Chapter-3 : Electrostatic Fields :

An electrostatic field is produced by a static charge distribution. A typical example of such a field is found on a cathode-ray tube.

Electrostatics is a fascinating subject that has grown up in ~~direction~~ diverse areas of application. Electric power transmission, X-ray machines, and lightening protection are associated with strong electric fields and will require knowledge of electrostatics to understand and design suitable equipments.

The devices used in solid-state electronics are based on electrostatics. These includes resistors, capacitors, and active devices such as BJT and FET s, which are based on control of electron motion by electrostatic fields.

Computer peripherals such as touch pads, capacitance keyboards, CRTs, LCDs, and electrostatic printers are based on electrostatic fields.

In medical work, electrostatic has application on electrocardiograms, electroencephalograms, and other recording of electrical activity of organs including eyes, ears and stomachs.

In industry electrostatic is applied in a variety of forms such as paint spraying, electrodeposition, electrochemical machining, and separation of fine particles.

In agriculture, electrostatics is used in applications such as sorting of seeds, for direct spraying of plants, to measure the moisture content of crops, to spin cotton and for speed baking bread and smoking meat.

We begin our study of electrostatics by investigating the two fundamental laws governing electrostatic fields: (1) Coulomb's law (2) Gauss's law. Both the laws are based on experimental studies and they are independent. Although Coulomb's law is applicable in finding the electric field due to any charge configuration, it is easier to use Gauss's law when charge distribution is symmetrical.

Coulomb's law and Field Intensity

Coulomb's law states that the force F betn two point charges Q_1 & Q_2 is

1. Along the line joining them
2. Directly proportional to the product $Q_1 Q_2$ of the charges.
3. Inversely proportional to the square of the distance 'R' between them.

Mathematically,

$$F = \frac{K\, Q_1 Q_2}{R^2} \quad \text{——— (3.1)}$$

Where K is the Proportionality Constant whose value depends on the choice of the system of units. In SI Units, charges Q_1 & Q_2 are in Coulombs (C), the distance R is in meters (m), and force F is in Newton (N) so that

$K = \frac{1}{4\pi\epsilon_0}$, The Constant ϵ_0 is known as the permittivity of free space (Farad/meter) and has value

$$\boxed{\begin{aligned} \epsilon_0 &= 8.854 \times 10^{-12} \simeq \frac{10^{-9}}{36\pi}\ \frac{F}{m} \\ \text{or} \quad K &= \frac{1}{4\pi\epsilon_0} \simeq 9 \times 10^{9}\ \frac{m}{F} \end{aligned}} \quad \text{—(3.2)}$$

The equation (3.1) becomes,

$$F = \frac{Q_1 Q_2}{4\pi\epsilon_0 R^2} \quad \text{——— (3.3)}$$

If point charges Q_1 & Q_2 are located at points having position vectors r_1 and r_2, then the force F_{12} on Q_2 due to Q_1, shown in figure 3.1, is given by

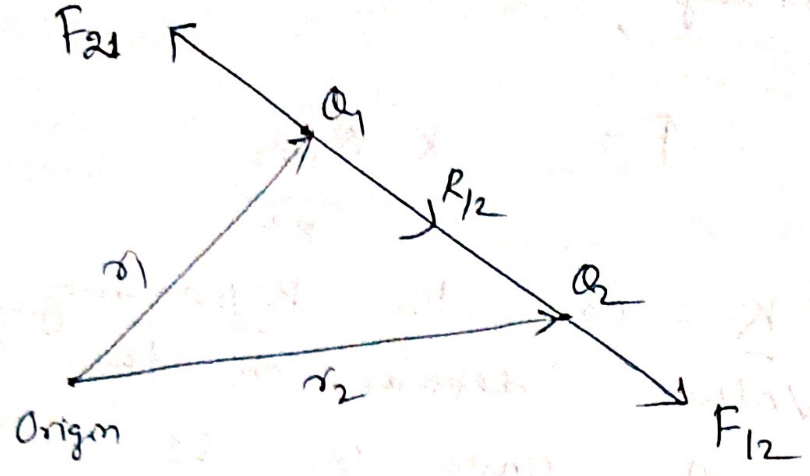


Fig 3.1 Coulombs vector force on point Charges Q_1 & Q_2

$$F_{12} = \frac{Q_1 Q_2}{4\pi\epsilon_0 R^2} a_{R_{12}} \qquad \text{—— (3.4)}$$

Where

$$R_{12} = r_2 - r_1 \qquad \text{—— 3.5 (a)}$$

$$R = |R_{12}| \qquad \text{—— 3.5 (b)}$$

$$a_{R_{12}} = \frac{R_{12}}{R} \qquad \text{—— 3.5 (c)}$$

By substituting eqn (3.5) onto eqn (3.4), we have

$$F_{12} = \frac{Q_1 Q_2}{4\pi\epsilon_0 R^2} \times \frac{R_{12}}{R} = \frac{Q_1 Q_2 R_{12}}{4\pi\epsilon_0 R^3} \qquad \text{— 3.6 (a)}$$

$$\text{or } F_{12} = \frac{Q_1 Q_2 (r_2 - r_1)}{4\pi\epsilon_0 |(r_2 - r_1)|^3} \qquad \text{—— 3.6 (b)}$$

Note :-

1) The force F_{21} on Q_1 due to Q_2 is given by

$$F_{21} = |F_{12}| \cdot a_{R_{21}}$$

$$= |F_{12}| \left(-a_{R_{12}}\right)$$

or $$F_{21} = -F_{12} \qquad (3.7)$$

2. Like charges (charges of same sign) repel each other, while unlike charges attract. This is illustrated in Figure 3.2.

Fig 3.2 (a), (b) Like charges repel.
(c) Unlike charges attract.

3. The distance R between the charged bodies Q_1 & Q_2 must be large compared with the linear dimensions of the bodies; that is Q_1 & Q_2 must be point charges.

4. Q_1 & Q_2 must be static (at rest).

5. The signs of Q_1 & Q_2 must be taken into account in equation (3.4) for like charges, $Q_1 Q_2 > 0$. For unlike charges, $Q_1 Q_2 < 0$.

If we have more than two point charges, we can use the principle of superposition to determine the force on a particular charge. The principle states that if there are N charges $Q_1, Q_2, \ldots, Q_N$ located, respectively, at points with position vectors $r_1, r_2, \ldots r_N$, the resultant force F on a charge Q located at point r is the vector sum of the forces exerted on Q by each of the charges $Q_1, Q_2, \ldots Q_N$. Hence,

$$F = \frac{QQ_1(r-r_1)}{4\pi\epsilon_0 |r-r_1|^3} + \frac{QQ_2(r-r_2)}{4\pi\epsilon_0 |r-r_2|^3} + \cdots + \frac{QQ_N(r-r_N)}{4\pi\epsilon_0 |r-r_N|^3}$$

or

$$F = \frac{Q}{4\pi\epsilon_0} \sum_{k=1}^{N} \frac{Q_k(r-r_k)}{|r-r_k|^3} \quad \text{— (3.8)}$$

Now, the electric field intensity or (electric field strength) 'E' is the force per unit charge when placed in an electric field.

Thus,
$$E = \lim_{Q \to 0} \frac{F}{Q} \quad \text{— (3.9)}$$

or simply
$$E = \frac{F}{Q} \quad \text{— (3.10)}$$

For $Q > 0$, the electric field intensity E is obviously in the direction of the force F' and is measured in Newtons per Coulomb or volts per meter.

The electric field intensity at point 'r' due to a point charge located at 'r'' is readily obtained from eqn. (3.6) and (3.10) as

$$E = \frac{Q}{4\pi\epsilon_0 R^2} a_R = \frac{Q (r - r')}{4\pi\epsilon_0 |r - r'|^3} \quad \text{---(3.11)}$$

$$\left(\because \; E = \frac{Q_1 Q_2}{4\pi\epsilon_0 R^2} a_R \text{ and } Q_1 = Q, \; Q_2 = 1 \text{ (Unit charge)} \right)$$

For N point charges $Q_1, Q_2, \ldots Q_N$ located at $r_1, r_2, \ldots r_N$, the electric field intensity at point r is obtained from equations (3.8) and (3.10) as

$$E = \frac{Q_1 (r - r_1)}{4\pi\epsilon_0 |r - r_1|^3} + \frac{Q_2 (r - r_2)}{4\pi\epsilon_0 |r - r_2|^3} + \cdots + \frac{Q_N (r - r_N)}{4\pi\epsilon_0 |r - r_N|^3}$$

or

$$E = \frac{1}{4\pi\epsilon_0} \sum_{K=1}^{N} \frac{Q_K (r - r_K)}{|r - r_K|^3} \quad \text{---(3.12)}$$

Example 3.1

Point charges 1 mC and −2 mC are located at (3, 2, −1) and (−1, −1, 4) respectively. Calculate the electric force on a 10 nC charge located at (0, 3, 1) and the electric field intensity at that point.

