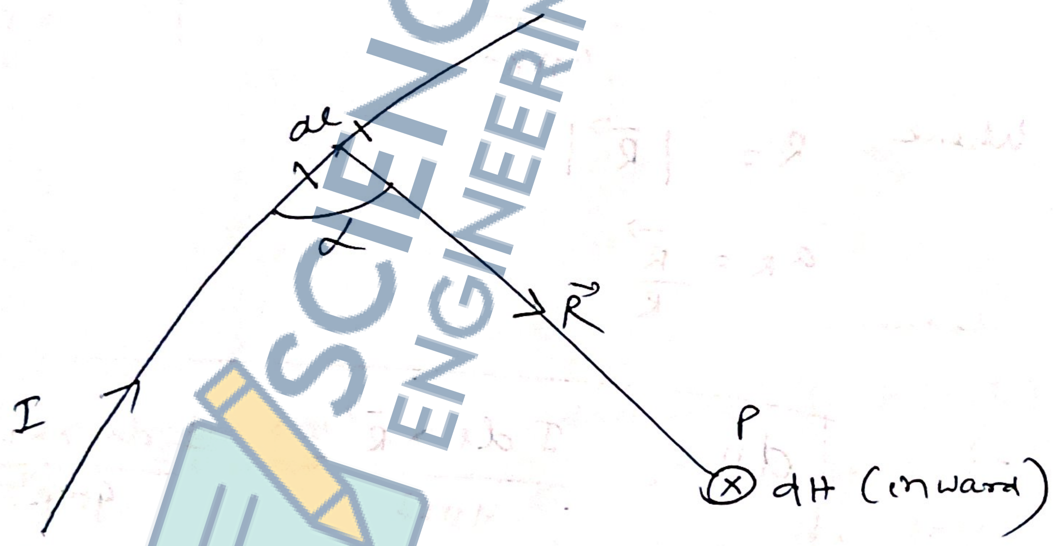


Fig 4.1 Magnetic field dH at P due to current element Idl

Mathematically

$$dH \propto \frac{I\,dl \sin\alpha}{R^2} \qquad \text{——— (4.1)}$$

$$\Rightarrow dH = \frac{K\,I\,dl \sin\alpha}{R^2} \qquad \text{——— (4.2)}$$

Where K is the constant of proportionality. In SI Units, $K = \frac{1}{4\pi}$

So eqn (4.2) becomes

$$dH = \frac{I\,dl \sin\alpha}{4\pi R^2} \quad \text{———} \quad (4.3)$$

From the definition of cross product, eqn (4.3) can be put in vector form as

$$dH = \frac{I\,dl \times a_R}{4\pi R^2}$$

$$= \frac{I\,dl \times \vec{R}}{4\pi R^3}$$

$\because$ $I\,dl \times a_R$

$= |I\,dl|\,|a_R| \sin\alpha$

$= I\,dl \cdot 1 \cdot \sin\alpha$

$= I\,dl \sin\alpha$

where $R = |\vec{R}|$

$$a_R = \frac{\vec{R}}{R}$$

$$\therefore \boxed{dH = \frac{I\,dl \times \vec{R}}{4\pi R^3} = \frac{I\,dl \times a_R}{4\pi R^2}} \quad \text{—} (4.4)$$

$\vec{R}$ and $\vec{dl}$ are illustrated in fig 4.1.

The direction of dH can be determined by the right-hand rule with the right-hand thumb pointing in the direction of the current and the right-hand fingers encircling the wire in the direction of dH as shown in figure 4.2 (a).

Alternatively, we can use the right-handed-screw rule to determine the direction of dH: with the screw placed along the wire and pointed in the direction of current flow, the direction of advance of the screw is the direction of dH as in figure 4.2(b).

It is customary to represent the direction of the magnetic field intensity H (or current I) by a small circle with a dot or cross sign depending on whether H (or I) is out of, or into the page as illustrated in figure 4.3.

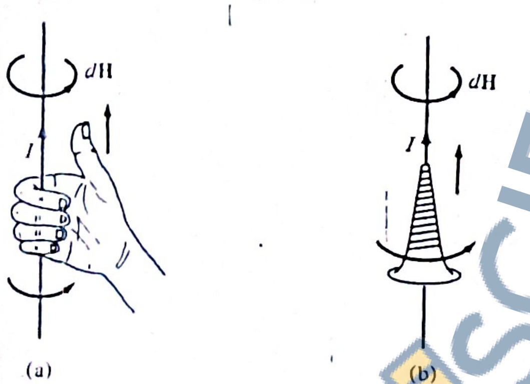


Figure 4.2 Determining the direction of dH using (a) the right-hand rule or (b) the right-handed-screw rule.

