Module-1 :- Coordinate Systems and Vector Calculus

(Chapter-1)

In general, the physical quantities we shall be dealing with in EM (Electromagnetic) are function of space and time. In order to describe the spatial variations of the quantities, we must be able to define all points uniquely in space in a suitable manner. This requires using an appropriate coordinate system.

A point or vector can be represented in any curvilinear coordinate system, which may be orthogonal or non orthogonal.

Non orthogonal systems are hard to work with, and they are of little or no practical use.

An orthogonal system is one in which the coordinates are mutually perpendicular. We shall restrict ourselves to the three best-known coordinate systems:

(i) Cartesian (or Rectangular)

(ii) The circular Cylindrical

(iii) The Spherical

Cartesian Coordinates (x, y, z)

A point 'P' can be represented as (x, y, z) as illustrated in Figure 1.1. The range of the coordinate variables x, y, z are

$$\left.\begin{aligned} -\infty < x < \infty \\ -\infty < y < \infty \\ -\infty < z < \infty \end{aligned}\right\} \quad \text{— (1.1)}$$

A vector 'A' in Cartesian (Rectangular) Co-ordinates can be written as

$$(A_x, A_y, A_z) \quad \text{or} \quad A_x a_x + A_y a_y + A_z a_z \quad \text{— (1.2)}$$

Where a_x, a_y, a_z are the unit vectors along x, y, and z directions as shown in figure 1.1.

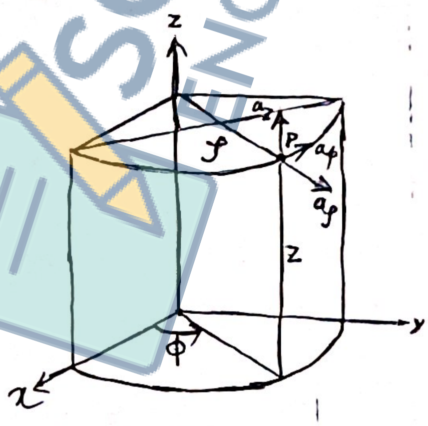


Fig 1.1:- Point 'P' and unit vectors in the cylindrical coordinate system

Circular Cylindrical Coordinates (ρ, ϕ, z)

The Circular cylindrical coordinate system is very convenient whenever we are dealing with problems having cylindrical symmetry.

A point 'p' in cylindrical coordinates is represented as (ρ, ϕ, z) and is shown in Fig. 1.1. Here, the space variables

$\rho \rightarrow$ radius of the cylinder passing through 'p' or radial distance from z-axis.

$\phi \rightarrow$ is called the azimuthal angle, and measured from the x-axis in the xy-plane.

$z \rightarrow$ is same as in the Cartesian system.

The range of variables are

$$\left.\begin{aligned} 0 &\leq \rho < \infty \\ 0 &\leq \phi < 2\pi \\ -\infty &< z < \infty \end{aligned}\right\} \quad (1.3)$$

A vector 'A' in cylindrical co-ordinate can be written as

$$(A_\rho, A_\phi, A_z) \text{ or } A_\rho a_\rho + A_\phi a_\phi + A_z a_z \quad (1.4)$$

where a_ρ, a_ϕ and a_z are unit vectors in the ρ-, ϕ-, and z- directions as illustrated in Figure 1.1.

Ex:- If force of 10 N act on a particle in circular motion, the force may be represented as, $\boxed{F = 10\, a_\phi\, N}$. In this case a_ϕ is in Newton.

Thus, a_ϕ is not degree, it assumes the unit of A.

→ The magnitude of A is

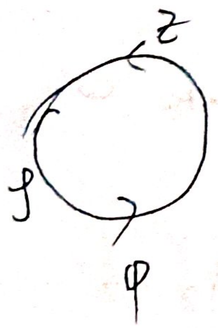
$$|A| = \sqrt{A_\rho^2 + A_\phi^2 + A_z^2} \qquad (1.5)$$

→ Notice that the unit vectors a_ρ, a_ϕ, a_z are mutually perpendicular because our coordinate system is orthogonal [See Figure 1.1].

a_ρ points in the direction of increasing ρ

a_ϕ in the direction of increasing ϕ,

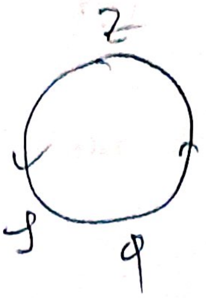
a_z in the +ve z-direction. Thus,



$$a_\rho \cdot a_\rho = a_\phi \cdot a_\phi = a_z \cdot a_z = 1 \qquad (1.6a)$$

$$a_\rho \cdot a_\phi = a_\phi \cdot a_z = a_z \cdot a_\rho = 0 \qquad (1.6b)$$

$$\left(\because \begin{array}{l} A \cdot B = AB \cos\theta \\ A \perp B = AB \cos 90^\circ = 0 \end{array} \right)$$



$a_\rho \times a_\phi = a_z$ —— (1.6 c)

$a_\phi \times a_z = a_\rho$ —— (1.6 d)

$a_z \times a_\rho = a_\phi$ —— (1.6 e)

(Like $a_x \times a_y = a_z$, $a_y \times a_z = a_x$)

→ Relation Between Cartesian Coordinate System and cylindrical Coordinate System

From figure 1.2, the relationship between the variables (x, y, z) of Cartesian Co-ordinate system and those of the cylindrical system (ρ, ϕ, z) are easily obtained.

$$\boxed{\rho = \sqrt{x^2+y^2}, \quad \phi = \tan^{-1}\left(\frac{y}{x}\right), \quad z = z}$$

Proof:-

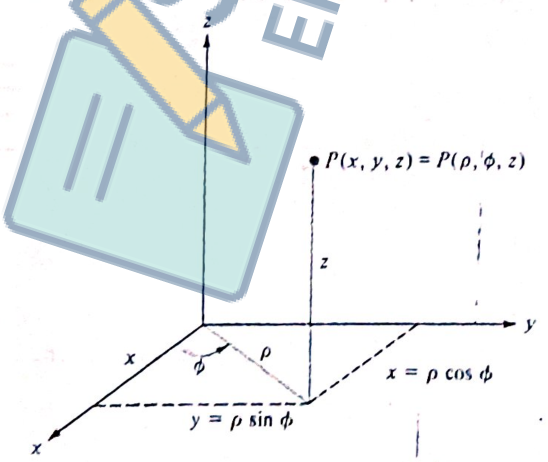
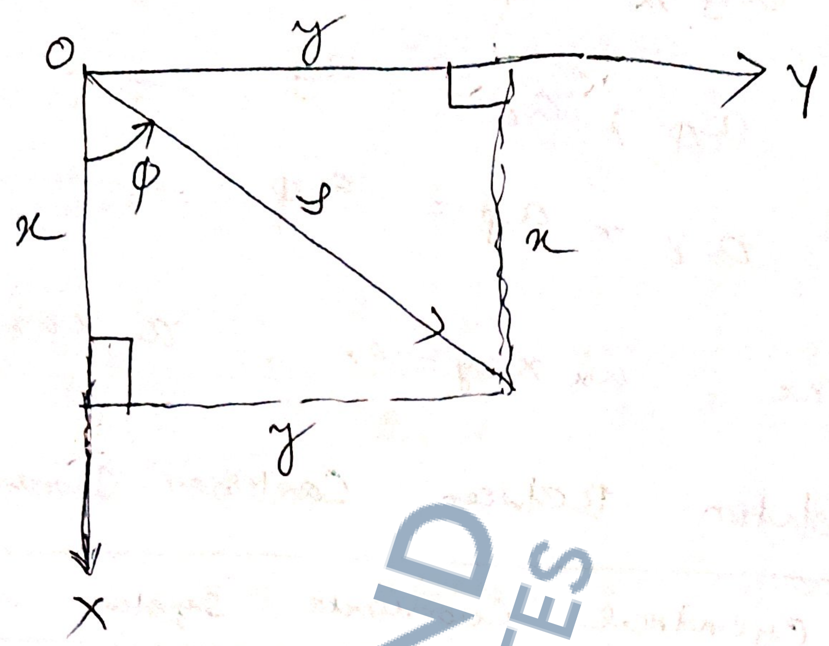


Figure:- 1.2 Relationship betn (x, y, z) and (ρ, ϕ, z)



$$\cos\phi = \frac{x}{\rho}, \quad \sin\phi = \frac{y}{\rho}, \quad \tan\phi = \frac{y}{x}$$

$$\Rightarrow \boxed{x = \rho\cos\phi, \quad y = \rho\sin\phi, \quad \tan\phi = \frac{y}{x}} \quad \text{—} (1)$$

From eqⁿ (1), $x^2 + y^2 = \rho^2\cos^2\phi + \rho^2\sin^2\phi$

$$= \rho^2(\cos^2\phi + \sin^2\phi)$$

$$\boxed{x^2 + y^2 = \rho^2} \quad \text{(Proved)}$$

$$\Rightarrow \boxed{\rho = \sqrt{x^2+y^2}} \quad \text{—} (2)$$

Thus,

$$\boxed{\rho = \sqrt{x^2+y^2}, \quad \phi = \tan^{-1}\left(\frac{y}{x}\right), \quad z = z} \quad \text{—} (1.7)$$

or

$$\boxed{x = \rho\cos\phi, \quad y = \rho\sin\phi, \quad z = z} \quad \text{—} (1.8)$$

Equation (1.7), is for transforming a point from Cartesian (x, y, z) to cylindrical (ρ, ϕ, z).

Eqn (1.8) is for $(\rho, \phi, z) \rightarrow (x, y, z)$.