Ans:-

$$F = \sum_{K=1,2} \frac{Q\,Q_K}{4\pi\epsilon_0 R^2}\, a_R$$

$$= \sum_{K=1,2} \frac{Q\,Q_K}{4\pi\epsilon_0} \frac{(r - r_K)}{|r - r_K|^3}$$

$$= \frac{Q}{4\pi\epsilon_0} \left\{ \frac{10^{-3} \times [(0,3,1) - (3,2,-1)]}{|(0,3,1) - (3,2,-1)|^3} + \frac{(-2)\times 10^{-3} \times [(0,3,1) - (-1,-1,4)]}{|(0,3,1) - (-1,-1,+4)|^3} \right\}$$

$$= \frac{10 \times 10^{-9}}{\cancel{4\pi} \times \frac{10^{-9}}{\cancel{36\pi}}} \times 10^{-3} \left[\frac{(-3,1,2)}{|(-3,1,2)|^3} - 2\frac{(1,4,-3)}{|(1,4,-3)|^3} \right]$$

$$= 9 \times 10^{-2} \left[\frac{(-3,1,2)}{(\sqrt{9+1+4})^3} - \frac{2(1,4,-3)}{(\sqrt{1+16+9})^3} \right]$$

$$\Rightarrow F = 9\times10^{-2}\left[\frac{(-3,1,2)}{(\sqrt{14})^3} - \frac{2(1,4,-3)}{(\sqrt{26})^3}\right]$$

$$= 9\times10^{-2}\left[\frac{(-3,1,2)}{(\sqrt{14})^2(\sqrt{14})} - \frac{2(1,4,-3)}{(\sqrt{26})^2(\sqrt{26})}\right]$$

$$= 9\times10^{-2}\left[\frac{(-3,1,2)}{14\sqrt{14}} - \frac{2(1,4,-3)}{26\sqrt{26}}\right]$$

$$= 90\times10^{-3}\left[\frac{(-3,1,2)}{52.38} - \frac{(2,8,-6)}{132.574}\right]$$

$$= 90\left[\left(\frac{-3}{52.38} - \frac{2}{132.574}\right)a_x + \left(\frac{1}{52.38} - \frac{8}{132.574}\right)a_y + \left(\frac{2}{52.38} + \frac{6}{132.574}\right)a_z\right] mN$$

$$F = \left\{-6.51 a_x - 3.71 a_y + 7.50 a_z\right\} mN$$

At that Point,

$$E = \frac{F}{Q} = \frac{-6.51 a_x - 3.71 a_y + 7.50 a_z \times 10^{-3}}{10\times10^{-9}}$$

$$E = -651 a_x - 371 a_y + 750 a_z \ \frac{KV}{m}$$

(Ans)

Example 3.2

Two point charges of equal mass 'm' and charge 'Q' are suspended at a common point by two threads of negligible mass and length 'l'. Show that at equilibrium the inclination angle 'α' of each thread to the vertical is given by

$$Q^2 = 16\pi\epsilon_0 mg l^2 \sin^2\alpha \tan\alpha$$

If α is very small, show that

$$\alpha = \sqrt[3]{\frac{Q^2}{16\pi\epsilon_0 mg l^2}}$$

Ans :-

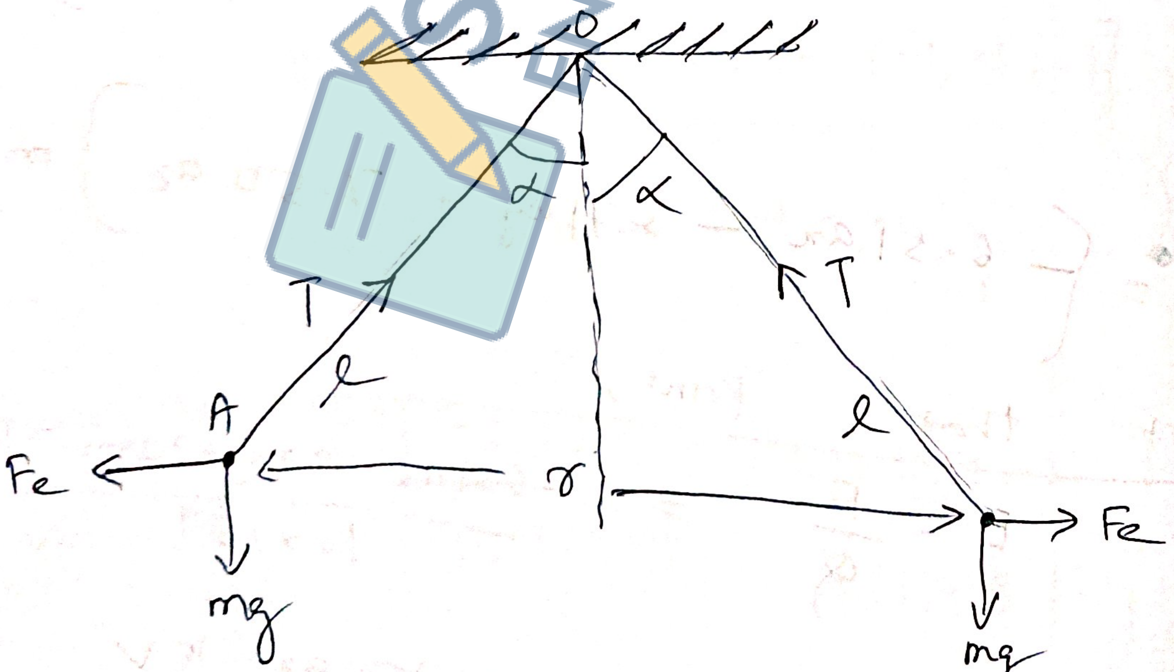
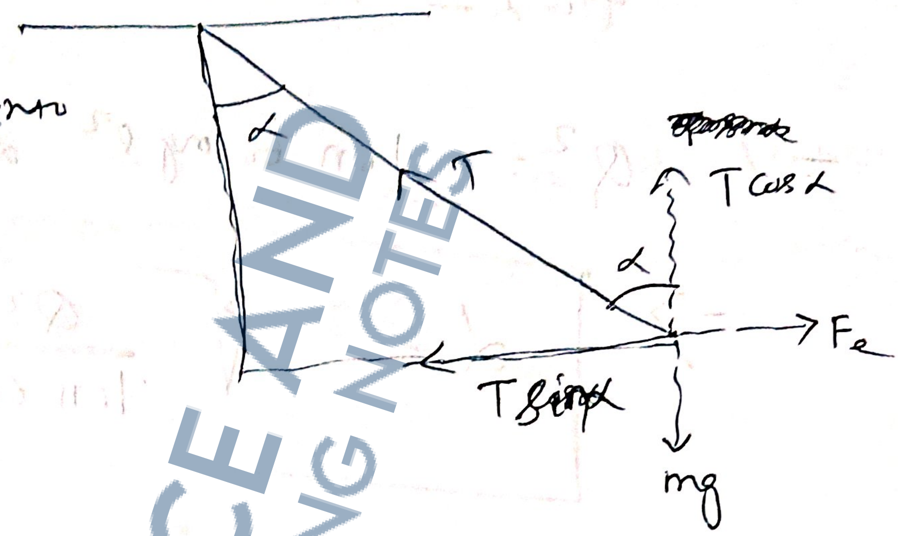


Fig:- 3.3 Suspended Charged particles; for example 3.2

Consider the system of charges as shown in figure 3.3, where Fe is the electric or Coulomb force, T is the tension in each thread, and mg is the weight of each charge.

Resolving 'T' into 2 rectangular components.