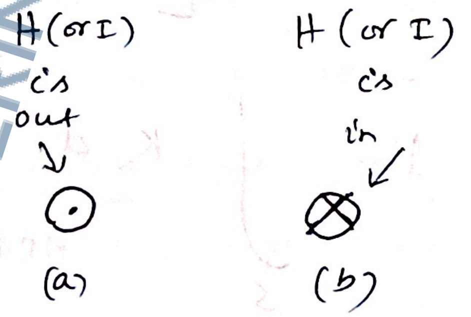


Fig 4.3: Conventional representation of H (or I)
(a) out of page
(b) into the page

- Just we have different charge distribution (point/ line/ surface/ volume), we can have different current distributions: line current, surface current, and volume current as shown in fig 4.4. If we define K as the surface current density in amperes per meter and

J is the volume current density in amperes per meter squared, the source elements are related as

$$I\,dl \equiv K\,ds \equiv J\,dv \qquad (4.5)$$

Thus in terms of distributed current sources, the Biot-Savart's law in eqn (4.4) becomes

$$H = \int_L \frac{I\,dl \times a_R}{4\pi R^2} \quad (\text{line current}) \qquad (4.6)$$

$$H = \int_S \frac{K\,ds \times a_R}{4\pi R^2} \quad (\text{surface current}) \qquad (4.7)$$

$$H = \int_V \frac{J\,dv \times a_R}{4\pi R^2} \quad (\text{volume current}) \qquad (4.8)$$

Where a_R is the unit vector pointing from the differential element of current to the point of interest.

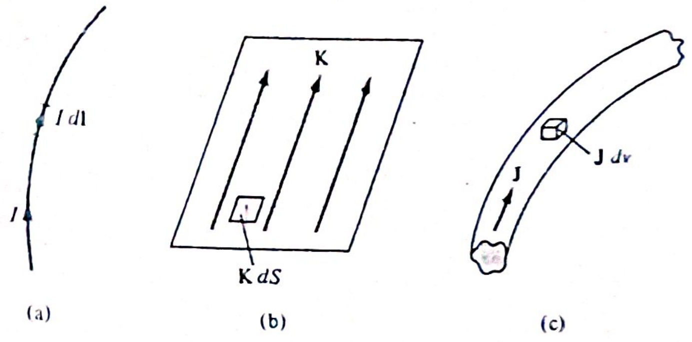


Fig: 4.4 — **Figure 4.4** Current distributions: (a) line current, (b) surface current, (c) volume current.

To Determine field due to Straight ~~Conductor~~ Current-carrying Conductor

Consider a current-carrying conductor AB as shown in figure 4.5. We assume that the conductor is along the Z-axis with its upper and lower ends, respectively, subtending angles α_2 and α_1 at P, the point at which H is to be determined.

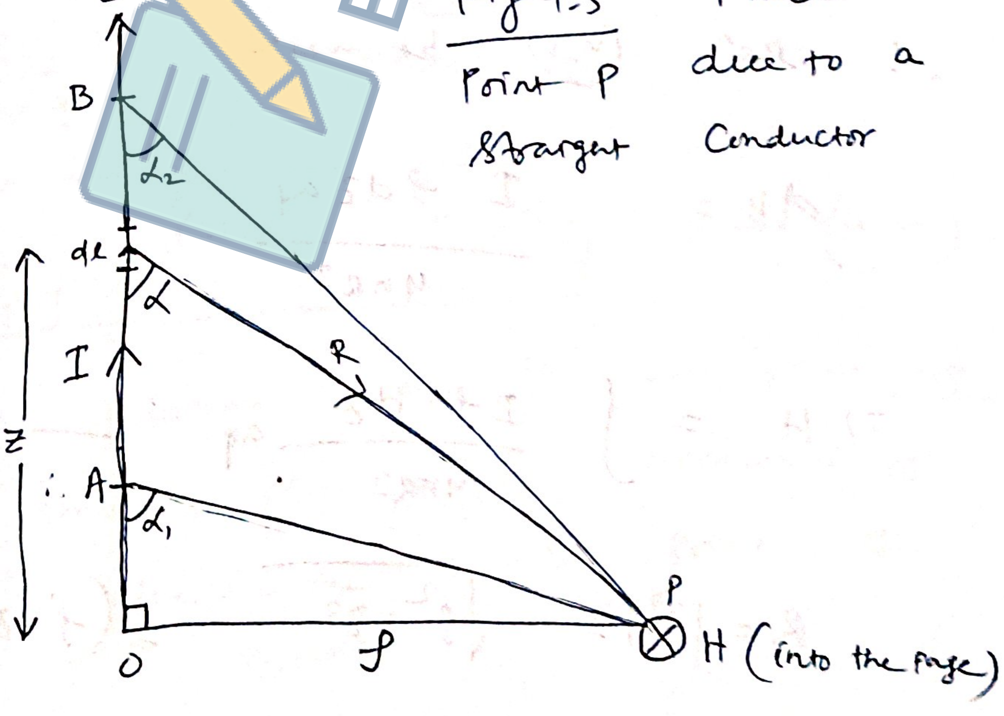


Fig 4:5 Field at Point P due to a Straight Conductor

Fig 4.5

Current flows from point A, where $\alpha = \alpha_1$ to point B, where $\alpha = \alpha_2$. If we consider the contribution $\underline{dH}$ at P due to an element dl $(0, 0, z)$,