Relationship between (a_x, a_y, a_z) and (a_ρ, a_ϕ, a_z)

(Relation betn betn the unit vectors of Cartesian & cylindrical coordinates)

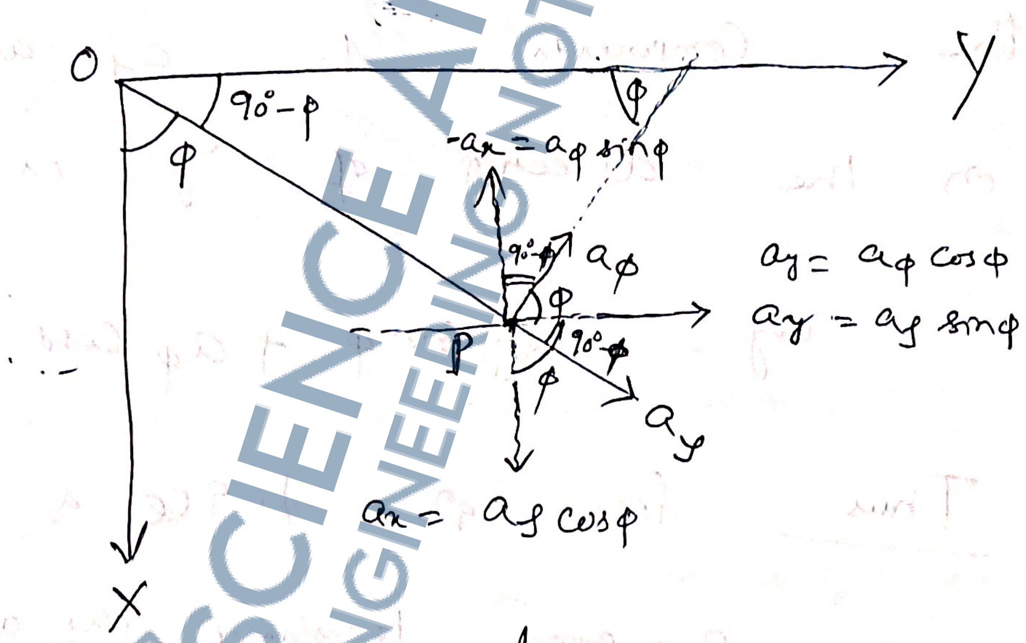


Fig 1.3 (a):- Unit vector transformation from cylindrical to cartesian

At point P,

If $\hat{a}_\rho$ is resolve into 2 components,

$$a_x = a_\rho \cos\phi \quad \text{——— } 1.9\,(a)$$

$$a_y = a_\rho \sin\phi \quad \text{——— } 1.9\,(b)$$

Similarly, If a_ϕ is resolve into 2 components,

$$a_y = a_\phi \cos\phi \quad \text{——— } 1.9\,(c)$$

$$-a_x = a_\phi \sin\phi$$

$$\Rightarrow a_x = -a_\phi \sin\phi \quad \text{——— } 1.9\,(d)$$

Combining eqn 1.9 (a) & 1.9 (d), the components of a_ρ & a_ϕ on the direction of 'x' is given by

$$a_x = a_\rho \cos\phi - a_\phi \sin\phi \quad \text{--- } 1.9(e)$$

Similarly, combining eqn 1.9(b) & 1.9(c), the components of a_ρ and a_ϕ on the direction of 'y' is given by

$$a_y = a_\rho \sin\phi + a_\phi \cos\phi \quad \text{--- } 1.9(f)$$

Thus from eqn 1.9(e) & 1.9(f)

~~$a_x = a_\rho \cos\phi$~~ transforming $(a_\rho, a_\phi, a_z) \rightarrow (a_x, a_y, a_z)$

$$\boxed{\begin{aligned} a_x &= \cos\phi\, a_\rho - \sin\phi\, a_\phi \\ a_y &= \sin\phi\, a_\rho + \cos\phi\, a_\phi \\ a_z &= a_z \end{aligned}} \quad \text{--- } 1.9$$

Similarly, the transformation of $(a_x, a_y, a_z) \rightarrow (a_\rho, a_\phi, a_z)$ can be done as follows.

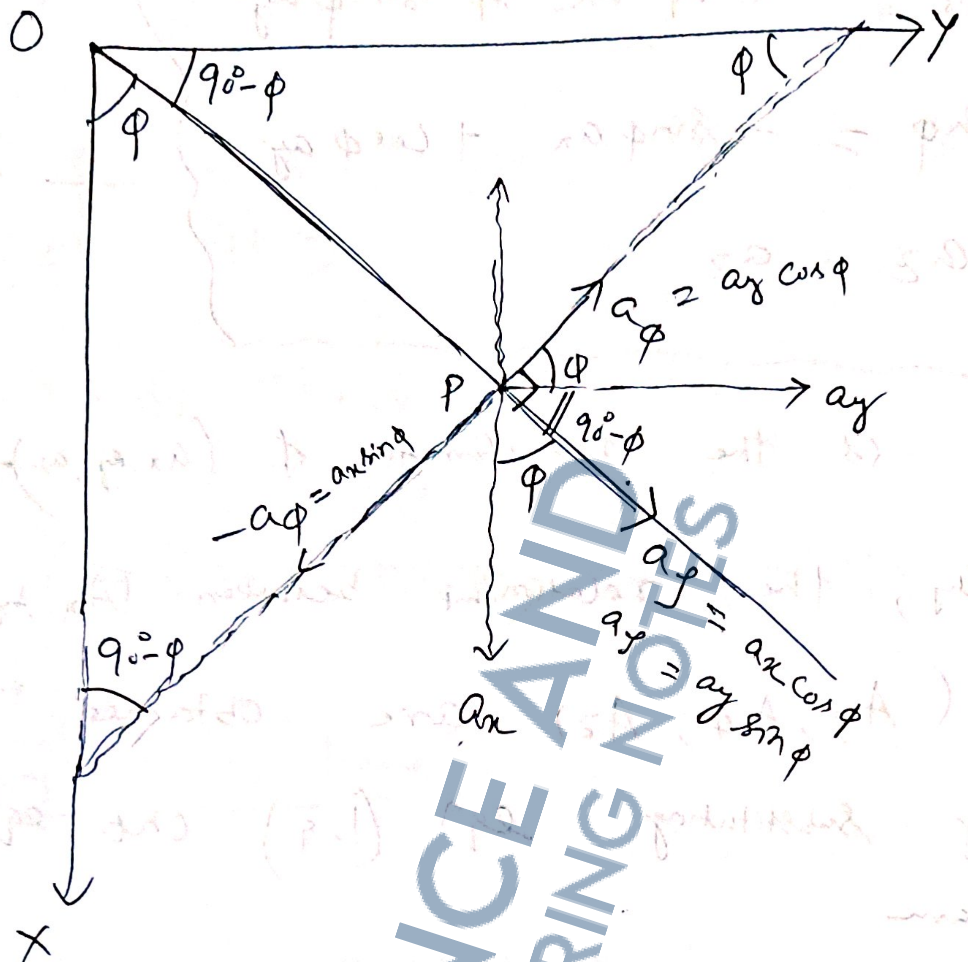


Fig 1.13(b) → Unit vector transformation from Cartesian to cylindrical.

If a_x is resolve into 2 components, along a_ρ and $-a_\phi$,

$$a_\rho = a_x \cos\phi \quad \text{—— } 1.10(a)$$

$$-a_\phi = a_x \sin\phi \quad \text{—— } 1.10(b)$$

If a_y is resolve into 2 rectangular components, along a_ρ and a_ϕ, we have

$$a_\rho = a_y \sin\phi \quad \text{—— } 1.10(c)$$

$$a_\phi = a_y \cos\phi \quad \text{—— } 1.10(d)$$

Combining 1.10 (a) & 1.10 (c), 1.10 (b) & 1.10 (d) } we have

$$\boxed{\begin{aligned} a_\rho &= \cos\phi \, a_x + \sin\phi \, a_y \\ a_\phi &= -\sin\phi \, a_x + \cos\phi \, a_y \\ a_z &= a_z \end{aligned}} \quad \text{——} \quad (1.10)$$

This is the transformation of $(a_x, a_y, a_z) \to (a_\rho, a_\phi, a_z)$

Finally, the relationship between (A_x, A_y, A_z) and (A_ρ, A_ϕ, A_z) are obtained by simply substituting eqn (1.9) into eqn (1.2), we have

From eqn (1.2), we have

$$A = A_x a_x + A_y a_y + A_z a_z$$

$$= A_x (\cos\phi \, a_\rho - \sin\phi \, a_\phi) + A_y (\sin\phi \, a_\rho + \cos\phi \, a_\phi) + A_z a_z$$

Rearranging, we have

$$A = (A_x \cos\phi + A_y \sin\phi) a_\rho + (-A_x \sin\phi + A_y \cos\phi) a_\phi + A_z a_z \quad \text{——} \quad (1.11a)$$

Similarly putting eqn (1.10) into eqn (1.4)

We have,

From eqn (1.4)

$$A = A_\rho a_\rho + A_\phi a_\phi + A_z$$

$$= A_\rho(\cos\phi\, a_x + \sin\phi\, a_y) + A_\phi(-\sin\phi\, a_x + \cos\phi\, a_y) + A_z$$

Rearranging, we have

$$A = (A_\rho \cos\phi - A_\phi \sin\phi)\, a_x + (A_\rho \sin\phi + A_\phi \cos\phi)\, a_y + A_z \quad \text{--- (1.11 b)}$$

From eqn eqn 1.11(a), we have

$$\left.\begin{aligned} A_\rho &= A_x \cos\phi + A_y \sin\phi \\ A_\phi &= -A_x \sin\phi + A_y \cos\phi \\ A_z &= A_z \end{aligned}\right\} \text{--- (1.12 a)}$$

Similarly from eqn 1.11 (b), we have

$$\left.\begin{aligned} A_x &= A_\rho \cos\phi - A_\phi \sin\phi \\ A_y &= A_\rho \sin\phi + A_\phi \cos\phi \\ A_z &= A_z \end{aligned}\right\} \text{--- (1.12 b)}$$

In Matrix form, we write the transformation of vector A form (A_x, A_y, A_z) to (A_ρ, A_ϕ, A_z) as [Refer eqn (1.12a)]

V. Imp

$$\boxed{\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}} \quad (1.13)$$

The inverse transformation $(A_\rho, A_\phi, A_z) \rightarrow (A_x, A_y, A_z)$ is obtained as

$$\Rightarrow \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} \quad (1.14)$$

or From eqn 1.12 (b)

V. Imp

$$\boxed{\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}} \quad (1.15)$$

Assignment 1:- Prove that

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} a_x\cdot a_\rho & a_x\cdot a_\phi & a_x\cdot a_z \\ a_y\cdot a_\rho & a_y\cdot a_\phi & a_y\cdot a_z \\ a_z\cdot a_\rho & a_z\cdot a_\phi & a_z\cdot a_z \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} \quad (1.16)$$

Hints: Start with R.H.S → Make dot product of equation (1.10) . a_x, a_y, a_z

Spherical Coordinates (r, θ, ϕ)

The spherical coordinate system is most appropriate when one is dealing with problems having a degree of spherical symmetry. A point 'P' can be represented as (r, θ, ϕ) and is illustrated in figure 1.4.