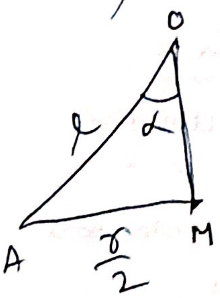


$$T \sin\alpha = F_e$$

$$T \cos\alpha = mg$$

$$\Rightarrow \frac{\sin\alpha}{\cos\alpha} = \frac{F_e}{mg} = \frac{1}{mg} \times \frac{Q^2}{4\pi\epsilon_0 r^2} \quad \text{—(1)}$$

But $\triangle OAM$, $\sin\alpha = \dfrac{\frac{r}{2}}{l} = \dfrac{r}{2l}$



$$\Rightarrow r = 2l\sin\alpha \quad \text{—(2)}$$

Putting eqn (2) in eqn (1)

$$\frac{\sin\alpha}{\cos\alpha} = \frac{Q^2}{mg \times 4\pi\epsilon_0 \, 4l^2\sin^2\alpha}$$

$$\Rightarrow \boxed{Q^2\cos\alpha = 16\pi\epsilon_0 \, mg\, l^2 \sin^3\alpha}$$

$$\Rightarrow \boxed{Q^2 = 16\pi\epsilon_0 mg l^2 \sin^2\alpha \tan\alpha} \text{ (Proved)}$$

$$\therefore \frac{\sin^3\alpha}{\cos\alpha} = \sin^2\alpha . \tan\alpha$$

When α is very small

$$\tan\alpha \simeq \alpha \simeq \sin\alpha$$

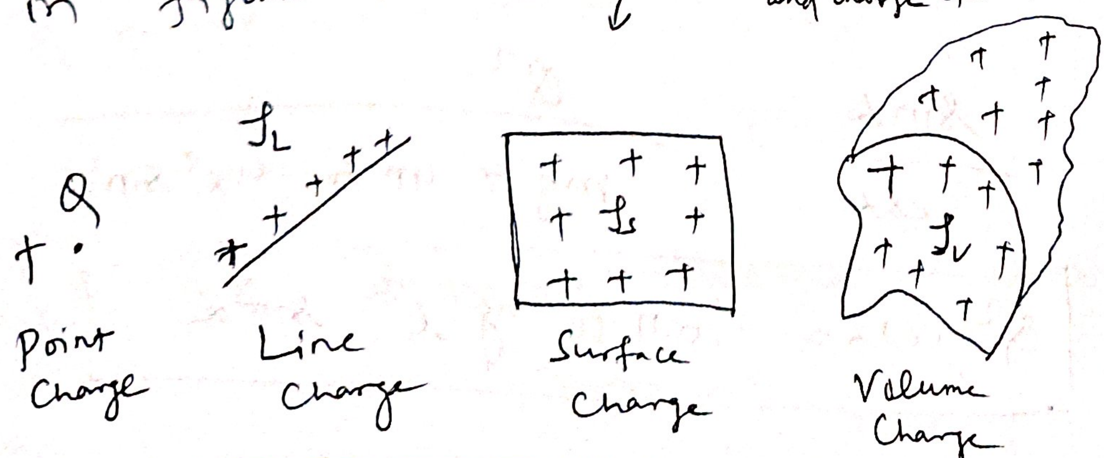
$$\Rightarrow Q^2 = 16\pi\epsilon_0 mg l^2 \alpha^3$$

$$\Rightarrow \boxed{\alpha = \sqrt[3]{\frac{Q^2}{16\pi\epsilon_0 mg l^2}}} \text{ (Proved)}$$

Electric fields due to continuous charge distributions

So far we have considered only forces and electric fields due to point charges, which are essentially charges occupying very small physical space. It is also possible to have continuous charge distribution along a line, on a surface or in a volume as illustrated in figure 3.5.

Fig 3.5: Various charge distributions and charge elements



We denote line charge density, surface charge density, and volume charge density by $\rho_L \left(\frac{C}{m}\right)$, $\rho_S \left(\frac{C}{m^2}\right)$, and $\rho_V \left(\frac{C}{m^3}\right)$, respectively.

<Note:- It should not be confused with 'ρ' used for radial distance in cylindrical coordinates>

The charge element dQ and the total charge Q due to these charge distributions are obtained form Figure 3.5 as

$$dQ = \rho_L \, dl$$

$$\Rightarrow \boxed{Q = \int_L \rho_L \, dl}$$ (line charge) — (3.13 a)

$$dQ = \rho_S \, dS$$

$$\Rightarrow \boxed{Q = \int_S \rho_S \, dS}$$ (surface charge) — (3.13 b)

$$dQ = \rho_V \, dV$$

$$\Rightarrow \boxed{Q = \int_V \rho_V \, dV}$$ (volume charge) — (3.13 c)

The electric field intensity due to each of the charge distributions ρ_L, ρ_S, and ρ_V may be regarded as the summation of the fields contributed by the numerous point charges making up the charge distribution.

Thus by replacing Q in eqn (3.11) with charge element $dQ = \rho_L dl$, $\rho_s ds$, $\rho_v dv$ and integrating, we get

$$E = \int_L \frac{\rho_L dl}{4\pi\epsilon_0 R^2} a_R \quad \text{(line charge)} \; -(3.14)$$

$$E = \int_S \frac{\rho_s ds}{4\pi\epsilon_0 R^2} a_R \quad \text{(Surface charge)} \; -(3.15)$$

$$E = \int_V \frac{\rho_v dv}{4\pi\epsilon_0 R^2} a_R \quad \text{(volume charge)} \; -(3.16)$$

Electric flux density

The electric field intensity is dependent on the medium in which the charge is placed (free space in this chapter). Suppose a new vector field 'D' is defined by

$$D = \epsilon_0 E \quad \text{——} \; (3.17)$$

The electric flux Ψ in terms of 'D', may be defined as

$$\Psi = \int_S D \cdot ds \quad \text{——} \; (3.18)$$

In SI units, one line of electric flux

emanates from <u>+1 C</u> and terminates on <u>−1 C</u>.

Therefore, the electric flux is measured in Coulombs. Hence, the vector field 'D' is called <u>electric flux density</u> and is measured in <u>Coulombs per square meter</u>. The electric flux density is also called <u>electric displacement</u>.

The formulas derived for <u>E</u> from Coulomb's law can be used in calculating D, except that we have to <u>multiply</u> those formulas by <u>ϵ_0</u>.

<u>For example:-</u>

For Volume Charge distribution,

$$D = \epsilon_0 \times E = \cancel{\epsilon_0} \times \int_V \frac{\rho_v \, dv}{4\pi \cancel{\epsilon_0} R^2} a_R \quad \left(\text{from eqn } 3.16\right)$$

$$\therefore \quad D = \int_V \frac{\rho_v \, dv}{4\pi R^2} a_R \qquad (3.19)$$

Thus D is a function of charge and position only; it is independent of the medium.

Gauss's law - Maxwell's Equation:-

Gauss's law Constitute one of the fundamental laws of electromagnetism.

It states that the total electric flux Ψ through any closed surface is equal to the total charge enclosed by that surface.

Thus

$$\Psi = Q_{enc} \qquad (3.20)$$

i.e

$$\Psi = \oint_S d\Psi = \oint_S D \cdot ds \qquad (3.21)$$

(from eqn 3.18)

Total Charge enclosed $(Q) = \int_V \rho_v \, dV$ (3.22)

(from eqn 3.13c)

By Gauss's law, equating (3.21) & (3.22)

$$\boxed{\oint_S D \cdot ds = \int_V \rho_v \, dV = Q} \qquad (3.23)$$

By applying divergence theorem, using eqn (2.31) i.e $\oint_S A \cdot ds = \int_V \nabla \cdot A \, dV$

$$\oint_S D \cdot ds = \int_V \nabla \cdot D \, dV \qquad (3.24)$$

Comparing the two volume integrals in eqn (3.23) & (3.24), we have

$$\boxed{\rho_v = \nabla \cdot D} \quad (3.25)$$

or

$$\boxed{\nabla \cdot D = \rho_v}$$

Which is the first of the four Maxwell's equations to be derived.

Equation (3.25) states that the volume charge density is same as the divergence of the electric flux density.

Note :-

1. Equation (3.23) to (3.25) are basically stating Gauss's law in different ways; eqn (3.23) is the integral form, whereas (3.25) is the differential or point form of Gauss's law.

2. Gauss's law is an alternative statement of Coulomb's law; proper application of the divergence theorem to Coulomb's law results in Gauss's law.

3. Gauss's law provides an easy means of finding E or D for symmetrical charge distributions such as a point charge, an infinite

line charge, ~~or~~ an infinite cylindrical surface charge, and a spherical distribution of charge.

4. A continuous charge distribution has rectangular symmetry if it depends only on x (or y or z), cylindrical symmetry if it depends only on ρ, spherical symmetry if it depends only on r (independent of θ and ϕ). It must be stressed that whether the charge distribution is symmetric or not, Gauss's law always holds.

5. Consider the charge distribution in figure 3.12. where V_1 and V_2 are closed surfaces (or volumes). The total flux leaving V_1 is $10-5 = 5\,nC$ because only 10 nC and −5 nC charges are enclosed by V_1.

Fig 3.12 : Illustration of Gauss's law: flux leaving V_1 is 5 nC and that leaving V_2 is 0 C.