$$dH = \frac{I\, dl \times \vec{R}}{4\pi R^3} \quad \text{—— (4.9)}$$

But $dl = dz\, a_z$, $\vec{R} = (\rho, \phi, 0) - (0, \phi, z)$

$$\vec{R} = \rho\, a_\rho - z\, a_z$$

$$\vec{dl} \times \vec{R} = \begin{vmatrix} a_\rho & a_\phi & a_z \\ 0 & 0 & dz \\ \rho & 0 & -z \end{vmatrix}$$

$$= a_\rho [0] - a_\phi (0 - \rho\, dz) + a_z (0 - 0)$$

$$dl \times \vec{R} = \rho\, dz\, a_\phi \quad \text{—— (4.10)}$$

$\therefore$ eqn (4.9) becomes,

$$dH = \frac{I\, \rho\, dz\, a_\phi}{4\pi R^3}$$

$$\Rightarrow H = \int \frac{I \rho\, dz}{4\pi R^3} a_\phi \quad \text{—— (4.11)}$$

$$R = |\vec{R}| = \sqrt{\rho^2 + z^2} = (\rho^2 + z^2)^{\frac{1}{2}}$$

$\therefore$ Eqn (4.11) becomes,

$$\Rightarrow H = \int \frac{I \rho \, dz}{4\pi (\rho^2+z^2)^{\frac{3}{2}}} a_\phi \quad \text{——— (4.12)}$$

Let $z = \rho \cot\alpha$, $dz = \rho . -\operatorname{cosec}^2\alpha \, d\alpha$

$\therefore$ eqn (4.12) becomes,

$$H = \frac{1}{4\pi} \int_{\alpha_1}^{\alpha_2} \frac{I \times \rho \, (-\rho \operatorname{cosec}^2\alpha) \cdot d\alpha}{\rho^3 \operatorname{cosec}^3\alpha} a_\phi$$

$$\Rightarrow H = -\frac{I}{4\pi} \int_{\alpha_1}^{\alpha_2} \frac{\cancel{\rho^2}}{\cancel{\rho^3}\,\rho} \sin\alpha \, d\alpha \; a_\phi$$

$$\Rightarrow H = \frac{-I}{4\pi\rho} \int_{\alpha_1}^{\alpha_2} \sin\alpha \, d\alpha \; a_\phi$$

$$\Rightarrow H = \frac{-I}{4\pi\rho} \Big[-\cos\alpha\Big]_{\alpha_1}^{\alpha_2} a_\phi$$

$$\Rightarrow \boxed{H = \frac{I}{4\pi\rho} (\cos\alpha_2 - \cos\alpha_1) a_\phi} \quad \text{——— (4.13)}$$

Note:- H is always along the unit vector a_ϕ (i.e along concentric circular paths) irrespective of the length of the wire or the point of interest P.

Case-I : Semi infinite Conductor

When the conductor is semi infinite, the point A is at (0,0,0), B is at $(0,0,\infty)$.

$\alpha_1 = 90^\circ$, $\alpha_2 = 0^\circ$, and eqn (4.13) becomes,

$$H = \frac{I}{4\pi\rho}\left[\cos 0^\circ - \cos 90^\circ\right] a_\phi$$

$$\Rightarrow \boxed{H = \frac{I}{4\pi\rho}\, a_\phi} \quad \text{—— (4.14)}$$

Case-II Infinite Conductor

For this case, point A is at $(0,0,-\infty)$ while B is at $(0,0,\infty)$; $\alpha_1 = 180^\circ$ and $\alpha_2 = 0^\circ$ and eqn (4.13) becomes

$$H = \frac{I}{4\pi\rho}\left[\cos 0^\circ - \cos 180^\circ\right] a_\phi$$

$$= \frac{\cancel{2}}{2\,\cancel{4}\pi\rho}\, a_\phi$$

$$\Rightarrow \boxed{H = \frac{I}{2\pi\rho}\, a_\phi} \quad \text{—— (4.15)}$$

where

$$\boxed{a_\phi = a_\ell \times a_\rho} \quad \text{—— (4.16)}$$

, where a_ℓ is a unit vector along the line current and a_ρ is a unit vector along the perpendicular line from the line current to the field point.

Example 4.1 The conducting triangular loop in fig 4.6 (a) carries a current of 10 A. Find H at (0,0,5) due to side 1 of the loop.