From figure 1.4, 'r' is defined as the distance from the origin to point P or radius of a sphere centered at origin.

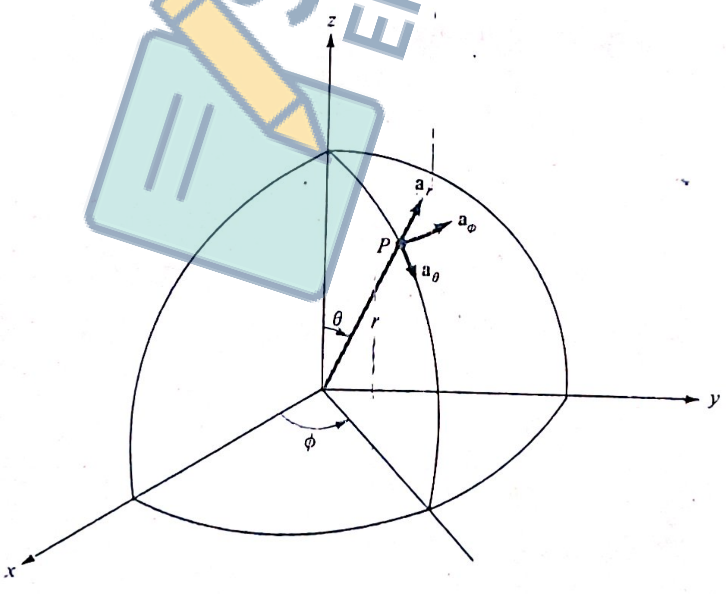


Figure 1.4 Point P and unit vectors in spherical coordinates.

'θ' (Called the Colatitude) is the angle between the Z-axis and the position vector of 'P'; 'ϕ' is ~~the~~ measured from X-axis (the same azimuthal angle in cylindrical coordinates). According to these definitions, the range of the variables are

$$\left.\begin{aligned} 0 &\leq r < \infty \\ 0 &\leq \theta \leq \pi \\ 0 &\leq \phi \leq 2\pi \end{aligned}\right\} \quad (1.17)$$

A vector A in spherical coordinates may be written as

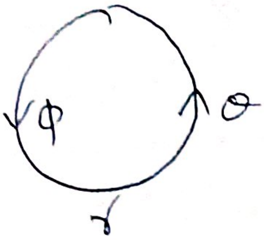
$$(A_r, A_\theta, A_\phi) \text{ or } A_r a_r + A_\theta a_\theta + A_\phi a_\phi \quad (1.18)$$

Where a_r, a_θ, and a_ϕ are unit vectors along r, θ, ϕ directions. The magnitude of A is

$$|A| = \sqrt{A_r^2 + A_\theta^2 + A_\phi^2} \quad (1.19)$$

The unit vectors a_r, a_θ, a_ϕ are mutually orthogonal, a_r being directed along the radius or in the direction of increasing 'r', a_θ in the direction of increasing 'θ', and a_ϕ in the direction of of increasing 'ϕ'. Thus,

$$\left.\begin{aligned} a_r \cdot a_r &= a_\theta \cdot a_\theta = a_\phi \cdot a_\phi = 1 \\ a_r \cdot a_\theta &= a_\theta \cdot a_\phi = a_\phi \cdot a_r = 0 \\ a_r \times a_\theta &= a_\phi \\ a_\theta \times a_\phi &= a_r \\ a_\phi \times a_r &= a_\theta \end{aligned}\right\} \text{— (1.20)}$$



The space variables (x, y, z) in Cartesian coordinates can be related to variables (r, θ, ϕ) of a spherical coordinate system. From figure 1.5

$$r = \sqrt{\rho^2 + z^2}$$

$$r = \sqrt{x^2+y^2+z^2} \quad \text{— 1.21 (a)}$$

$(\because \rho^2 = x^2+y^2$ Eqn (1.7))

$$\tan\theta = \frac{\rho}{z} = \frac{r\sin\theta}{r\cos\theta} \quad \text{(verified)}$$

$\therefore$

$$\tan\theta = \frac{\rho}{z} = \frac{\sqrt{x^2+y^2}}{z} \quad \text{— 1.21 (b)}$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right) \quad \text{— 1.21 (c)}$$

(verify $\frac{y}{x} = \frac{\rho\sin\phi}{\rho\cos\phi} = \tan\phi$)

Thus,

$$\left.\begin{aligned} r &= \sqrt{x^2+y^2+z^2} \\ \theta &= \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right) \\ \phi &= \tan^{-1}\left(\frac{y}{x}\right) \end{aligned}\right\} \text{— (1.21)}$$

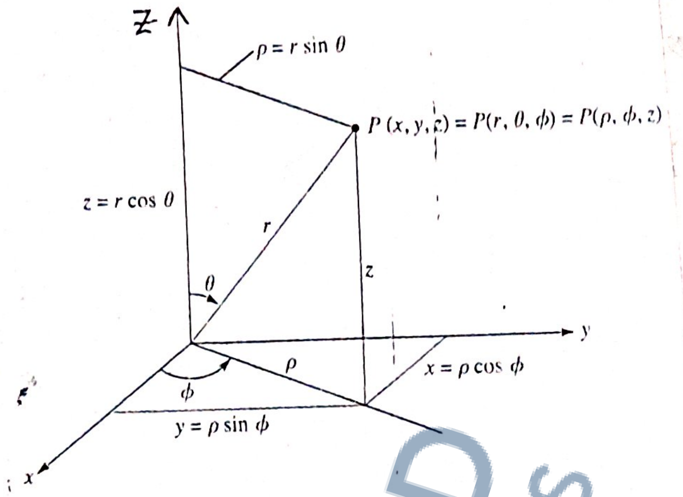


Figure 1.5 Relationships between space variables (x, y, z), (r, θ, ϕ), and (ρ, ϕ, z).

From figure 1.5,

$$x = \rho\cos\phi = r\sin\theta\cos\phi \qquad (\because \rho = r\sin\theta)$$

$$y = \rho\sin\phi = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$

Thus

$$\boxed{x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta} \quad \text{— (1.22)}$$

In eqn (1.21) we have $(x, y, z) \rightarrow (r, \theta, \phi)$ transformation and in eqn (1.22), we have $(r, \theta, \phi) \rightarrow (x, y, z)$ point transformation.

As we have done the conversion of unit vectors of Cartesian & cylindrical co-ordinates, similarly unit vectors of Cartesian and spherical coordinates can be derived as follows.

Assignment 2:- Prove that

$$\left.\begin{aligned} a_x &= \sin\theta\cos\phi\, a_r + \cos\theta\cos\phi\, a_\theta - \sin\phi\, a_\phi \\ a_y &= \sin\theta\sin\phi\, a_r + \cos\theta\sin\phi\, a_\theta + \cos\phi\, a_\phi \\ a_z &= \cos\theta\, a_r - \sin\theta\, a_\theta \end{aligned}\right\} \quad (1.23)$$

Assign 3:- Prove that

$$\left.\begin{aligned} a_r &= \sin\theta\cos\phi\, a_x + \sin\theta\sin\phi\, a_y + \cos\theta\, a_z \\ a_\theta &= \cos\theta\cos\phi\, a_x + \cos\theta\sin\phi\, a_y - \sin\theta\, a_z \\ a_\phi &= -\sin\phi\, a_x + \cos\phi\, a_y \end{aligned}\right\} \quad (1.24)$$

The component of vector $A = (A_x, A_y, A_z)$ and $A = (A_r, A_\theta, A_\phi)$ are related by substituting eqn (1.23) into eqn (1.2) and collecting terms.

$$A = A_x a_x + A_y a_y + A_z a_z$$

$$= A_x(\sin\theta\cos\phi\, a_r + \cos\theta\cos\phi\, a_\theta - \sin\phi\, a_\phi) + A_y(\sin\theta\sin\phi\, a_r + \cos\theta\sin\phi\, a_\theta + \cos\phi\, a_\phi) + A_z(\cos\theta\, a_r - \sin\theta\, a_\theta)$$

Rearranging

$$A = (A_x \sin\theta\cos\phi + A_y \sin\theta\sin\phi + A_z\cos\theta)\, a_r + (A_x\cos\theta\cos\phi + A_y\cos\theta\sin\phi - A_z\sin\theta)\, a_\theta + (-A_x\sin\phi + A_y\cos\phi)\, a_\phi \quad (1.25)$$

From this we obtain

$$\left.\begin{aligned} A_r &= A_x\sin\theta\cos\phi + A_y\sin\theta\sin\phi + A_z\cos\theta \\ A_\theta &= A_x\cos\theta\cos\phi + A_y\cos\theta\sin\phi - A_z\sin\theta \\ A_\phi &= -A_x\sin\phi + A_y\cos\phi \end{aligned}\right\} (1.26)$$

In Matrix form, the $(A_x, A_y, A_z) \rightarrow (A_r, A_\theta, A_\phi)$ vector transformation is performed according to

V.Imp

$$\boxed{\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}} \quad (1.27)$$

$$\Rightarrow \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix}^{-1} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}$$

Find the inverse of above matrix,

we have, $(A_r, A_\theta, A_\phi) \to (A_x, A_y, A_z)$ Conversion.

V. Imp

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} \quad (1.28)$$

Assignment 4:- Prove that eqn (1.28) Can be Obtained using dot product For example:

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} a_r \cdot a_x & a_r \cdot a_y & a_r \cdot a_z \\ a_\theta \cdot a_x & a_\theta \cdot a_y & a_\theta \cdot a_z \\ a_\phi \cdot a_x & a_\phi \cdot a_y & a_\phi \cdot a_z \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} \quad (1.29)$$

Hints:- Start with R.H.S, make dot product of eqn

$(1.24) \cdot a_x \to$ 3 times

$(1.24) \cdot a_y \to$ 3 times

$(1.24) \cdot a_z \to$ 3 times

Relation betn Cylindrical & Spherical Coordinates

From eqn (1.21) (r, θ, ϕ) (ρ, ϕ, z)

$$r = \sqrt{x^2+y^2+z^2} = \sqrt{\rho^2+z^2}$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right) = \tan^{-1}\left(\frac{\rho}{z}\right)$$

ϕ is same for both

Thus

$$\boxed{\begin{aligned} r &= \sqrt{\rho^2+z^2} \\ \theta &= \tan^{-1}\left(\frac{\rho}{z}\right) \\ \phi &= \phi \end{aligned}} \quad \text{--- (1.30)}$$

Similarly from figure (1.5)

$$\boxed{\begin{aligned} \rho &= r\sin\theta \\ \phi &= \phi \\ z &= r\cos\theta \end{aligned}} \quad \text{--- (1.31)}$$

Distance between 2 points

The distance (d) between 2 points (x_2, y_2, z_2) and (x_1, y_1, z_1) or distance (d) between 2 points having position vectors (r_1) & (r_2) is given by

$$d = |r_2 - r_1| \quad \text{--- (1.32)}$$

In Cartesian

$$d^2 = (x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2 \quad \text{--- (1.33)}$$

For finding the distance between 2 points in cylindrical & spherical co-ordinates substitute the Cartesian with corresponding cylindrical & spherical co-ordinates & simplify.