Although charges 20nC and 15nC outside V_1 do contribute to the flux crossing V_1, the net flux crossing V_1, according to Gauss's law, is irrespective of those charges outside V_1.

Similarly, the total flux leaving V_2 is zero because no charge is enclosed by V_2. Thus we see that Gauss's law, $\psi = Q_{enclosed}$, is still obeyed even though the charge distribution is not symmetric.

Applications of Gauss's law

The procedure for applying Gauss's law to calculate the electric field involves first knowing whether symmetry exists. Once it has been found that symmetric charge distribution exists, we construct a mathematical closed surface (known as a Gaussian surface). The surface is chosen such that $\underline{D}$ is normal or tangential to the Gaussian surface.

A. Point Charge

Suppose a point charge Q is located at the origin. To determine 'D' at a point P, it is easy to see that choosing a

Spherical surface containing P will satisfy symmetry conditions. Thus, a spherical surface centered at the origin is the Gaussian surface in this case and is shown in Figure 3.13.

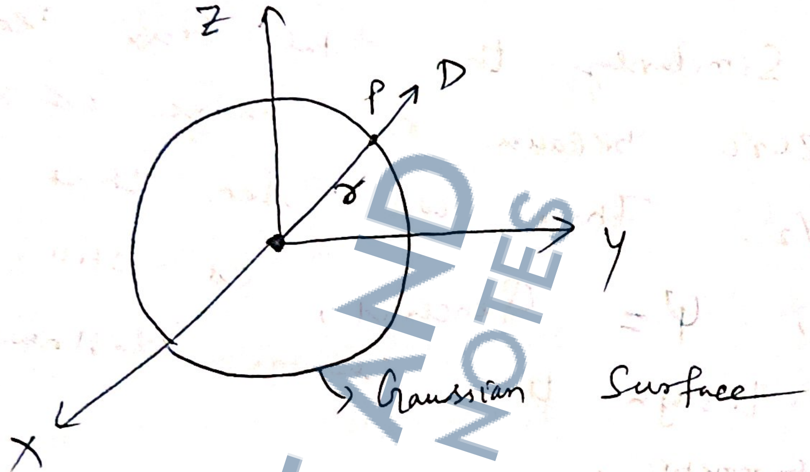


Fig 3.13 Gaussian surface about a point charge.

Since D is everywhere normal to the Gaussian surface, that is $D = D_r a_r$, applying Gauss's law, ($\psi = Q_{enclosed}$) gives

$$Q = \oint_S D \cdot ds = D_r \oint_S ds \quad (\because D_r \text{ is a constant everywhere})$$

$$= D_r \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} r^2 \sin\theta \, d\theta \, d\phi \quad (\text{Refer eqn } 2A)$$

$$= D \cdot r^2 \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta \quad (\text{Let } D_r = D)$$

$$= D \times r^2 \times 2\pi \times -\cos\theta \Big]_0^{\pi}$$

$$= 2\pi r^2 D \times [2] = 4\pi r^2 D$$

Imp

$$\Rightarrow \boxed{D = \frac{Q}{4\pi r^2} a_r} \quad \text{--- (3.26)}$$

As expected from eqn (3.11) and (3.12).

B. Infinite Line Charge

Suppose the infinite line of uniform charge $\rho_L \frac{C}{m}$ lies along z axis. To determine D at a point P, we choose a cylindrical surface containing P to satisfy the symmetry condition as shown in Figure 3.14

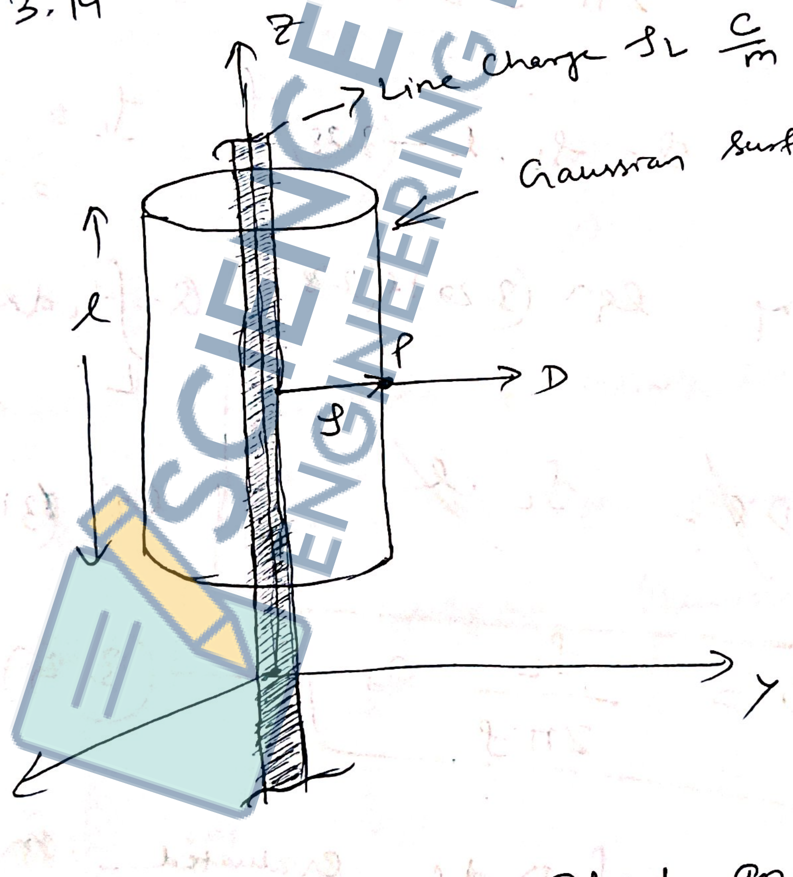


Fig 3.14 Gaussian surface about an infinite line charge

The electric flux density D is constant on and normal to the cylindrical Gaussian surface; that is, $D = D_\rho a_\rho$. If we apply

Gauss's law to an arbitrary length 'l' of the line

$$Q = \oint_S D \cdot ds = D_\rho \oint_S ds = D_\rho \oint_S \rho \, d\phi \, dz \, a_\rho$$

$$= D_\rho \int_{\phi=0}^{2\pi} \int_0^l \rho \, d\phi \, dz \, a_\rho$$

$$= D \times \rho \quad d\phi \Big]_0^{2\pi} \, dz \Big]_0^l$$

$$Q = 2\pi \rho D l \quad \text{— (3.27)}$$

But $Q = \rho_L \cdot l$ — (3.28)

ρ_L = line charge density $= \frac{C}{m}$

$Q = \int_L \rho_L \, dl = \rho_L \int_0^l dl$

$= \rho_L \cdot l$

eqn (3.13 a) → Refer

equating eqn (3.27) & (3.28)

$$2\pi \rho D \cancel{l} = \rho_L \cdot \cancel{l}$$

Line

$$\Rightarrow \boxed{D = \frac{\rho_L}{2\pi\rho} a_\rho} \quad \text{— (3.29)}$$

Note :- $\int D \cdot ds$ evaluated on the top a bottom surfaces of the cylinder is zero, since D has no Z - component. Only it has ~~comp~~ component along a_ρ direction.

C. Infinite Sheet of Charge

Consider an infinite sheet of uniform charge $\rho_s \frac{C}{m^2}$ lying on the $Z=0$ plane. To determine D at point P, we choose a rectangular box that is cut symmetrically by sheet of charge and has two of its faces parallel to the sheet as shown in Figure 3.15.