Ans :-

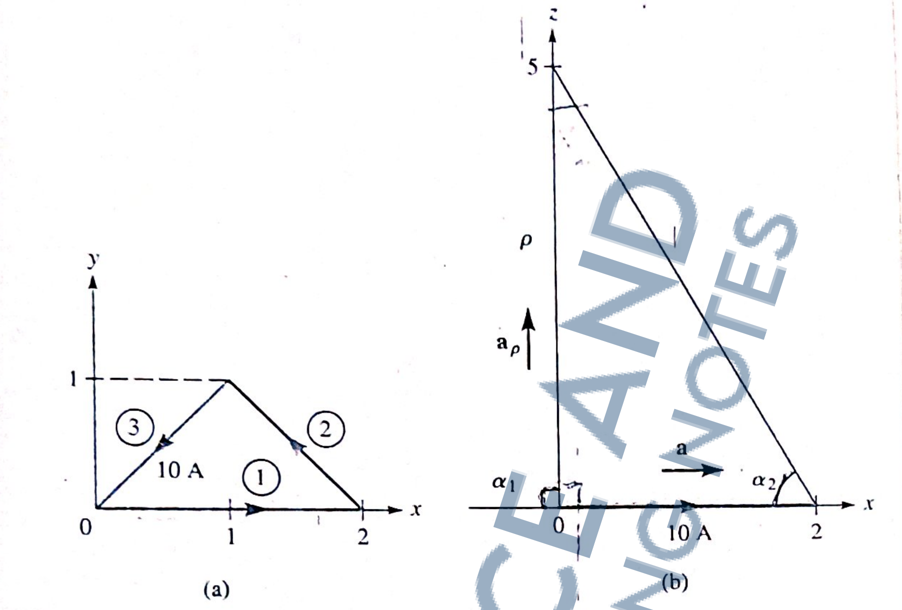


Figure 4.6 For Example 4.1: (a) conducting triangular loop. (b) side 1 of the loop.

- To find H at (0,0,5) due to side 1 of the loop, in fig 4.6 (a), consider fig 4.6 (b), where side 1 is treated as a straight conductor.

→ Notice at we join the point of interest (0,0,5) to the begining and end of the line current.

$\therefore \alpha_1 = 90^\circ$ [α_1 & α_2 and ρ are assigned in the same manner as in fig 4.5 on which eqn (4.13) is based]

$\cos\alpha_1 = 0$.

Similarly

$$\cos\alpha_2 = \frac{2}{\sqrt{5^2+2^2}} = \frac{2}{\sqrt{29}}, \quad \text{and } \rho = 5$$

$$a_\phi = a_\ell \times a_\rho = a_x \times a_z = -a_y$$

$$\therefore H = \frac{10}{4\pi\rho}\left[\cos\alpha_2 - \cos\alpha_1\right] a_\phi$$

$$= \frac{10}{4\pi \times 5}\left[\frac{2}{\sqrt{29}} - 0\right](-a_y)$$

$$= \frac{-10}{20\pi}\left[\frac{2}{\sqrt{29}}\right] a_y$$

$$\Rightarrow \boxed{H = 59.1 \frac{mA}{m}}$$

Ampere's Circuit law - Maxwell's Equation

Ampere's circuit law states that the line integral of H around a closed path is the same as the net current $\underline{I_{enc}}$ enclosed by the path.

In otherwords, the circulation of H equals I_{enc}; that is

$$\boxed{\oint H \cdot dl = I_{enc}} \quad \text{——} \quad (4.17)$$

→ Ampere's law is similar to Gauss's law, since Ampere's law is easily applied to determine H when the current distribution is symmetrical.

→ Ampere's law is a special case of Biot-Savart's law.

By applying Stoke's theorem to the left-hand side of eqn (4.17), we obtain

$$I_{enc} = \oint H \cdot dl = \int_S (\nabla \times H) \cdot ds \quad \text{[Stoke's thm eqn (2.41)]} \quad \text{——— (4.18)}$$

But

$$I_{enc} = \int_S J \cdot ds \quad \text{——— (4.19)}$$

Comparing the surface integral in eqn (4.18) & (4.19) clearly reveals that

$$\boxed{\nabla \times H = J} \quad \text{——— (4.20)}$$

This is the third Maxwell's equation, It is essentially Ampere's law in differential form. where eqn (4.17) is the integral form.

→ Note that $\nabla \times H = J \neq 0$; that is magnetostatic field is not conservative.

Applications of Ampere's law

We now apply Ampere's circuit law to determine 'H' for some symmetrical current distributions as we did for Gauss's law. We will consider an infinite line current. We apply $\boxed{\oint_L H \cdot dl = I_{enc}}$

Infinite line current

Consider an infinitely long current $\underline{I}$ along the z-axis as in figure 4.7. To determine $\underline{H}$ at an observation point P, we allow a closed path to pass through P. This path, on which Ampere's law is to be applied, is known as an Amperian path (analogous to the term "Gaussian surface").

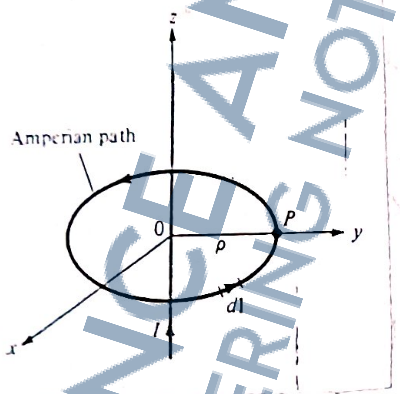


Fig: 4.7: Ampere's law applied to an infinite line current

We choose a concentric circle as the Amperian path in view of eqn (4.15) which shows that H is constant provided 'ρ' is constant. Since this path encloses the whole current I, according to Ampere's law

$$I = \int H_\phi a_\phi \cdot \rho \, d\phi \, a_\phi$$

$$= H_\phi \int_0^{2\pi} \rho \, d\phi$$

dl in cylindrical co-ordinate system

$dl = d\rho\, a_\rho + \rho\, d\phi\, a_\phi + dz\, a_z$

for a_ϕ component

$dl = \rho\, d\phi\, a_\phi$

$\Rightarrow \quad I = H_\phi \times \rho \times 2\pi$

$$\Rightarrow \boxed{H_\phi = \frac{I}{2\pi\rho} a_\phi} \quad \text{———} \quad (4.21)$$