For example (spherical from eq^n 1.22) $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

$x_2 \to r_2\sin\theta_2\cos\phi_2$ | $x_1 \to r_1\sin\theta_1\cos\phi_1$

$y_2 \to r_2\sin\theta_2\sin\phi_2$ | $y_1 \to r_1\sin\theta_1\sin\phi_1$

$z_2 = r_2\cos\theta_2$ | $z_1 \to r_1\cos\theta_1$

Substitute these in eq^n (1.33) & simplify then you will get

$$\boxed{d^2 = r_2^2 + r_1^2 - 2r_1r_2\cos\theta_2\cos\theta_1 - 2r_1r_2\sin\theta_2\sin\theta_1\cos(\phi_2-\phi_1)} \quad (1.34)$$

Similarly in (cylindrical) $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$

$$\boxed{d^2 = \rho_2^2 + \rho_1^2 - 2\rho_1\rho_2\cos(\phi_2-\phi_1) + (z_2-z_1)^2} \quad (1.35)$$

Problems

1) Given point $P(-2,6,3)$ and vector $A = y a_x + (x+z) a_y$, express P and A in cylindrical and spherical coordinates. Evaluate A at P in the Cartesian, cylindrical, and spherical systems.

Ans:- At point P: $x=-2,\ y=6,\ z=3$

Hence,

$$\rho = \sqrt{x^2+y^2} = \sqrt{4+36} = 6.32$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{6}{-2}\right) = 108.43^\circ$$

$$z = 3$$

(Ans)

$$r = \sqrt{x^2+y^2+z^2} = \sqrt{4+36+9} = \sqrt{49} = 7$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right) = \tan^{-1}\left(\frac{\sqrt{40}}{3}\right) = 64.62^\circ$$

$$\phi = 108.43^\circ \text{ (As derived)}$$

(Ans)

Thus,

$$P = (-2, 6, 3) = P(6.32, 108.43^\circ, 3) = P(7, 64.62^\circ, 108.43^\circ)$$

In the Cartesian system, A at P is

$$A = y a_x + (x+z) a_y$$

$$\Rightarrow A = 6 a_x + (-2+3) a_y$$

$$\Rightarrow \boxed{A = 6 a_x + a_y}$$

(Ans)

For vector A, $A_x = 6$, $A_y = 1$, $A_z = 0$. Hence,

on the cylindrical system

From eqn (1.13)

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} 6\cos\phi + \sin\phi \\ -6\sin\phi + \cos\phi \\ 0 \end{bmatrix}$$

$$\Rightarrow A_\rho = 6\cos\phi + \sin\phi$$

$$A_\phi = -6\sin\phi + \cos\phi$$

$$A_z = 0$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{6}{-2}\right)$$

$$\Rightarrow \tan\phi = \frac{6}{-2} = \left(\frac{p}{b}\right)$$

$$\sin\phi = \frac{6}{\sqrt{6^2+2^2}} = \frac{6}{\sqrt{40}}$$

$$\cos\phi = \frac{-2}{\sqrt{6^2+2^2}} = \frac{-2}{\sqrt{40}}$$

$$\sin\phi = \frac{p}{h}$$

$$\cos\phi = \frac{b}{h}$$

$\because A_\rho = 6\cos\phi + \sin\phi$

$$= 6\times\left(\frac{-2}{\sqrt{40}}\right) + \frac{6}{\sqrt{40}}$$

$$= \frac{-12}{\sqrt{40}} + \frac{6}{\sqrt{40}}$$

$$\boxed{A_\rho = -\frac{6}{\sqrt{40}} = -0.9487}$$

$A_\phi = -6\sin\phi + \cos\phi$

$$= -6\times\frac{6}{\sqrt{40}} + \frac{-2}{\sqrt{40}}$$

$$\boxed{A_\phi = \frac{-38}{\sqrt{40}} = -6.008}$$

$A = A_\rho a_\rho + A_\phi a_\phi + A_z a_z$

$\Rightarrow A = (-0.9487)a_\rho + (-6.008)a_\phi + 0$

$\Rightarrow \boxed{A = -0.9487 a_\rho - 6.008 a_\phi}$ (Ans)

We know,

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow A_r = 6\sin\theta\cos\phi + \sin\theta\sin\phi$$

$$A_\theta = 6\cos\theta\cos\phi + \cos\theta\sin\phi$$

$$A_\phi = -6\sin\phi + \cos\phi$$

$$\tan\theta = \frac{\sqrt{x^2+y^2}}{z} = \frac{\sqrt{(-2)^2+6^2}}{z} = \frac{\sqrt{40}}{3} \quad \left(\frac{p}{b}\right)$$

$$\tan\phi = \frac{y}{x} = \frac{6}{-2}, \quad \sin\phi = \frac{6}{\sqrt{40}}, \quad \cos\phi = \frac{-2}{\sqrt{40}}$$

$$\sin\theta = \frac{\sqrt{40}}{\sqrt{(\sqrt{40})^2+3^2}} = \frac{\sqrt{40}}{\sqrt{40+9}} = \frac{\sqrt{40}}{7} \quad \left(\frac{p}{h}\right)$$

$$\cos\theta = \frac{3}{7} \quad \left(\frac{b}{h}\right)$$

Substituting

$$A_r = 6\times\frac{\cancel{\sqrt{40}}}{7}\times\frac{-2}{\cancel{\sqrt{40}}} + \frac{\cancel{\sqrt{40}}}{7}\times\frac{6}{\cancel{\sqrt{40}}} = \frac{-6}{7}$$

$$A_\theta = 6\times\frac{3}{7}\times\frac{-2}{\sqrt{40}} + \frac{3}{7}\times\frac{6}{\sqrt{40}} = \frac{-18}{7\sqrt{40}}$$

$$A_\phi = -6\times\frac{6}{\sqrt{40}} + \frac{-2}{\sqrt{40}} = \frac{-38}{\sqrt{40}}$$

$$A = \frac{-6}{7}a_r \; \frac{-18}{7\sqrt{40}}a_\theta \; \frac{-38}{\sqrt{40}}a_\phi = -0.857\,a_r - 0.4066\,a_\theta - 6.008\,a_\phi \quad \text{(Ans)}$$

Thus,

$A = 6a_x + a_y$

$= -0.9487 a_\rho - 6.008 a_\phi$

$= -0.8571 a_r - 0.4066 a_\theta - 6.008 a_\phi$

Note $|A|$ is the same in the three systems i.e.

$|A(x,y,z)| = |A(\rho,\phi,z)| = |A(r,\theta,\phi)| = \sqrt{A_x^2 + A_y^2 + A_z^2} = \sqrt{A_\rho^2 + A_\phi^2 + A_z^2}$

$= \sqrt{A_r^2 + A_\theta^2 + A_\phi^2} = 6.083$

2) Express the vector

$$B = \frac{10}{r} a_r + r\cos\theta\, a_\theta + a_\phi$$

in Cartesian & cylindrical coordinates.

Find $B(-3,4,0)$ & $B\left(5, \frac{\pi}{2}, -2\right)$

Assignment :- 5 Prove that (Transformation of Spherical → cylindrical)

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}$$

Ans :- <u>Given</u>

$$B = \frac{10}{r} a_r + r\cos\theta\, a_\theta + a_\phi$$

$\because B_r = \frac{10}{r}, \quad B_\theta = r\cos\theta$

$B_\phi = 1$

<u>From eqn</u> (1.28)

$$\begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} B_r \\ B_\theta \\ B_\phi \end{bmatrix}$$

$$= \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} \frac{10}{r} \\ r\cos\theta \\ 1 \end{bmatrix}$$

$$\Rightarrow B_x = \frac{10}{r}\sin\theta\cos\phi + r\cos^2\theta\cos\phi - \sin\phi$$

$$B_y = \frac{10}{r}\sin\theta\sin\phi + r\cos^2\theta\sin\phi + \cos\phi$$

$$B_z = \frac{10}{r}\cos\theta - r\cos\theta\sin\theta$$

But, $r = \sqrt{x^2+y^2+z^2}, \quad \theta = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right),$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right)$$

<u>Hence</u>, $\sin\theta = \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}, \quad \cos\theta = \frac{z}{\sqrt{x^2+y^2+z^2}}$

$\left(\because \tan\theta = \frac{p}{b}, \ \sin\theta = \frac{p}{h}, \ \cos\theta = \frac{b}{h}\right)$

$$\sin\phi = \frac{p}{h} = \frac{y}{\sqrt{x^2+y^2}}$$

$$\cos\phi = \frac{b}{h} = \frac{x}{\sqrt{x^2+y^2}}$$

Substituting all these values

$$B_x = \frac{10}{r}\sin\theta\cos\phi + r\cos^2\theta\cos\phi - \sin\phi$$

$$= \frac{10}{\sqrt{x^2+y^2+z^2}} \times \frac{\cancel{\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2+z^2}} \times \frac{x}{\cancel{\sqrt{x^2+y^2}}}$$

$$+ \cancel{\sqrt{x^2+y^2+z^2}} \times \frac{z^2}{(x^2+y^2+z^2)\cancel{\sqrt{x^2+y^2+z^2}}} \times \frac{x}{\sqrt{x^2+y^2}}$$

$$- \frac{y}{\sqrt{x^2+y^2}}$$

$$\Rightarrow B_x = \frac{10x}{x^2+y^2+z^2} + \frac{xz^2}{\sqrt{(x^2+y^2)}(x^2+y^2+z^2)} - \frac{y}{\sqrt{x^2+y^2}}$$

$$B_y = \frac{10}{r}\sin\theta\sin\phi + r\cos^2\theta\sin\phi + \cos\phi$$

$$= \frac{10}{\sqrt{x^2+y^2+z^2}} \times \frac{\cancel{\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2+z^2}} \times \frac{y}{\cancel{\sqrt{x^2+y^2}}} + \sqrt{x^2+y^2+z^2} \times \frac{z^2}{x^2+y^2+z^2} \times \frac{y}{\sqrt{x^2+y^2}}$$