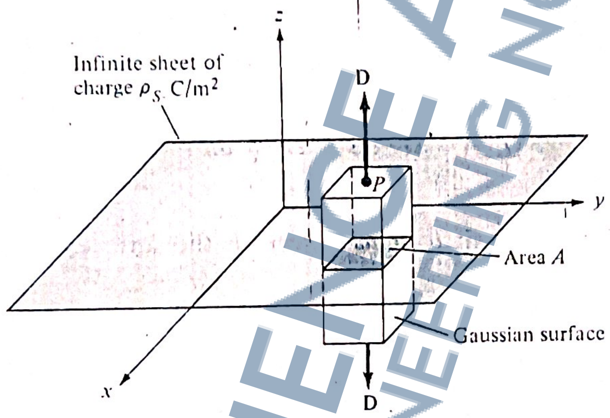


Fig 3.15: Gaussian Surface about an infinite line sheet of charge

- As D is normal to the sheet,

$D = D_z a_z$, and applying Gauss's law gives

$$Q = \oint_S D \cdot ds$$

L.H.S

$$Q = \int_S \rho_s \, ds \qquad \left(\text{from eqn } (3.13\,b)\right)$$

$$Q = \rho_s \times A \qquad \left[\because \text{Area of the sheet is } A\right] \quad \text{—} (3.30)$$

R.H.S

$$\oint_S D.ds = D_z\left[\int_{Top} ds + \int_{Bottom} ds\right] \quad -(3.31)$$

Because $D.ds$ evaluated on the sides of the box is zero as D has no components along a_x and a_y. ~~If the~~ top and bottom area of the box, each has area 'A',

eqn (3.31) becomes

$$\oint_S D.ds = D_z [A+A] = 2AD. \quad -(3.32)$$

We know $Q = \oint_S D.ds$, from Gauss's law

Thus equating eqn (3.30) & (3.32)

$$\rho_s \cancel{A} = 2\cancel{A}D$$

$$\Rightarrow \boxed{D = \frac{\rho_s}{2} a_z} \quad -(3.33)$$

or

$$\boxed{E = \frac{D}{\epsilon_0} = \frac{\rho_s}{2\epsilon_0} a_z} \quad -(3.34)$$

D. Uniformly Charged Sphere

Consider a sphere of radius 'a' with a uniform charge $\rho_0 \frac{C}{m^3}$. To determine D

everywhere, we construct Gaussian surfaces for cases $r \le a$ and $r \ge a$ separately.

Since the charge has spherical symmetry, it is obvious that a spherical surface is an appropriate Gaussian surface.

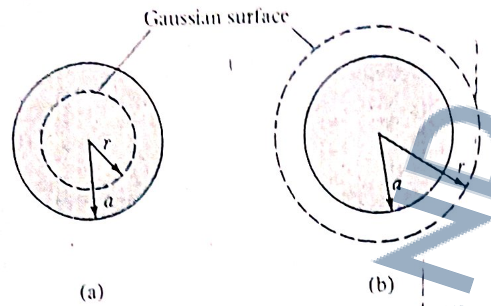


Fig 3.16: Gaussian surface for a uniformly charged sphere when (a) $r \ge a$ and (b) $r \le a$

- Case-I $(r \le a)$ [i.e the point 'p' is inside the charged sphere]

The total charge enclosed by the spherical surface of radius r, as shown in Figure 3.16(a), is

$$Q_{enc} = \int_v \rho_v \, dv = \int_v \rho_0 \, dv = \rho_0 \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{r} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \rho_0 \times \phi\Big]_0^{2\pi} \times r^2 dr\Big]_0^{r} \times \sin\theta \, d\theta\Big]_0^{\pi}$$

$$= \rho_0 \times (2\pi) \times \frac{r^3}{3}\Big]_0^{r} \times -\cos\theta\Big]_0^{\pi}$$

$$= \rho_0 \times 2\pi \times \frac{1}{3} \times r^3 \times 2$$

$$Q_{enc} = \rho_0 \frac{4}{3} \pi r^3 \qquad (3.35)$$

and

$$\psi = \oint_S D \cdot ds = D_r \oint_S ds = D_r \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} r^2 \sin\theta \, d\theta \, d\phi$$

$$\Rightarrow \psi = D_r \times r^2 \times \phi \Big]_0^{2\pi} \sin\theta \, d\theta \Big]_0^{\pi}$$

$$\Rightarrow \psi = D_r \times r^2 \times 2\pi \times 2$$

$$\Rightarrow \psi = D_r \times 4\pi r^2 \quad \text{——— (3.36)}$$

Equating (3.35) & (3.36) by Gauss's law

$$\rho_0 \frac{4}{3} \pi r^3 = D_r \times 4\pi r^2$$

$$\Rightarrow D_r = \frac{r}{3} \rho_0 \, a_r$$

$$\Rightarrow \boxed{D = \frac{r}{3} \rho_0 \, a_r, \quad 0 < r \leq a} \quad \text{—(3.37)}$$

Case-II $(r > a)$ [The point 'p' is outside the charged sphere]

The Gaussian surface for this case is as shown in Figure 3.16(b). The charge enclosed by the surface is the entire charge in this case, that is

$$Q_{enc} = \int_V \rho_v \, dv = \rho_0 \int_V dv = \rho_0 \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{a} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$Q_{enc} = \rho_0 \times 2\pi \times \int_0^a r^2 dr \times \int_0^{\pi} \sin\theta \, d\theta$$

$$= \rho_0 \times 2\pi \times \frac{r^3}{3} \Big]_0^a \times \Big[-\cos\theta\Big]_0^{\pi}$$

$$\Rightarrow Q_{enc} = \rho_0 \times 2\pi \times \frac{a^3}{3} \times 2$$

$$\Rightarrow Q_{enc} = \frac{4}{3}\pi a^3 \rho_0 \quad \text{—— (3.38)}$$

and

$$\Psi = \oint_S D \cdot ds = D_r \times 4\pi r^2 \quad \left[\text{As derived in eq}^n\ (3.36)\right] \quad \text{—— (3.39)}$$

Equating (3.38) and (3.39)

$$\frac{\cancel{4}}{3}\cancel{\pi} a^3 \rho_0 = D_r \times \cancel{4\pi} r^2$$

$$\Rightarrow D_r = \frac{a^3}{3r^2}\rho_0\, a_r, \quad r \geq a$$

$$\Rightarrow \boxed{D = \frac{a^3}{3r^2}\rho_0\, a_r, \quad r \geq a} \quad \text{—— (3.40)}$$

From eq^n (3.37) and (3.40), D everywhere is given by

$$D = \begin{cases} \frac{r}{3}\rho_0\, a_r, & 0 < r \leq a \\ \frac{a^3}{3r^2}\rho_0\, a_r, & r \geq a \end{cases}$$

and $|D|$ is as sketched in figure 3.17.

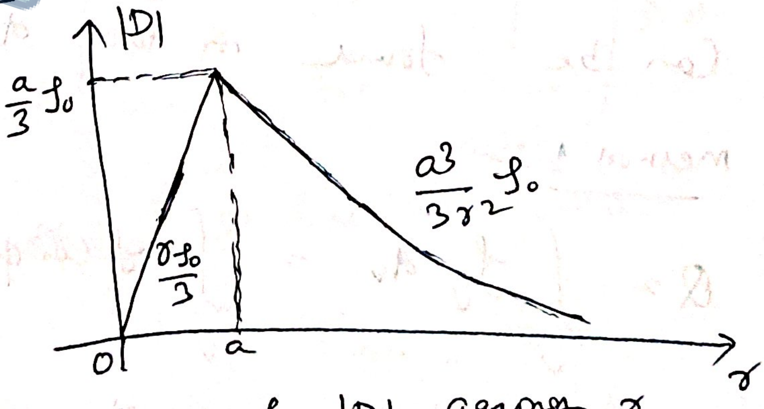


Fig 3.17 Sketch of $|D|$ against r for a uniformly charged sphere.

Example 3.8 Given that $D = z \rho \cos^2\phi \, a_z \, \frac{C}{m^2}$.

Calculate the charge density at $(1, \frac{\pi}{4}, 3)$ and total charge enclosed by the cylinder of radius 1m with $-2 \leq z \leq 2m$.

Solution :- From eqn (3.25)

$$\rho_v = \nabla \cdot D$$

$$= \frac{\partial}{\partial z} D_z$$

$$= \frac{\partial}{\partial z} (z \rho \cos^2\phi)$$

$$\rho_v = \rho \cos^2\phi$$

At $(1, \frac{\pi}{4}, 3)$, $\rho_v = 1 \cdot \cos^2\frac{\pi}{4} = 1 \times \left(\frac{1}{\sqrt{2}}\right)^2$

$$\Rightarrow \rho_v = \frac{1}{2} \frac{C}{m^3}$$

$$\Rightarrow \boxed{\rho_v = 0.5 \frac{C}{m^3}}$$

The total charge enclosed by the cylinder can be found in two different ways

Method 1 :-

$$Q = \int_v \rho_v \, dv = \int_v \rho \cos^2\phi \, \rho \, d\rho \, d\phi \, dz$$

$$(\because dv = \rho \, d\rho \, d\phi \, dz)$$

$$\Rightarrow Q = \int_{z=-2}^{+2} dz \int_{\phi=0}^{2\pi} \cos^2\phi \, d\phi \int_{\rho=0}^{1} \rho^2 \, d\rho$$

$$= \left. z \right]_{-2}^{2} \times \pi \times \left. \frac{\rho^3}{3} \right]_0^1$$

$$\left(\because \int_0^{2\pi} \cos^2\phi \, d\phi = \int_0^{2\pi} \left(\frac{1+\cos 2\phi}{2} \right) d\phi \right.$$

$$= \frac{1}{2} \left[\phi + \frac{\sin 2\phi}{2} \right]_0^{2\pi}$$

$$= \frac{1}{2} \left[(2\pi + 0) - (0 + 0) \right]$$

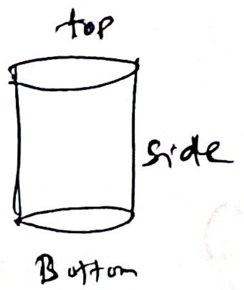
$$\left. = \pi \right)$$

$$\Rightarrow Q = 4 \times \pi \times \frac{1}{3}$$

$$\Rightarrow \boxed{Q = \frac{4\pi}{3} \, C}$$

Method 2:- Using Gauss's law

$$Q = \Psi = \oint D \cdot ds = \left[\int_{side} + \int_{top} + \int_{bottom} \right] D \cdot ds$$