As expected from eqn (4.15), (Proved)

Magnetic flux density - Maxwell's Equation

The magnetic flux density B is similar to the electric flux density D. As $D = \epsilon_0 E$ in free space, the magnetic flux density B is related to the magnetic field intensity H according to

$$\boxed{B = \mu_0 H} \quad \text{———} \quad (4.22)$$

Where μ_0 is a constant known as the permeability of free space. The constant is in Henry / meter $\left(\frac{H}{m}\right)$ and has the value of

$$\boxed{\mu_0 = 4\pi \times 10^{-7} \frac{H}{m}} \quad \text{———} \quad (4.23)$$

The magnetic flux through a surface S is given by

$$\boxed{\Psi = \int_S B \cdot ds} \quad \text{———} \quad (4.24)$$

(Similar to eqn (3.18) i.e $\Psi = \int_S D \cdot ds$)

150

$\Rightarrow \quad I = H_\phi \times \rho \times 2\pi$

$$\Rightarrow \quad \boxed{H_\phi = \frac{I}{2\pi\rho}\, a_\phi} \quad \text{--- (4.21)}$$

As expected form eqn (4.15). (Proved)

B. Infinite Sheet of Current

Consider an infinite current sheet on the $Z=0$ plane (XY plane). If the sheet has a uniform current density $K = K_y a_y \ \frac{A}{m}$ as shown in figure 1. Applying Ampere's law to the rectangular closed path 1-2-3-4-1 (Amperian path) gives

$$\oint H \cdot dl = I_{enc} = K_y b \quad \text{—— (1)}$$

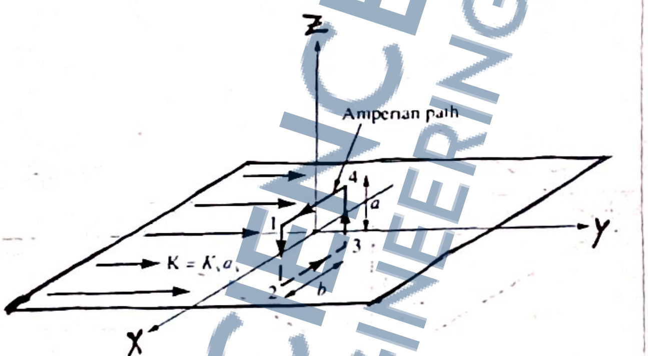


Fig.1 Application of Ampere's law to an infinite sheet (Amperian path 1-2-3-4-1)

Since it is a infinite sheet, a number of amperian path we have to take to accomodate the surface current. We have considered one path. We can consider 2 extra amperian paths on both the sides of of the Amperian path considered in figure 1.

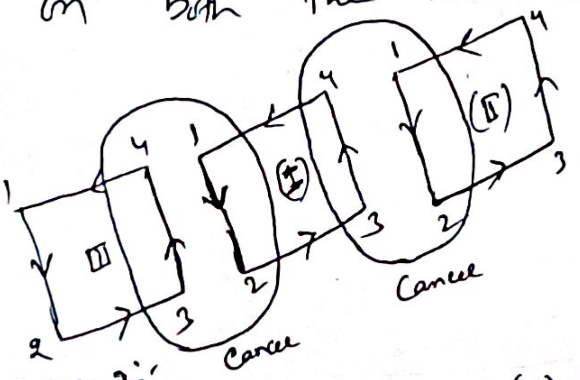


Fig 2:

We can observe, for closed path (I) (Fig 2), as (1-2) & (3-4) are cancelled by 2 other Amperian Path. So 'H' due to (1-2) & (3-4) of amperian path (I) is 0 (zero)

→ only we have path (2-3) and (4-1) of amperian path (I) to calculate the 'H' field intensity.

For path 4 to 1, if we place the thumb along Y-axis, rest fingers upward direction ⟨because path above XY plane⟩ the curling of fingers gives the direction of H along +ve X-axis.

$$H_{4-1} = H_0 \, a_x \quad \text{——— (2)}$$

For path 2 to 3 if we place thumb along Y-axis, rest fingers downward ⟨because path below XY plane⟩ direction, the curling of fingers gives the direction of H along −ve X-axis.

$$H_{2-3} = H_0 (-a_x) \quad \text{——— (3)}$$

Combining (2) & (3)

$$H = \begin{cases} H_0 \, a_x, & Z > 0 \\ -H_0 \, a_x, & Z < 0 \end{cases} \quad \text{——— (4)}$$

Where H_0 is yet to be determined

$$\therefore \oint H \cdot dl = \left[\int_1^2 + \int_2^3 + \int_3^4 + \int_4^1 \right] H \cdot dl$$

$$\Rightarrow \oint H \cdot dl = 0 \times (-a) + (-H_0)(-b) + 0 \cdot a + H_0(b)$$