$$+ \frac{x}{\sqrt{x^2+y^2}}$$

$$\Rightarrow B_y = \frac{10y}{x^2+y^2+z^2} + \frac{yz^2}{\sqrt{(x^2+y^2)(x^2+y^2+z^2)}} + \frac{x}{\sqrt{x^2+y^2}}$$

$$B_z = \frac{10}{r}\cos\theta - r\cos\theta\sin\theta$$

$$= \frac{10}{\sqrt{x^2+y^2+z^2}} \times \frac{z}{\sqrt{x^2+y^2+z^2}} - \sqrt{x^2+y^2+z^2} \times \frac{z}{\sqrt{x^2+y^2+z^2}} \times \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}$$

$$\Rightarrow B_z = \frac{10z}{x^2+y^2+z^2} - \frac{z\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}$$

$$B = B_x a_x + B_y a_y + B_z a_z$$

At $P(-3,4,0)$

$$B_x = \frac{10\times(-3)}{25} + \frac{(-3)\times 0}{(\ )} - \frac{4}{5} = \frac{-30}{25} - \frac{4}{5}$$

$$\Rightarrow B_x = \frac{-30-20}{25} = \frac{-50}{25} = -2$$

$$B_y = \frac{10\times 4}{25} + \frac{4\times 0}{(\ )} + \frac{-3}{5} = \frac{+40}{25} - \frac{3}{5} = \frac{25}{25} = 1$$

$$B_z = \frac{10\times 0}{(\ )} - \frac{0}{(\ )} = 0$$

Thus,

$$B = B_x a_x + B_y a_y$$

$$\Rightarrow \boxed{B = -2 a_x + a_y}$$ (Ans)

For Spherical to Cylindrical vector transformation

< See Assignment → >

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}$$

Thus,

$$\begin{bmatrix} B_\rho \\ B_\phi \\ B_z \end{bmatrix} = \begin{bmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} \frac{10}{r} \\ r\cos\theta \\ 1 \end{bmatrix}$$

$$\Rightarrow B_\rho = \frac{10}{r}\sin\theta + r\cos^2\theta$$

$$B_\phi = 1$$

$$B_z = \frac{10}{r}\cos\theta - r\cos\theta\sin\theta$$

But

$$r = \sqrt{x^2+y^2+z^2} = \sqrt{\rho^2+z^2}$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right) = \tan^{-1}\left(\frac{\rho}{z}\right)$$

$$\phi = \phi$$

$\therefore$ ~~sin θ =~~ $\tan\theta = \frac{\rho}{z}$

$$\sin\theta = \frac{\rho}{\sqrt{\rho^2+z^2}}, \quad \cos\theta = \frac{z}{\sqrt{\rho^2+z^2}}$$

$$\therefore B_\rho = \frac{10}{\sqrt{\rho^2+z^2}} \times \frac{\rho}{\sqrt{\rho^2+z^2}} + \sqrt{\rho^2+z^2} \times \frac{z^2}{\rho^2+z^2}$$

$$\Rightarrow B_\rho = \frac{10\rho}{\rho^2+z^2} + \frac{z^2}{\sqrt{\rho^2+z^2}}$$

$$B_\phi = 1$$

$$B_z = \frac{10}{\sqrt{\rho^2+z^2}} \times \frac{z}{\sqrt{\rho^2+z^2}} - \sqrt{\rho^2+z^2} \times \frac{z}{\sqrt{\rho^2+z^2}} \times \frac{\rho}{\sqrt{\rho^2+z^2}}$$

$$\Rightarrow B_z = \frac{10z}{(\rho^2+z^2)} - \frac{\rho z}{\sqrt{\rho^2+z^2}}$$

At $P(5, \frac{\pi}{2}, -2)$, $\rightarrow$ $\rho = 5$, $\phi = \frac{\pi}{2}$, $z = -2$

$$B_\rho = \frac{10\times 5}{25+4} + \frac{4}{\sqrt{25+4}} = \frac{50}{29} + \frac{4}{\sqrt{29}} = 2.467$$

$B_\phi = 1$

$$B_z = \frac{10 \times (-2)}{29} - \frac{5(-2)}{\sqrt{29}}$$

$$= \frac{-20}{29} + \frac{10}{\sqrt{29}}$$

$$= 1.167$$

$$\therefore \boxed{B = 2.467\, a_\rho + a_\phi + 1.167\, a_z} \quad \text{(Ans)}$$

Note that At (5, $\frac{\pi}{2}$, −2)

$$|B(x,y,z)| = |B(\rho,\phi,z)| = |B(r,\theta,\phi)| = \underline{2.907}$$

< To verify Answer is Correct >

$$< \because \sqrt{2.467^2 + 1 + 1.167^2} = 2.907$$

At P (−3, 4, 0), $B = -2a_x + a_y$

$$|B(x,y,z)| = |B(\rho,\phi,z)| = |B(r,\theta,\phi)|$$

$$= \sqrt{2^2 + 1} = \sqrt{4+1} = \underline{2.236}$$

— 0 —

Chapter-2: Vector Calculus

This Chapter deals with Vector Calculus - integration and differentiation of vectors.

Differential length, area and volume:-

Differential elements in length, area & volume are useful in vector Calculus. They are defined in the Cartesian, cylindrical, and Spherical Coordinate systems.

A. Cartesian Coordinate Systems

From Figure 2.1, we notice that the differential displacement at 'dl' is the vector from Point $S(x, y, z)$ to Point $B(x+dx, y+dy, z+dz)$.

1. Differential displacement is given by

$$\boxed{dl = dx\, a_x + dy\, a_y + dz\, a_z} \quad (2.1)$$

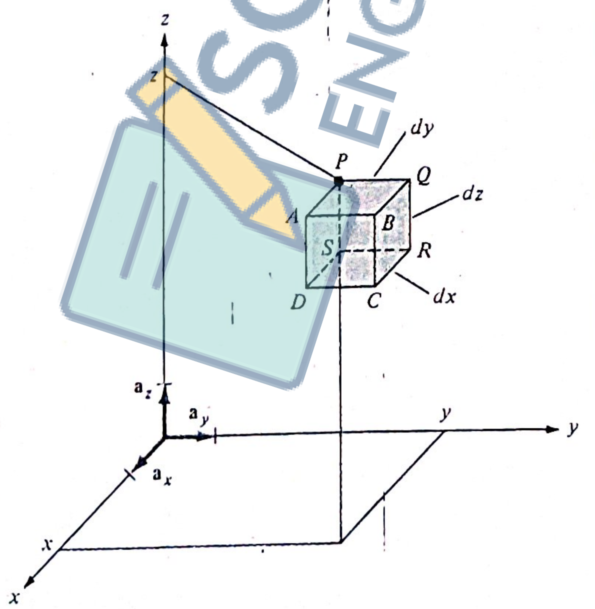


Fig 2.1 - Differential elements in the right-handed Cartesian Coordinate system.

2. Differential normal surface area is given by and illustrated in Fig 2.2

$$\left.\begin{aligned} ds &= dy\,dz\,a_x \\ \text{or} \quad ds &= dx\,dz\,a_y \\ \text{or} \quad ds &= dx\,dy\,a_z \end{aligned}\right\} \quad \text{—} \; (2.2)$$

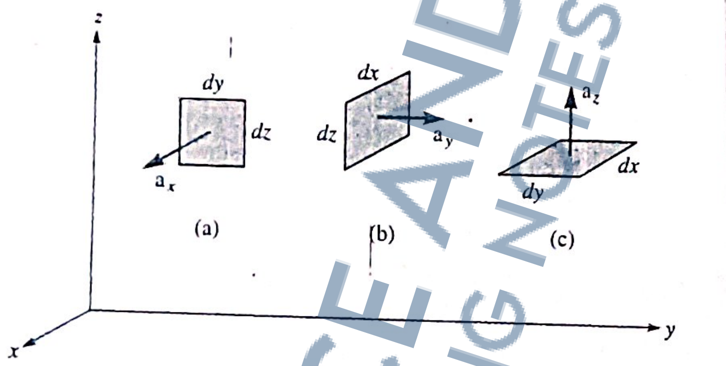


Figure 2.2 Differential normal surface areas in Cartesian coordinates: (a) $dS = dy\,dz\,a_x$, (b) $dS = dx\,dz\,a_y$, (c) $dS = dx\,dy\,a_z$.

3. Differential volume is given by

$$dv = dx\,dy\,dz \quad \text{———} \; (2.3)$$

Note :- 1) dl and ds are vectors but dv is a scalar.

2) From Fig 2.1, observe that when we move from P to Q, $\boxed{dl = dy\,a_y}$, because we are moving in the Y-direction by dy amount.

3) If we move from S to Q,

$$\boxed{dl = dy\,a_y + dz\,a_z}$$

4) Similarly, If we move from D to Q

$dl = dx\, a_x + dy\, a_y + dz\, a_z$

5) The differential surface (area) element ds may generally be defined as

$$\boxed{ds = ds\, a_n} \qquad (2.4)$$

where ds is the area of the surface element and a_n is a unit vector normal to the surface ds (and directed away from the volume of ds is part of the surface describing the volume)

Ex:- (1) In Figure 2.1, for the surface ABCD,

$ds = dy\, dz\, a_x$, where as

for the surface PQRS,

$ds = -dy\, dz\, a_x$ because

$a_n = -a_x$ is normal to PQRS.

6) Note that → when 'dl' is remembered 'ds' and 'dv' can be easily found out. E.g. ds along a_x can be obtained from 'dl' in eqn (2.1) by multiplying the components of dl along a_y and a_z; that is $dy\, dz\, a_x$. Similarly ds along a_z is the product of component along a_x and a_y; that $dx\, dy\, a_z$. Also dv can be obtained from dl as the product

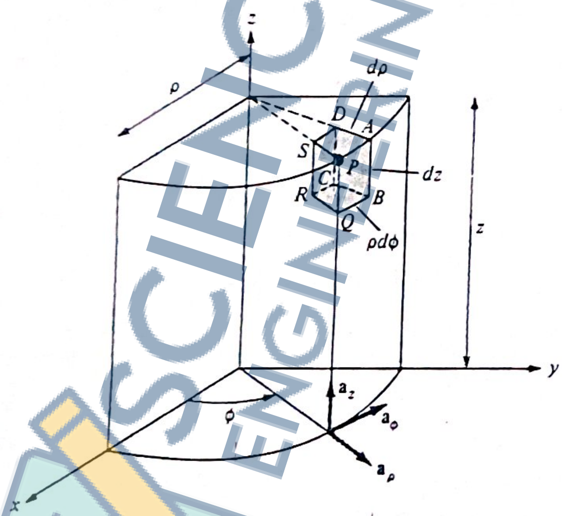
of the three components of dl, that
is $dx\,dy\,dz$. The idea developed here for Cartesian coordinates will now be extended to other coordinate systems.