Since D has no component along a_ρ, $\Psi_{side} = 0$

For Ψ_{top}, $ds = \rho \, d\phi \, d\rho \, a_z$

Ψ_{bottom}, $ds = -\rho \, d\phi \, d\rho \, a_z$

$$\Psi_{top} = \int_{\rho=0}^{1}\int_{\phi=0}^{2\pi} z\,\rho\cos^2\phi\;\rho\,d\phi\,d\rho\Big]_{z=2}$$

$$= 2\int_0^1 \rho^2\,d\rho \times \int_0^{2\pi}\cos^2\phi\,d\phi$$

$$= 2\times\frac{\rho^3}{3}\Big]_0^1 \times \pi$$

$$= \frac{2\pi}{3}$$

For

Ψ_{bottom}, $\quad d\mathbf{s} = -\rho\,d\phi\,d\rho\,\mathbf{a}_z$

$$\Psi_{bottom} = -\int_{\rho=0}^{1}\int_{\phi=0}^{2\pi} z\cos^2\phi\;\rho\,d\phi\,d\rho\Big]_{z=-2}$$

$$= 2\times\int_0^1 \rho^2\,d\rho \times \int_0^{2\pi}\cos^2\phi\,d\phi$$

$$= \frac{2\pi}{3}$$

$\therefore\ \Psi_{total} = \Psi_{side} + \Psi_{top} + \Psi_{bottom}$

$$= 0 + \frac{2\pi}{3} + \frac{2\pi}{3}$$

$\therefore\ Q = \Psi_{total} = \frac{4\pi}{3}\ C$, (Ans)

Example 3.9 A charge distribution with spherical symmetry has density

$$\rho_v = \begin{cases} \dfrac{\rho_0 r}{R}, & 0 \le r \le R \\ 0, & r > R \end{cases}$$

Determine E everywhere

Ans:- The charge distribution is similar to Figure 3.16. Since symmetry exists, we can apply Gauss's law to find E.

$$\oint_S D \cdot ds = \int_V \rho_v \, dv$$

$$\Rightarrow \epsilon_0 \oint_S E \cdot ds = \int_V \rho_v \, dv$$

(a) Case-I $(0 \le r \le R)$

$$\epsilon_0 \times E_r \times 4\pi r^2 = \int_0^r \int_0^{\pi} \int_0^{2\pi} \frac{\rho_0 r}{R} \cdot r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \frac{\rho_0}{R} \times \int r^3 dr \Big]_0^r \times \int_0^{\pi} \sin\theta \, d\theta \times d\phi \Big]_0^{2\pi}$$

$$= \frac{\rho_0}{R} \times \frac{r^4}{\cancel{4}} \times \cancel{2} \times \cancel{2\pi} = \frac{\rho_0 \pi r^4}{R}$$

$$\Rightarrow \epsilon_0 \times E_r \times \cancel{4\pi} \cancel{r^2} = \frac{\rho_0 \cancel{\pi} \cancel{r^4} r^2}{R} \Rightarrow \boxed{E_r = \frac{\rho_0 r^2}{4 \epsilon_0 R} a_r}$$

(b) Case-II $(r > R)$

$$\epsilon_0 \times E_r \times 4\pi r^2 = \int_0^{\pi}\int_0^{2\pi}\int_0^{R} \rho_v \, r^2 \sin\theta \, d\theta \, d\phi \, dr$$

$$+ \int_0^{\pi}\int_0^{2\pi}\int_R^{r} 0 \cdot r^2 \sin\theta \, d\theta \, d\phi \, dr$$

$$= \int_0^{\pi} \int_0^{2\pi} \int_0^{R} \frac{\rho_0 r}{R} \cdot r^2 \sin\theta \, d\theta \, d\phi \, dr$$

$$= \int \sin\theta \, d\theta \Big]_0^{\pi} \int d\phi \Big]_0^{2\pi} \times \frac{\rho_0}{R} \int r^3 dr \Big]_0^{R}$$

$$= \cancel{2} \times \cancel{2}\pi \times \frac{\rho_0}{\cancel{R}} \times \frac{\cancel{R^4} \, R^3}{\cancel{4}}$$

$$= \pi \rho_0 R^3$$

$$\Rightarrow \epsilon_0 \times E_r \times \cancel{4\pi} r^2 = \cancel{\pi} \rho_0 R^3$$

$$\Rightarrow \boxed{E_r = \frac{\rho_0 R^3}{4 \epsilon_0 r^2} \, a_r}$$

(Ans)

Electric Potential

In the last section, we have found the electric field intensity 'E' from the electric flux density. $\left(\because E = \frac{D}{\epsilon_0}\right)$

Another way of obtaining E is from the electric scalar potential (V). In the sense, this way of finding E is easier because it is easier to handle scalar than vectors.

→ Suppose we wish to move a point charge Q from point A to point B in an electric field E as shown in figure 3.18. From Coulomb's law, the force on Q is $F = QE$ so that the work done in displacing the charge by 'dl' is

$$dW = -F.dl = -QE.dl \quad (3.41)$$

The -ve sign indicates that the work is being done by an external agent. Thus the total work done, or the potential energy required in moving Q from A to B is

$$W = -Q\int_A^B E.dl \quad (3.42)$$

Dividing W by Q in eqn (3.42) gives the potential energy per unit charge. This quantity, denoted by V_{AB}, is known as potential difference between points A and B. (119)

Thus,

$$V_{AB} = \frac{W}{Q} = -\int_A^B E \cdot dl \qquad (3.43)$$

Note: -

1. In determining V_{AB}, A is the initial point while B is the final point.

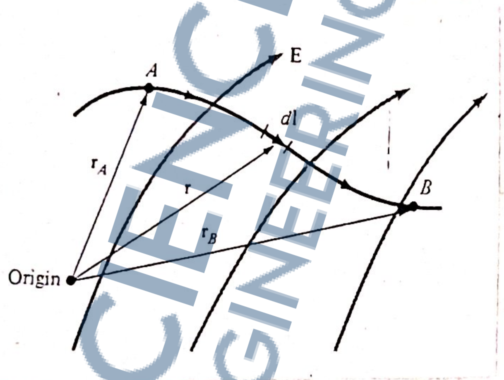


Fig 3.18: Displacement of point charge Q in an electrostatic field E.

2. If V_{AB} is -ve, there is loss in potential energy in moving Q from A to B; this implies the work is being done by the field.

However, if V_{AB} is +ve, there is a gain in potential energy, in the movement; an external agent performs this work.

3. V_{AB} is independent of the path taken.

4. V_{AB} is measured in Joules per Coulomb,

(∵ $W = VQ \Rightarrow V = \frac{W}{Q} = \frac{\text{Joule}}{\text{Coulomb}}$)

or Commonly referred to as volts (V).

As an example, if the E field in Figure 3.18 is due to a point charge Q located at the origin, then

$$E = \frac{Q}{4\pi\epsilon_0 r^2} a_r \quad \text{— (3.44)}$$

[Refer eqn (3.11)]

So eqn (3.43) becomes,

$$V_{AB} = -\int_{r_A}^{r_B} \frac{Q}{4\pi\epsilon_0 r^2} a_r \cdot dr\, a_r \quad \text{— (3.45 a)}$$

$$= -\int_{r_A}^{r_B} \frac{Q}{4\pi\epsilon_0 r^2} dr$$

$$= -\frac{Q}{4\pi\epsilon_0} \int_{r_A}^{r_B} r^{-2} dr$$

$$= \frac{-Q}{4\pi\epsilon_0} \left[-r^{-1} \right]_{r_A}^{r_B}$$

$$= +\frac{Q}{4\pi\epsilon_0} \left[\frac{1}{r_B} - \frac{1}{r_A} \right]$$

$$\therefore V_{AB} = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{r_B} - \frac{1}{r_A}\right]$$

$$\Rightarrow V_{AB} = V_B - V_A \qquad \text{--- (3.45 b)}$$

where V_B and V_A are the potentials (or absolute potentials) at B and A, respectively. Thus the potential difference V_{AB} may be regarded as the potential at B with reference to A.