$\because$ Path $\rightarrow$ $-ve$ Z-axis

$1-2 \rightarrow dl \rightarrow -a$

$2-3 \rightarrow H \rightarrow -H_0$ (Eqn (3))

& $dl = -b$ ($-ve$ x axis)

$3-4 \rightarrow dl \rightarrow a$

$4-1 \rightarrow H \rightarrow H_0$ $\rightarrow dl \rightarrow b$

$$\Rightarrow \oint H \cdot dl = 2 H_0 b \quad \text{--- (5)}$$

Equating (1) & (5)

$$K_y \cancel{b} = 2 H_0 \cancel{b}$$

$$\Rightarrow H_0 = \frac{K_y}{2} \quad \text{--- (6)}$$

Putting eqn (6) in eqn (4)

$$\boxed{H = \begin{cases} \dfrac{K_y}{2} a_x, & Z > 0 \\ -\dfrac{K_y}{2} a_x, & Z < 0 \end{cases}} \quad \text{--- (7)}$$

Magnetic flux density – Maxwell's Equation

The magnetic flux density $\underline{B}$ is similar to the electric flux density $\underline{D}$. As $\underline{D = \epsilon_0 E}$ in free space, the magnetic flux density $\underline{B}$ is related to the magnetic field $\underline{\text{intensity } H}$ according to

$$\boxed{B = \mu_0 H} \quad \text{—— (4.22)}$$

Where $\underline{\mu_0}$ is a constant known as the permeability of free space. The constant is in Henry / meter $\left(\frac{H}{m}\right)$ and has the value of

$$\boxed{\mu_0 = 4\pi \times 10^{-7} \frac{H}{m}} \quad \text{—— (4.23)}$$

The magnetic flux through a surface S is given by

$$\boxed{\Psi = \int_S B \cdot ds} \quad \text{—— (4.24)}$$

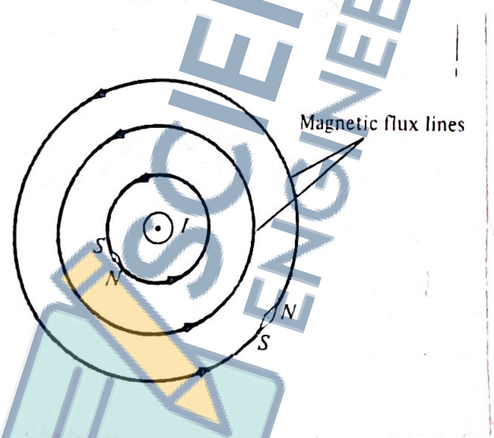
(Similar to eqn (3.18) i.e. $\Psi = \int_S D \cdot ds$)

Where the magnetic flux (ψ) is in webers (Wb) and the magnetic flux density is in webers per square meter $\left(\frac{Wb}{m^2}\right)$ or Teslas (T).

Note:-

1) A magnetic flux line is a path to which B is tangential at every point on the line. It is a line along which the needle of a magnetic compass will orient itself if placed in the presence of a magnetic field.

2) The magnetic flux lines due to a straight long wire are shown in Fig. 4.8.



(Fig 4.8) Magnetic flux lines due to a straight wire with current coming out of the page

3) The flux lines are ~~determi~~ closed and has no beginning or end. Also they don't cross each other regardless of current distribution. The direction of 'B' is taken as that indicated as "north" by the needle of the magnetic compass.

4) It is not possible to have isolated magnetic poles (or magnetic charge). For example, if we desire to have an isolated magnetic pole by dividing a magnetic bar successively into two, we end up with pieces each having north and south poles as illustrated in Figure 4.9. We find it impossible to separate the north pole from the south pole.

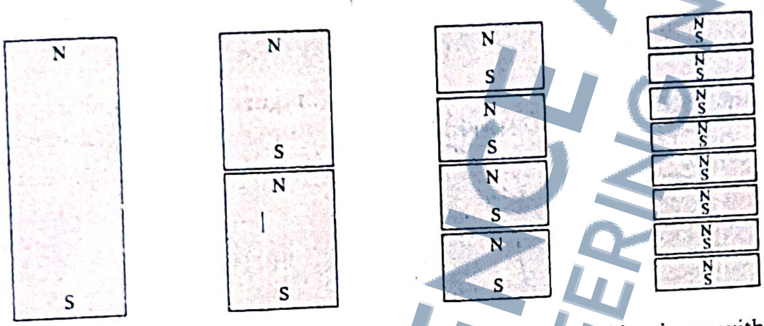


Figure ~~6.18~~ 4.9 Successive division of a bar magnet results in pieces with north and south poles, showing that magnetic poles cannot be isolated.

6.4

5) Thus isolated magnetic charge does not exist.

Thus the total flux through a closed surface in a magnetic field must be zero; that is

$$\boxed{\oint B \cdot dS = 0} \qquad \text{—— (4.25)}$$

This eq^n is referred to as the law of conservation of magnetic flux or Gauss's law for magnetostatic fields, Just as $\oint D \cdot ds = Q$ is Gauss's law for electrostatic fields.

6) Although the magnetostatic field (H) is not conservative, magnetic flux (B) is conserved.