B. Cylindrical Coordinate Systems

From figure 2.3, the differential elements in cylindrical coordinates can be found as follows

1. Differential displacement is given by

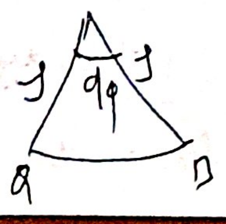
$$dl = d\rho\, a_\rho + \rho\, d\phi\, a_\phi + dz\, a_z \qquad (2.5)$$



(Figure 2.3) Differential elements in cylindrical coordinates

Suppose, we want to move from 'R' to A, differential displacement along 'ρ' is $\boxed{d\rho}$ (RQ)

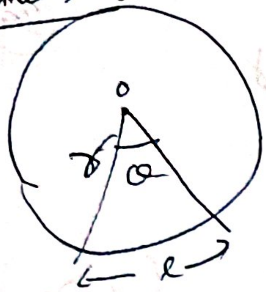
along 'φ' is $\rho\, d\phi$ because on a arc



$$d\phi = \frac{BQ}{\rho}$$

Remember:- $\theta = \frac{l}{r}$ in arc of circle

Remember:-



$$\theta = \frac{l}{r}$$

$\cancel{\Rightarrow d\phi =}$

$$\Rightarrow BQ = \rho\, d\phi$$

∴ differential displacement along 'φ' direction is

$$\boxed{\rho\, d\phi}$$

And from B point, we have to move to A. Differential displacement along 'z' direction is dz

∴ Total differential displacement

$$dl = d\rho\, a_\rho + \rho\, d\phi\, a_\phi + dz\, a_z$$

2. Differential normal surface is given by

From Fig. 2.4:-

$$\left.\begin{aligned} dS &= \rho\, d\phi\, dz\, a_\rho \\ \text{or} &= d\rho\, dz\, a_\phi \\ \text{or} &= \rho\, d\rho\, d\phi\, a_z \end{aligned}\right\} \text{— (2.6)}$$

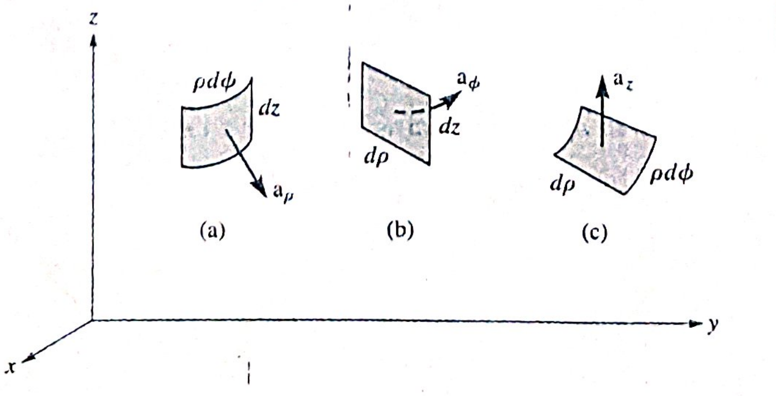


Figure 2.4 Differential normal surface areas in cylindrical coordinates: **(a)** $dS = \rho\, d\phi\, dz\, \mathbf{a}_\rho$, **(b)** $dS = d\rho\, dz\, \mathbf{a}_\phi$, **(c)** $dS = \rho\, d\rho\, d\phi\, \mathbf{a}_z$.

3. Differential Volume is given by

$$dv = d\rho\,(\rho\, d\phi)\,.\,dz$$

$$\Rightarrow \boxed{dv = \rho\ d\rho\, d\phi\, dz} \quad \text{—— (2.7)}$$

C. Spherical Coordinate System

From figure 2.5, the differential elements in spherical coordinates can be found as follows.

1. The differential displacement

$$dl = \underline{dr}\, a_r + \underline{r\,d\theta}\, a_\theta + \underline{r\sin\theta\, d\phi}\, a_\phi \quad \text{—— (2.8)}$$

From fig 2.5,

Along 'r' differential displacement → $\underline{dr}$.

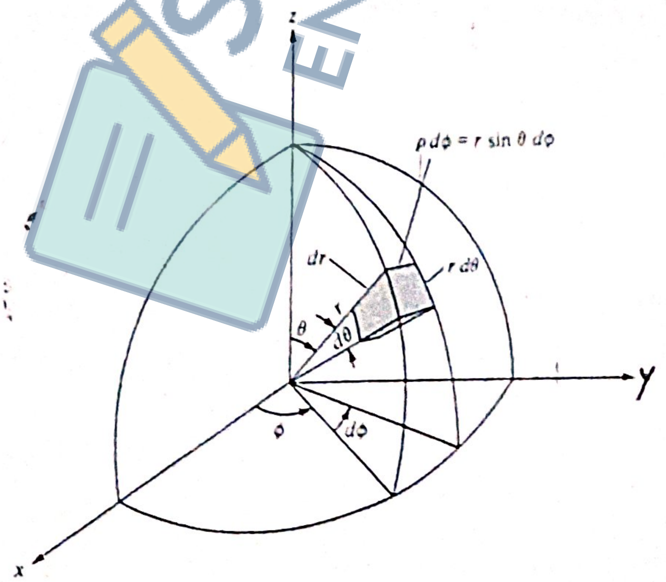


Fig: 2.5 Differential elements in the spherical Coordinate system

Along 'θ', differential displacement

$\underline{r\,d\theta}$

$\left(\because \quad \theta = \frac{l}{r}, \Rightarrow d\theta = \frac{l}{r} \Rightarrow l = r\,d\theta \right)$

Along ϕ, differential displacement,

$\rho\, d\phi$ as discussed earlier

But $\rho = r \sin\theta$ in spherical coordinates.

ϕ is differential displacement along ϕ is

$\underline{\rho}\, d\phi$ i.e $\underline{r \sin\theta\, d\phi}$.

$\therefore$ Combining all,

$$\boxed{dl = dr\, a_r + r\,d\theta\, a_\theta + r\sin\theta\, d\phi\, a_\phi}$$

2. **<u>Differential normal surface area</u>**

Applying the same concept, the differential normal surface can be obtained by multiplying the components of dl, we have

$$ds = (r\,d\theta).(r\sin\theta\,d\phi)\,a_r = r^2 \sin\theta\, d\theta\, d\phi\, a_r$$

or $$ds = (dr).(r\sin\theta\, d\phi)\,a_\theta = r\sin\theta\, dr\, d\phi\, a_\theta$$

$\underline{\text{or}}$ $$ds = (dr).(r\,d\theta)\,a_\phi = r\,dr\,d\theta\,a_\phi$$

Summarizing,

$$\left.\begin{aligned} ds &= r^2 \sin\theta \, d\theta \, d\phi \, a_r \\ &= r \sin\theta \, dr \, d\phi \, a_\theta \\ &= r \, dr \, d\theta \, a_\phi \end{aligned}\right\} \quad \text{—(2.9)}$$

Refer fig 2.6

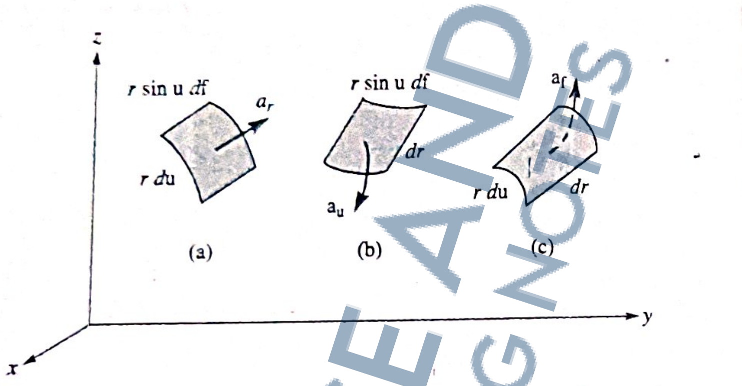


Figure 2.6 Differential normal surface areas in spherical coordinates: (a) $dS = r^2 \sin\theta \, d\theta \, d\phi \, \mathbf{a}_r$, (b) $dS = r \sin\theta \, dr \, d\phi \, \mathbf{a}_\phi$, (c) $dS = r \, dr \, d\theta \, \mathbf{a}_\phi$.

3. The differential volume is

$$dv = (dr).(r\,d\theta).(r\sin\theta\,d\phi)$$

$$\Rightarrow \boxed{dv = r^2 \sin\theta \, dr \, d\theta \, d\phi} \quad \text{—(2.10)}$$

Problem:- 2.1) Consider the object shown in fig 2.7. Calculate

(a) The length BC

(b) The length CD

(c) The surface area ABCD

(d) The surface area ABO

(e) The surface area AOFD

(f) The volume ABDCFO

Answer :-

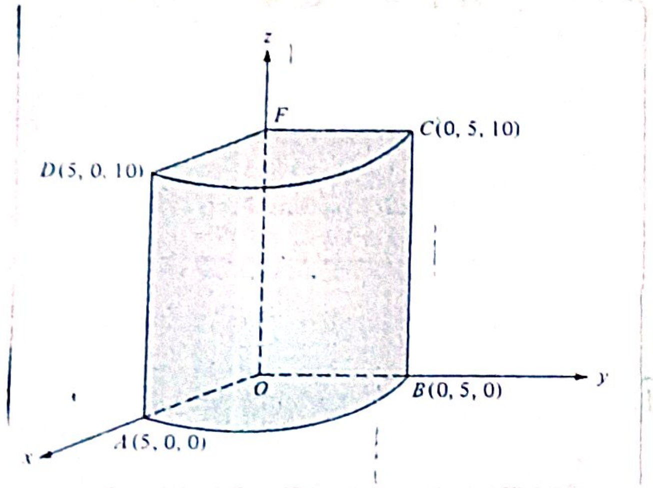


Fig 2.7 For example 2.1

Although the point A, B, C, D are given in Cartesian coordinates, it is obvious that the object has cylindrical symmetry. Hence, we solve the problem in cylindrical coordinates. The points are transformed from Cartesian to cylindrical coordinates as follows.