In problem involving point charges, it is customary to choose infinity as reference; that is, we assume the potential at infinity is zero. Thus if $V_A = 0$ as $r_A \to \infty$ in eqn (3.45), the potential at any point ($r_B \to r$) due to a point charge Q located at the origin is

$$\boxed{V = \frac{Q}{4\pi\epsilon_0 r}} \qquad \text{--- (3.46)}$$

Note:-

1) Note from eqn 3.45(a) that because E points in the radial direction, any contribution from a displacement in the θ or ϕ direction is wiped out by the dot product $E \cdot dl = E\cos\alpha\, dl = E\, dr$, where α is angle

between E and dl. Hence the potential difference V_{AB} is independent of path as asserted earlier. In general, vectors whose line integral does not depend on the path of integration are called conservative. This E is conservative.

~~point is the~~

2. The potential at any point ~~is the~~ point and potential difference between that point and a chosen point (or reference point) at which the potential is zero.

3. In other words, by assuming zero potential at infinity, the potential at a distance 'r' from the point charge is the work done per unit charge by an external agent in transferring a test charge from infinity to that point. Thus

$$V = -\int_{\infty}^{r} E \cdot dl \qquad (3.47)$$

4. If the point charge 'Q' is not located at the origin but at a point whose position vector is r', the potential $V(x,y,z)$ or simply $V(r)$ at r becomes

$$V(r) = \frac{Q}{4\pi\epsilon_0 |r - r'|} \qquad (3.48)$$

5) For n point charges $Q_1, Q_2 \ldots Q_n$ located at points with position vectors $r_1, r_2 \ldots, r_n$, the potential at r is

$$V(r) = \frac{Q_1}{4\pi\epsilon_0 |r-r_1|} + \frac{Q_2}{4\pi\epsilon_0 |r-r_2|} + \ldots + \frac{Q_n}{4\pi\epsilon_0 |r-r_n|}$$

$$\Rightarrow V(r) = \frac{1}{4\pi\epsilon_0} \sum_{k=1}^{n} \frac{Q_k}{|r-r_k|} \quad \text{——— (3.49)}$$

6) For Continuous Charge distribution

We replace Q_k in eq (3.49) with charge element $\rho_L\,dl$, $\rho_s\,ds$, $\rho_v\,dv$ and the summation becomes an integration.

$$\therefore V(r) = \frac{1}{4\pi\epsilon_0} \int_L \frac{\rho_L(r')\,dl'}{|r-r'|} \quad \text{(line charge)} \quad (3.50)$$

$$V(r) = \frac{1}{4\pi\epsilon_0} \int_S \frac{\rho_s(r')\,ds'}{|r-r'|} \quad \text{(surface charge)} \quad -(3.51)$$

$$V(r) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho_v(r')\,dv'}{|r-r'|} \quad \text{(volume charge)} \quad -(3.52)$$

Where the primed coordinates (r') are used ~~to~~ customarily to denote source point and the unprimed (r) coordinates refer to field point

(the point at which V is to be determined)

7) If reference point is chosen other than infinity, then eqn (3.48) becomes

$$V = \frac{Q}{4\pi\epsilon_0 r} + C \quad \text{——— (3.53)}$$

where C is constant that is determined at the chosen point of reference. The same idea applicable to eqn (3.46) to (3.52)

8) If charge distribution is known, the potential at a point can be determined using eqn (3.48) to (3.53)

9) If E is known, we simply use

$$V = -\int E \cdot dl + C \quad \text{——— (3.54)}$$

10) The potential difference V_{AB} can be found generally from

$$V_{AB} = V_B - V_A = -\int_A^B E \cdot dl = \frac{W}{Q} \quad \text{——— (3.55)}$$

Example :- (3.10) Two point charges $-4\mu C$ and $5\mu C$ are located at (2,−1,3) and (0,4,−2), respectively. Find the potential at (1,0,1) assuming zero potential at infinity.

Ans:- Let

$$Q_1 = -4\mu C, \quad Q_2 = 5\mu C$$

$$V(r) = \frac{Q_1}{4\pi\epsilon_0 |r - r_1|} + \frac{Q_2}{4\pi\epsilon_0 |r - r_2|} + C_0$$

If $V(\infty) = 0$, $C_0 = 0$

$$|r - r_1| = |(1,0,1) - (2,-1,3)|$$

$$= |(-1, +1, -2)|$$

$$= \sqrt{1+1+4}$$

$$|r - r_1| = \sqrt{6}$$

$$|r - r_2| = |(1,0,1) - (0,4,-2)|$$

$$= |(1,-4,+3)|$$

$$= \sqrt{1+16+9}$$

$$|r - r_2| = \sqrt{26}$$

Hence, $V(1,0,1) = \dfrac{10^{-6}}{4\pi \times \frac{10^{-9}}{36\pi}} \left[\dfrac{-4}{\sqrt{6}} + \dfrac{5}{\sqrt{26}} \right]$

$\Rightarrow V(1,0,1) = 9 \times 10^3 (-1.633 + 0.9806)$

$\Rightarrow V(1,0,1) = -5.872 \text{ KV}$ (Ans)

Relationship between (E and V) - Maxwell's eqn

In the preceding section, we have discussed the potential difference

$$V_{AB} = -V_{BA}$$

$$\Rightarrow V_{AB} + V_{BA} = 0$$

$$\Rightarrow \oint_L E \cdot dl = 0$$

[$\oint_L$ = line integral on a closed path]

$$\therefore \boxed{\oint_L E \cdot dl = 0} \quad (3.56)$$

This shows that the line integral of E along a closed path as shown in Figure 3.19 must be zero.

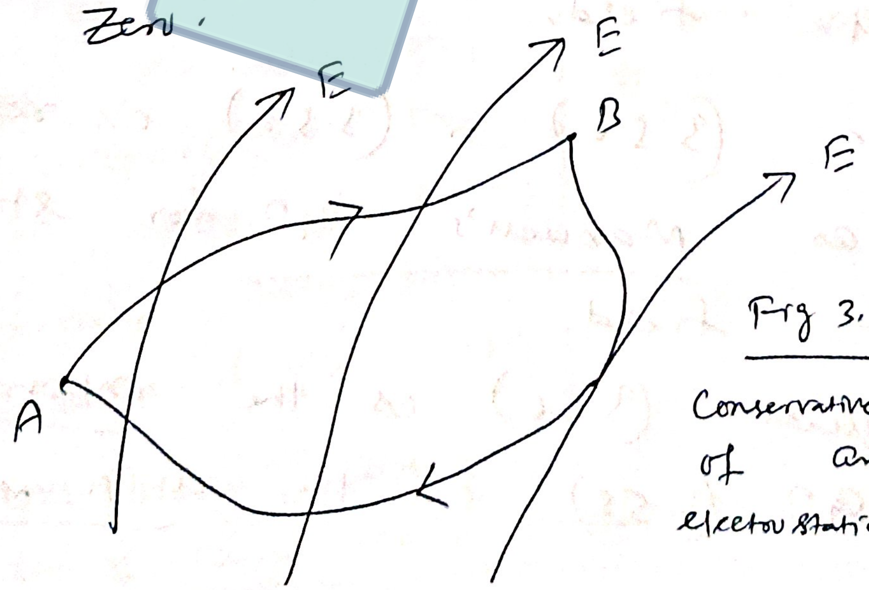


Fig 3.19 The Conservative nature of an electrostatic field.

Physically, this implies that no net work is done in moving a charge along a closed path in an electrostatic field. Applying Stoke's theorem to eqn (3.56)

$$\oint_L E.dl = \int_S (\nabla \times E).ds = 0 \quad \text{--- (3.57)}$$

$$\Rightarrow \boxed{\nabla \times E = 0} \quad \text{--- (3.58)}$$

Any vector field that satisfies eqn (3.56) or (3.58) is said to be conservative or irrotational, as discussed earlier. (In property of curl)

→ In other words, vectors whose line integral does not depend on the path of integration are called conservative vectors.

→ Thus an electrostatic field is a conservative field.

→ Eqn (3.56) or (3.58) is referred to as Maxwell's eqn for static electric field.

→ Equation (3.56) is the integral form and eqn (3.58) is the differential form.