Applying Divergence theorem to eqn (4.25), we have

$$\oint B \cdot ds = \int_V \nabla \cdot B \, dv = 0$$

or

$$\boxed{\nabla \cdot B = 0} \quad (4.26)$$

This equation is the 4th Maxwell's Equation. Equation (4.25) or (4.26) shows that magnetostatic fields have no sources or sinks. Equation (4.26) suggests that magnetic field lines are always continuous. [Since divergence of B is zero]

Maxwell's Equations for Static fields

We have already derived Maxwell's equations for static fields. Now we put them together in a Table 1. The choice bet^n differential & integral form of the equations depends on a given problem. From the table 1, it can be seen that a vector field is defined completely by specifying its curl and its divergence.

→ A field can be electric or magnetic only if it satisfies the corresponding Maxwell's equations.

→ Note that Maxwell's equations as in Table 1, are only for static electric and magnetic fields. For ~~dipo~~ time-varying EM fields, the divergence eq^n^s will remain same and curl eq^n^s will be modified. [will be derived later]

Table 1: Maxwell's eq^n^s for static electric & magnetic fields.

Differential or Point form	Integral form	Remarks
$\nabla \cdot D = \rho_v$	$\oint_S D \cdot ds = \int_V \rho_v \, dV$	Gauss's law
$\nabla \cdot B = 0$	$\oint_S B \cdot ds = 0$	Nonexistence of magnetic monopole
$\nabla \times E = 0$	$\oint_L E \cdot dl = 0$	Conservative nature of electrostatic field
$\nabla \times H = J$	$\oint_L H \cdot dl = \int_S J \cdot ds$	Ampere's law

Magnetic Scalar & Vector Potentials

We recall that some electrostatic problems were simplified by relating the electric potential 'V' to the electric field intensity (E) ($E = -\nabla V$). Similarly, we can define a potential associated with magnetostatic field B. In fact, the magnetic potential could be scalar V_m or vector 'A'. To define V_m and A involves two important identities [curl of gradient = 0, Divergence of curl = 0]

$$\nabla \times (\nabla V) = 0 \quad \text{——— (4.27)}$$

$$\nabla \cdot (\nabla \times A) = 0 \quad \text{——— (4.28)}$$

which must always hold for any scalar field 'V' and vector field 'A'.

Just as $E = -\nabla V$, we define the magnetic scalar potential V_m (in Amperes) as related to H according to

$$\boxed{H = -\nabla V_m}, \quad \text{if } J = 0 \text{ — (4.29)}$$

The condition attached to this equation is important and will be explained.

From Maxwell's eq^n

$$\nabla \times H = J \quad \text{— (4.30)}$$

But from eq^n (4.27)

$$\nabla \times (\nabla V) = 0 \quad \text{— (4.31)}$$

Similarly to

If $\nabla \times E = 0$

$E = -\nabla V$

because curl of grad = 0

$\nabla \times (-\nabla V) = 0$

$\Rightarrow \nabla \times H = J$, if $J = 0$, $H = -\nabla V_m$

If $J = 0$ in eqn (4.30), then

$$\nabla \times H = 0 \qquad \text{—— (4.32)}$$

Equating (4.31) & (4.32),

$$H = \nabla V \qquad \text{—— (4.33)}$$

Similar to (the logic explained in) electrostatics, in the magnetostatic

$$H = -\nabla V_m \qquad \text{—— (4.34)}$$

where $V_m \rightarrow$ Magnetic scalar potential.

Also $\underline{V_m}$ satisfies $\underline{\text{Laplace's equation}}$ just as V does for electrostatic fields; hence

$$\nabla^2 V_m = 0 \;,\; (J = 0) \qquad \text{—— (4.35)}$$

→ From Maxwell's equation,

$$\nabla \cdot B = 0 \qquad \text{—— (4.36)}$$

From eqn (4.28),

$$\nabla \cdot (\nabla \times A) = 0 \qquad \text{—— (4.37)}$$

$\underline{\text{Equating (4.36) \& (4.37)}}$, we can define Vector magnetic potential $\underline{'A'}$ (in Wb/m) such that

$$\boxed{B = \nabla \times A} \qquad \text{—— (4.38)}$$

Just as we have defined

$$V = \int \frac{dQ}{4\pi\epsilon_0 R} \qquad \text{—— (4.39)}$$

We can define

$$\boxed{A = \int_L \frac{\mu_0 I\, dl}{4\pi R}}$$ for line current — (4.40)

$$\boxed{A = \int_S \frac{\mu_0 K\, ds}{4\pi R}}$$ for surface current — (4.41)

$$\boxed{A = \int_V \frac{\mu_0 J\, dv}{4\pi R}}$$ for volume current — (4.42)

(Derivations not required)

→ From eqn (4.24), we have

$$\psi = \int_S B \cdot ds \quad \text{— (4.43)}$$

Putting eqn (4.38) into eqn (4.43), we have

$$\psi = \int_S (\nabla \times A)\, ds \quad \text{— (4.44)}$$

Applying Stoke's thm in eqn (4.44)

$$\boxed{\psi = \int_S (\nabla \times A)\, ds = \oint_L A \cdot dl} \quad \text{— (4.45)}$$

or $$\boxed{\psi = \oint_L A \cdot dl} \quad \text{— (4.46)}$$

Thus the magnetic flux through a given area can be found by using eqn (4.43) or eqn (4.46). Also magnetic field (H) can be determined either V_m or A;

$$\left[\because H = -\nabla V_m \quad , \quad H = \frac{1}{\mu}B = \frac{1}{\mu}(\nabla \times A)\right]$$

The choice is dictated by the nature of the given problem except that V_m can be used only in a source-free region ($J=0$).