Cartesian		Cylindrical
(x, y, z)	$\rightarrow$	(ρ, ϕ, z)
(x, y, z)	$\rightarrow$	$\left(\rho = \sqrt{x^2+y^2},\ \phi = \tan^{-1}\left(\frac{y}{x}\right),\ z\right)$
$A(5, 0, 0)$	$\rightarrow$	$(5, 0^\circ, 0)$
$B(0, 5, 0)$	$\rightarrow$	$(5, \frac{\pi}{2}, 0$
$C(0, 5, 10)$	$\rightarrow$	$(5, \frac{\pi}{2}, 10)$
$D(5, 0, 10)$	$\rightarrow$	$(5, 0^\circ, 10)$

(a) Along BC, $dl = dz$;

Hence

$$BC = \int dl = \int_0^{10} dz = 10 \quad \text{(Ans)}$$

(b) Along CD, $dl = \rho\, d\phi$ and $\rho = 5$

$$\left(\because \rho = \sqrt{x^2+y^2} = \sqrt{5^2+0^2} = 5 \right)$$

$$\therefore CD = \int_0^{\pi/2} \rho\, d\phi = \int_0^{\pi/2} 5\, d\phi = 5 \times \phi \Big|_0^{\frac{\pi}{2}}$$

$$CD = 5 \times \frac{\pi}{2} = 2.5\pi \quad \text{(Ans)}$$

(c) The surface area of ABCD

Here for ABCD surface area the normal vector is $\underline{a_\rho}$. So the differential normal surface area

$$dS = \rho\, d\phi\, dz\, a_\rho .$$

$$\therefore \text{Area ABCD} = \int dS = \iint \rho\, d\phi\, dz$$

$$= \int_{\phi=0}^{\pi/2} \int_{z=0}^{10} 5 \cdot d\phi\, dz$$

$$= 5 \times \frac{\pi}{2} \times 10 = 25\pi$$

(Ans)

(d) The surface area ABO

For ABO surface, normal is a_z, So

$$ds = \rho\, d\rho\, d\phi\, a_z$$

$$S = \iint \rho\, d\rho\, d\phi$$

$$= \int_{\phi=0}^{\pi/2} d\phi \int_0^5 \rho\, d\rho$$

$$= \left(\frac{\pi}{2}\right)\left[\frac{\rho^2}{2}\right]_0^5$$

$$= \frac{\pi}{2} \times \left(\frac{25}{2}\right)$$

$$= 6.25\pi \text{ (Ans)}$$

(e) The surface area AOFD

For AOFD surface, ϕ is a constant, $\phi = 0°$.

Variation along ρ & z.

$$\therefore\ ds = d\rho\, dz\, a_\phi$$

$$S = \int_{\rho=0}^{5} \int_{z=0}^{10} d\rho\, dz = 5 \times 10 = 50$$

(f) The volume ABCDFO

$$dv = \rho \, d\rho \, d\phi \, dz$$

$$V = \int_{\rho=0}^{5} \int_{\phi=0}^{\pi/2} \int_{z=0}^{10} \rho \, d\rho \, d\phi \, dz$$

$$= \int_{z=0}^{10} dz \int_{\phi=0}^{\pi/2} d\phi \int_{0}^{5} \rho \, d\rho$$

$$= 10 \times \frac{\pi}{2} \times \left.\frac{\rho^2}{2}\right]_0^5$$

$$= \cancel{10}^{5} \times \frac{\pi}{\cancel{2}} \times \frac{25}{2}$$

$$= 62.5\pi \quad \text{(Ans)}$$

Line, Surface, and Volume integrals

The familiar concept of integration will now be extended to cases in which the integrand involves a vector. By "line" we mean the path along a curve in space. We shall use terms such as line, curve, and contour interchangeably.

Defn :- The line integral $\boxed{\int_L A \cdot dl}$

is the integral of the tangential component of A along curve L.

Given a vector field 'A' and a curve 'L' we define the integral

$$\int_L A \cdot dl = \int_a^b |A| \cos\theta \, dl \quad \text{——} (2.11)$$

as the line integral of 'A' around 'L' (see fig 2.9).

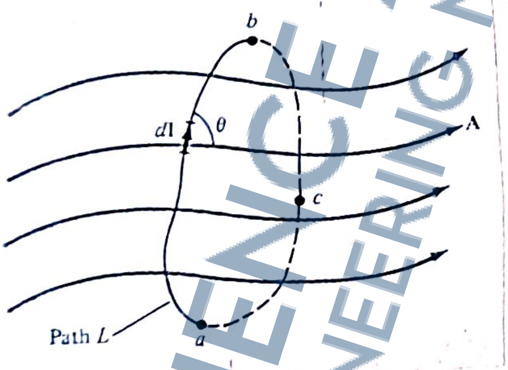


Fig 2.9 Path of Integration of vector field 'A'

→ If the path of integration is a closed curve such as abca in figure 2.9, eq^n (2.11) becomes a closed contour integral

$$\oint_L A \cdot dl \quad \text{——} (2.12)$$

Which is called the circulation of 'A' around 'L'.

Surface Integral

Given a vector field A, continuous on a region containing the smooth surface S, we define the surface integral or the flux of A through S (fig 2.10) as

$$\Psi = \int_S |A| \cos\theta \, ds = \int_S A \cdot a_n \, ds$$

or simply

$$\boxed{\Psi = \int_S A \cdot ds} \qquad (2.13)$$

Fig. 2.10: The flux of a vector field 'A' through surface S.

Where, at any point on S, $\hat{a}_n$ is the unit normal to S. For a closed surface (defining a volume), eq^n (2.13) becomes

$$\Psi = \oint_S A \cdot ds \qquad (2.14)$$

which is referred to as the net outward flux of A from S.

Volume integral

We define the integral

$$\boxed{\int_V \rho_v \, dv} \qquad (2.15)$$

as the volume integral of the scalar ρ_v over the volume V.

Example 2.2) Given that $F = x^2 a_x - xz a_y - y^2 a_z$, Calculate the circulation of F around the (closed) path shown in Figure 2.11.

Ans:

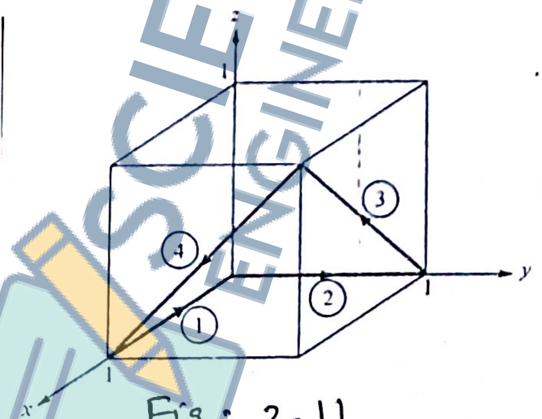


Fig: 2.11

The circulation of F around path 'L' is given by

$$\oint_L F \cdot dl = \left(\int_{①} + \int_{②} + \int_{③} + \int_{④} \right) F \cdot dl$$

where the path is broken into segments numbered 1 to 4 as shown in Figure 2.11.

For segment ①, $y = 0 = z$

$$F = \underline{x^2 a_x} - xz\, a_y - y^2 a_z \quad , \quad dl = \underline{dx\, a_x}$$

$$F \cdot dl = \int_1^0 x^2\, dx$$

1 → 0 (limit)

$$= \frac{x^3}{3}\Big]_1^0$$

$$= \frac{1}{3}(0-1)$$

$$= -\frac{1}{3}$$

dl is always taken as along $+a_x$ so that the direction on segment ① is taken care of by the limits of integration

For Segment ②, $x = 0 = z$,

$$F = x^2 a_x - xz\, a_y - y^2 a_z \quad , \quad dl = dy\, a_y$$

$$F \cdot dl = -xz \quad \& \quad x = 0 = z$$

$$\therefore \quad F \cdot dl = 0$$

For segment ③ $y = 1$,

$$F = x^2 a_x - xz\, a_y - y^2 a_z = x^2 a_x - xz\, a_y - a_z \Big|_{y=1}$$

$$dl = dx\, a_x + dz\, a_z$$

(Since no variation along y, it is constant)

So,

$$F \cdot dl = \int (x^2\, dx - dz)$$

But on ③, $z = x$ ($\because$ The line is on XZ Plane plane and $x = z$)

$\Rightarrow dz = dx$

$\therefore \quad F \cdot dl = \int x^2 dx - dx$

$= \int_0^1 (x^2 - 1) dx$

$= \left.\frac{x^3}{3} - x\right]_0^1$

$= \left(\frac{1}{3} - 1\right) = \frac{1-3}{3} = -\frac{2}{3}$

For Segment ④

$x = 1, \quad F = x^2 a_x - xz a_y - y^2 a_z$

$\Rightarrow F = a_x - z a_y - y^2 a_z$ | for $x = 1$

($\because x$ is const.)

$dl = dy\, a_y + dz\, a_z$

$F \cdot dl = -z\,dy - y^2 dz$

But on ④ $z = y \Rightarrow dz = dy$

$\therefore \quad F \cdot dl = \cancel{z} = -y\,dy - y^2 dy$

$\Rightarrow F \cdot dl = \int_1^0 (-y - y^2) dy$

$= -\left.\frac{y^2}{2} - \frac{y^3}{3}\right]_1^0 = (0) - \left(-\frac{1}{2} - \frac{1}{3}\right) = \frac{5}{6}$

By putting all these together, we get

$$\oint_L F \cdot dl = -\frac{1}{3} + 0 - \frac{2}{3} + \frac{5}{6}$$

$$= \frac{-2-4+5}{6}$$

$$= \frac{-1}{6} \quad \text{(Ans)}$$

Del Operator

The del operator, written ∇, is the vector differential operator. In Cartesian coordinates,

$$\nabla = \frac{\partial}{\partial x} a_x + \frac{\partial}{\partial y} a_y + \frac{\partial}{\partial z} a_z \quad \text{—— (2.16)}$$

This vector differential operator, otherwise known as the gradient operator, is not a vector in itself but when it operates on a scalar function, for example, a vector ensues. The operator is useful in defining
(results)

1. The gradient of a scalar V, written as ∇V
2. The divergence of a vector A, written as $\nabla \cdot A$
3. The curl of a vector A, written as $\nabla \times A$
4. The Laplacian of a scalar V, written as $\nabla^2 V$

Del operator in Cylindrical Co-ordinate

$$\nabla = a_\rho \frac{\partial}{\partial \rho} + a_\phi \frac{1}{\rho} \frac{\partial}{\partial \phi} + a_z \frac{\partial}{\partial z}$$

or

$$\boxed{\nabla = \frac{\partial}{\partial \rho} a_\rho + \frac{1}{\rho} \frac{\partial}{\partial \phi} a_\phi + \frac{\partial}{\partial z} a_z} \quad -(2.17)$$

Del operator in Spherical Coordinate

$$\boxed{\nabla = \frac{\partial}{\partial r} a_r + \frac{1}{r} \frac{\partial}{\partial \theta} a_\theta + \frac{1}{r\sin\theta} \frac{\partial}{\partial \phi} a_\phi} \quad -(2.18)$$

Gradient of Scalar (∇V)

The gradient of a scalar field V is a vector represents both the magnitude and the direction of maximum space rate of increase of V.