We know,

$$V = -\int E \cdot dl \,, \quad \text{[As derived earlier]}$$

It follows that

$$dV = -E \cdot dl = -E_x dx - E_y dy - E_z dz \quad (3.59)$$

But from Calculus of multivariables, a total change in $V(x, y, z)$ is the sum of partial changes w.r.to x, y, z variables:

$$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz \quad (3.60)$$

Equating (3.59) and (3.60), we have

$$E_x = -\frac{\partial V}{\partial x}, \quad E_y = -\frac{\partial V}{\partial y}, \quad E_z = -\frac{\partial V}{\partial z} \quad (3.61)$$

Thus:

$$\boxed{E = -\nabla V} \quad (3.62)$$

$$\because -\nabla V = -\left(\frac{\partial V}{\partial x} a_x + \frac{\partial V}{\partial y} a_y + \frac{\partial V}{\partial z} a_z\right)$$

$$= -\frac{\partial V}{\partial x} a_x - \frac{\partial V}{\partial y} a_y - \frac{\partial V}{\partial z} a_z$$

$$= E_x a_x + E_y a_y + E_z a_z$$

$$= E$$

That is the electric field intensity is the gradient of V. The -ve sign shows that the direction of E is opposite to the direction in which V increases; E is directed from higher to lower level of V. (↓)

Thus, if the potential field V is known, the E can be found using eqn (3.62).

Example 3.12

Given the potential $V = \frac{10}{r^2} \sin\theta \cos\phi$,

(a) Find the electric flux density D at $(2, \frac{\pi}{2}, 0)$.

(b) Calculate the work done in moving a $10\ \mu C$ charge from point $A(1, 30^\circ, 120^\circ)$ to $B(4, 90^\circ, 60^\circ)$

Ans:- (a) $D = \epsilon_0 E$

and $E = -\nabla V$

$$= -\left[\frac{\partial V}{\partial r} a_r + \frac{1}{r}\frac{\partial V}{\partial \theta} a_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi} a_\phi\right]$$

$$= -\left[10\sin\theta\cos\phi \times \frac{-2}{r^3} a_r + \frac{1}{r} \times \frac{10}{r^2}\cos\phi \times \cos\theta\, a_\theta + \frac{1}{r\cancel{\sin\theta}} \times \frac{10}{r^2}\cancel{\sin\theta}(-\sin\phi) a_\phi\right]$$

$$E = \frac{20\sin\theta\cos\phi}{r^3} a_r - \frac{10\cos\theta\cos\phi}{r^3} a_\theta + \frac{10\sin\phi}{r^3} a_\phi$$

$$E\Big|_{(2,\frac{\pi}{2},0)} = \frac{20 \times 1 \times 1}{8} a_r - \frac{10 \times 0 \times 1}{8} a_\theta + \frac{10 \times 0}{8} a_\phi$$

$$E = 2.5\, a_r \ \frac{\text{Volt}}{\text{meter}}$$

$$D = \epsilon_0 \epsilon_r E$$

$$= \frac{+10^{-9}}{36\pi} \times 2.5$$

$$D = 22.1 \ a_r \ \frac{pC}{m^2}$$

(b) $W = VQ$

$$= QV$$

$$= Q\left[-\int_A^B E \cdot dl\right]$$

$$= Q\,(V_B - V_A) \quad [\text{From eq}^n\ 3.58]$$

$$= 10 \times 10^{-6}\left[\frac{10}{r^2}\sin\theta\cos\phi\Big|_{(4,90^\circ,60^\circ)} - \frac{10}{r^2}\sin\theta\cos\phi\Big|_{(1,30^\circ,120^\circ)}\right]$$

$$= 10 \times 10^{-6}\left[\frac{10}{16} \times 1 \times \frac{1}{2} - \frac{10}{1} \times \frac{1}{2} \times \left(-\frac{1}{2}\right)\right]$$

$$= 10 \times 10^{-6}\left(\frac{10}{32} + \frac{5}{2}\right)$$

$$\boxed{W = 28.125\ \mu J}$$

(Ans)

Note:- 1) An electric flux line is an imaginary path or line drawn in such a way that its direction at any point is the direction of electric field at that point.

2) In other words, they are the lines to which the electric flux 'D' is tangential at every point.

3) Any surface on which the potential is same through out is known as equipotential surface. The intersection of an equipotential surface and a plane results in a path or line known as equipotential line.

4) No work is done in moving a charge from one point to another along an equipotential line or surface $(V_A - V_B = 0)$ hence

$$\int_L E \cdot dl = 0 \qquad (3.63)$$

on the line or surface.

5) From eqn (3.63), we may conclude that

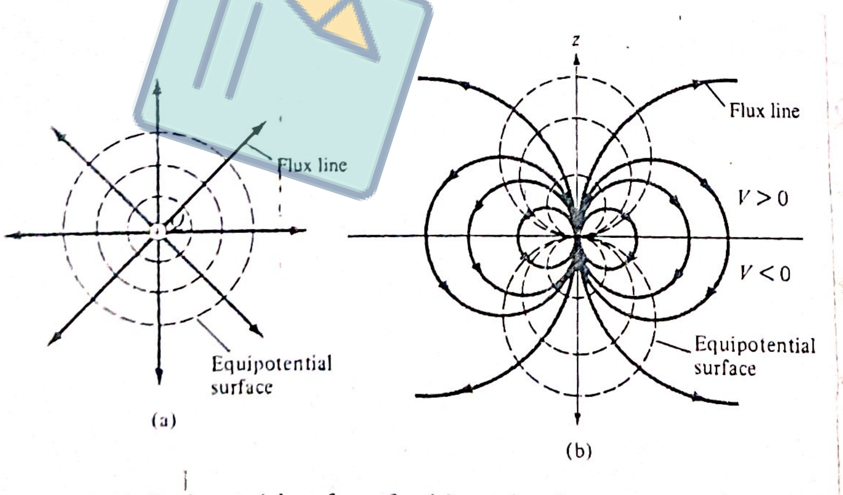


Figure 3.21 Equipotential surfaces for **(a)** a point charge and **(b)** an electric dipole.

Fig 3.21

the lines of force or flux line (or direction of E) are always normal to the equipotential surface. $(\because E \cdot dl = 0)$

the
(b) Example of equipotential surface for point charge and a dipole are shown in figure 3.21. Note from these examples that the direction of E is everywhere normal to the equipotential lines.

Poisson's and Laplace's Equations

Poisson's and Laplace's equations are easily derived from Gauss's law.

We know

$$\nabla \cdot D = \nabla \cdot \epsilon E = \rho_v \quad \text{——— (3.64)}$$

and

$$E = -\nabla V \quad \text{——— (3.65)}$$

Substituting eqn (3.65) into eqn (3.64)

$$\nabla \cdot (-\epsilon \nabla V) = \rho_v \quad \text{——— (3.66)}$$

for an inhomogeneous medium. For a homogeneous medium, eqn (3.66) becomes

$$\boxed{\nabla^2 V = -\frac{\rho_v}{\epsilon}} \quad \text{——— (3.67)}$$

This is known as Poisson's Equation.

A special case of this equation occurs when $\rho_v = 0$ (i.e. for a charge-free region). Eqn (3.67) becomes

$$\boxed{\nabla^2 V = 0} \longrightarrow (3.68)$$

which is known as Laplace's equation.

Note

1) Note that in taking 'ϵ' out of the left-hand side of eqn (3.66) to obtain eqn (3.67), we have assumed that 'ϵ' is a constant throughout the region in which V is defined;

2) For an inhomogeneous region, ϵ is not constant and eqn (3.67) does not follow eqn (3.66). Eqn (3.66) is Poisson's equation for an inhomogeneous medium; it becomes Laplace's equation for an inhomogeneous medium when $\rho_v = 0$.

3) Recall that the Laplacian operator ∇^2 derived in eqn (2.42), (2.43) and (2.44), Thus Laplace's eqn in Cartesian, cylindrical, or spherical coordinates respectively is given by

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \quad \text{--- } (3.69)$$

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0 \quad \text{--- } (3.70)$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = 0 \quad \text{--- } (3.71)$$

depending on the coordinate variables used to express V, that is, $V(x,y,z)$, $V(\rho,\phi,z)$ or $V(r,\theta,\phi)$. Poisson's equation in those coordinate system may be obtained by simply replacing zero on the right-hand side of eqn (3.69), (3.70), and (3.71) with

$$\boxed{-\frac{\rho_v}{\epsilon}}$$

4) Laplace's eqn is of primary importance in solving electrostatic problems involving a set of conductors maintained at different potentials. Examples of such problems include capacitors and vacuum tube diodes.

5) Laplace & Poissions eqn are used in various fields. For example, V would be treated as magnetic potential in magnetostatics, as temperature in heat conduction, as stress function in fluid flow, etc.