Example 4.2

Given the magnetic vector potential $\boxed{A = -\frac{\rho^2}{4} a_z \frac{Wb}{m}}$

Calculate the total magnetic flux crossing the surface $\phi = \frac{\pi}{2}$, $1 \le \rho \le 2m$, $0 \le z \le 5m$.

Ans:- Method 1

$$\psi = \int_S B \cdot ds = \int_S (\nabla \times A)\, ds$$

$$\nabla \times A = \frac{1}{\rho}\begin{vmatrix} a_\rho & \rho a_\phi & a_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ 0 & 0 & -\frac{\rho^2}{4} \end{vmatrix}$$

$$= \frac{1}{\rho}\left[a_\rho(0-0) - \rho a_\phi\left(-\frac{1}{4}\cdot 2\rho\right) + a_z(0)\right]$$

$$\nabla \times A = \frac{\rho}{2} a_\phi$$

$$ds = d\rho\, dz\, a_\phi \quad \left(\because B = \frac{\rho}{2} a_\phi \right)$$

$$\therefore \Psi = \int_S B \cdot ds = \int_{z=0}^{5} \int_{\rho=1}^{2} \frac{\rho}{2}\, d\rho\, dz$$

$$= \frac{1}{2} \left[\frac{\rho^2}{2} \right]_1^2 \left[z \right]_0^5$$

$$= \frac{1}{4} \left(2^2 - 1 \right) \times 5$$

$$= \frac{3 \times 5}{4}$$

$$\Psi = \frac{15}{4}$$

$$\Rightarrow \boxed{\Psi = 3.75 \text{ Wb}} \quad \text{(Ans)}$$

Method 2:-

We use

$$\Psi = \oint_L A \cdot dl = \Psi_1 + \Psi_2 + \Psi_3 + \Psi_4$$

Where L is the path bounding surface S; $\Psi_1, \Psi_2, \Psi_3, \Psi_4$ are, respectively, the evaluation of $\int A \cdot dl$ along the segments of L labelled 1 to 4 in fig 4.10. Since A has only a z-component

$$\Psi_1 = 0 = \Psi_3$$

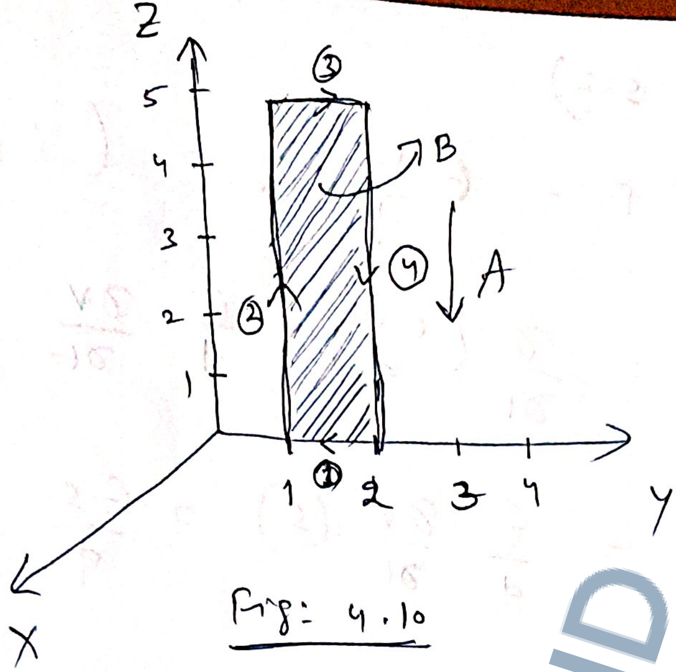


Fig: 4.10

[$\because$ The surface has $\phi = \frac{\pi}{2}$, $\therefore$ the surface is in yz plane and $1 \le \rho \le 2$
$0 \le z \le 5$]

$$\therefore \quad \psi = \psi_2 + \psi_4$$

$$= \int_0^5 -\frac{\rho^2}{4}\, dz + \int_5^0 -\frac{\rho^2}{4}\, dz$$

$$= \int_0^5 -\frac{1}{4}\, dz + \int_5^0 -\frac{4}{4}\, dz$$

$\because$ $\rho = 1$, for ψ_2
$\rho = 2$ for ψ_4

$$= -\frac{1}{4}[5-0] - 1(0-5)$$

$$= -\frac{5}{4} + 5$$

$$= 5 - \frac{5}{4}$$

$$\psi = \frac{15}{4}$$

$$\Rightarrow \boxed{\psi = 3.75 \text{ Wb}}$$ (Ans)

—o—