Proof:- A mathematical expression for the gradient can be obtained by evaluating the difference in the field dV between points P_1 and P_2 of figure 2.13, where V_1, V_2 and V_3 are contours on which V is constant. From Calculus,

$$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz$$

$$= \left(\frac{\partial V}{\partial x} a_x + \frac{\partial V}{\partial y} a_y + \frac{\partial V}{\partial z} a_z \right) \cdot \left(dx\, a_x + dy\, a_y + dz\, a_z \right)$$

For Convenience, Let

$$G = \frac{\partial V}{\partial x} a_x + \frac{\partial V}{\partial y} a_y + \frac{\partial V}{\partial z} a_z$$

Then

$$dV = G \cdot dl = G \cos\theta \, dl \quad \text{—— (2.18a)}$$

or

$$\frac{dV}{dl} = G \cos\theta \quad \text{—— (2.19)}$$

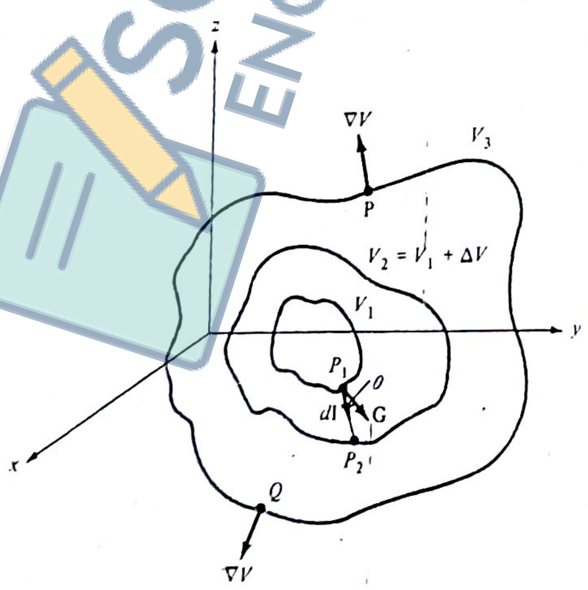


Fig. 2.13. Gradient of a Scalar

Where dl is the differential displacement from P_1 to P_2 and θ is the angle between G and dl. From eqn (2.19),

$\frac{dV}{dl}$ is maxm when $\theta = 0$, i.e. when dl is in the direction G. Hence

$$\left.\frac{dV}{dl}\right|_{max} = \frac{dV}{dn} = G$$

Where $\frac{dV}{dn}$ is the normal derivative. Thus G has its magnitude and direction as those of the maximum rate of change of V. (Proved).

By definition, G is the gradient of V.

Therefore,

$$\text{grad } V = \nabla V = \frac{\partial V}{\partial x} a_x + \frac{\partial V}{\partial y} a_y + \frac{\partial V}{\partial z} a_z$$

Thus gradient of 'V' can be expressed in Cartesian, cylindrical, & spherical coordinates as,

$$\nabla V = \frac{\partial V}{\partial x} a_x + \frac{\partial V}{\partial y} a_y + \frac{\partial V}{\partial z} a_z \qquad (2.20)$$

$$\nabla V = \frac{\partial V}{\partial \rho} a_\rho + \frac{1}{\rho} \frac{\partial V}{\partial \phi} a_\phi + \frac{\partial V}{\partial z} a_z \qquad (2.21)$$

$$\nabla V = \frac{\partial V}{\partial r} a_r + \frac{1}{r} \frac{\partial V}{\partial \theta} a_\theta + \frac{1}{r \sin\theta} \frac{\partial V}{\partial \phi} a_\phi \qquad (2.22)$$

Note:-

(i) $\nabla(V+U) = \nabla V + \nabla U$ — (2.23)

(ii) $\nabla(VU) = V\nabla U + U\nabla V$ — (2.24)

(iii) $\nabla\left[\frac{V}{U}\right] = \frac{U\nabla V - V\nabla U}{U^2}$ — (2.25)

(iv) $\nabla V^n = n V^{n-1}\nabla V$ — (2.26)

where U & V are scalars and n is an integer.

Fundamental Properties of Gradient of scalar field V

1. The magnitude of ∇V equals the maximum rate of change in V per unit distance.
2. ∇V points in the direction of the maximum rate of change in V.
3. ∇V at any point is perpendicular to the constant V surface that passes through that point. (see points P & Q in Figure 2.13)
4. The projection (or component) of ∇V in the direction of unit vector 'a' is $\nabla V \cdot a$ and is called the directional derivative of 'V' along 'a'. For example, $\frac{dV}{dl}$ in eqn (2.19) is the directional derivative of V

along P_1P_2 (Figure 2.13). Thus the gradient of a scalar function V provides us with both the direction in which V changes most rapidly and the magnitude of maximum directional derivative of V.

5. If $A = \nabla V$, V is said to be the scalar potential of A.

Ex: 2.3) Find the gradient of the following scalar fields.

(a) $V = e^{-z} \sin 2x \cosh y$

(b) $U = \rho^2 z \cos 2\phi$

(c) $W = 10 r \sin^2\theta \cos\phi$

Ans:- (a) $\nabla V = \frac{\partial V}{\partial x} a_x + \frac{\partial V}{\partial y} a_y + \frac{\partial V}{\partial y} a_z$

$\Rightarrow \nabla V = 2e^{-z} \cosh y \cos 2x \, a_x + e^{-z} \sin 2x \sinh y \, a_y$

$- \sin 2x \cosh y \, e^{-z}$ (Ans)

(b) $\nabla V = \frac{\partial U}{\partial \rho} a_\rho + \frac{1}{\rho} \frac{\partial u}{\partial \phi} a_\phi + \frac{\partial u}{\partial z} a_z$

$\Rightarrow \nabla U = z \cos 2\phi \cdot 2\rho \, a_\rho + \frac{1}{\cancel{\rho}} (-\cancel{\rho^2}^{\rho} z \sin 2\phi) \times (2) \, a_\phi$

$+ \rho^2 \cos 2\phi \quad a_z$

$\Rightarrow \nabla U = 2\rho z \cos 2\phi \, a_\rho - 2\rho z \sin 2\phi \, a_\phi$

$+ \rho^2 \cos 2\phi \, a_z$

(Ans)

(c) $\nabla W = \frac{\partial W}{\partial r} a_r + \frac{1}{r}\frac{\partial W}{\partial \theta} a_\theta + \frac{1}{r\sin\theta}\frac{\partial W}{\partial \phi} a_\phi$

$\Rightarrow \nabla W = \sin^2\theta \cdot \cos\phi \cdot (10)\, a_r + \frac{1}{\cancel{r}} \times (10\cancel{r}\cos\phi) \times (\sin 2\theta)\, a_\theta$

$+ \frac{1}{\cancel{r\sin\theta}} \times (10\cancel{r}\sin^2\theta) \times (-\sin\phi)\, a_\phi$

$\Rightarrow \nabla W = 10\sin^2\theta\cos\phi\, a_r + 10\sin 2\theta\cos\phi\, a_\theta$

$- 10\sin\theta\sin\phi\, a_\phi$

(Ans)

Ex:- 2.4) Given $W = x^2y^2 + xyz$, Compute ∇W and the directional derivative $\frac{dW}{dl}$ in the direction $3a_x + 4a_y + 12a_z$ at $(+2, -1, 0)$.

Ans:-

$\nabla W = \frac{\partial W}{\partial x} a_x + \frac{\partial W}{\partial y} a_y + \frac{\partial W}{\partial z} a_z$

$\nabla W = (y^2 \times (2x) + yz)\, a_x + (x^2 \times 2y + xz)\, a_y$

$+ (xy)\, a_z$

$\Rightarrow \nabla W = (2xy^2 + yz)\, a_x + (2x^2y + xz)\, a_y + (xy)\, a_z$

At $(+2, -1, 0)$, $\nabla W = (2 \cdot 2 \cdot 1 + 0)\, a_x + (2 \cdot 4(-1) + 0)\, a_y$

$+ (-2)\, a_z$

$\Rightarrow \nabla W = (4a_x - 8a_y - 2a_z)$

From eqn 2.18 (a)

$$\frac{dV}{dl} = G\cos\theta$$

$$\Rightarrow \frac{dV}{dl} = G \cdot a_l$$

$$\Rightarrow \frac{dV}{dl} = \nabla V \cdot a_l$$

$\because$ $G \cdot a_l = |G| \cdot |a_l| \cdot \cos\theta$
$= G \cdot 1 \cdot \cos\theta$
$= G\cos\theta$

Thus $\frac{dW}{dl} = \nabla W \cdot a_l$

$$= (4a_x - 8a_y - 2a_z) \cdot \frac{a_l}{|a_l|}$$

$$= (4a_x - 8a_y - 2a_z) \cdot \frac{(3a_x + 4a_y + 12a_z)}{\sqrt{3^2 + 4^2 + 12^2}}$$

$$= (4a_x - 8a_y - 2a_z) \cdot \frac{(3a_x + 4a_y + 12a_z)}{13}$$

$$= \frac{12 - 32 - 24}{13}$$

$$= \frac{-44}{13}$$

$$\therefore \left.\frac{dW}{dl}\right|_{(2,-1,0)} = -\frac{44}{13} \quad \text{(Ans)}$$

Divergence of a Vector and Divergence Theorem

From eqn (2.14), we have noticed that the net outflow of the flux of a vector field $\underline{A}$ from a closed surface 'S' is obtained from the integral $\oint A \cdot ds$.

The divergence of A at a given point P is the outward flux per unit volume as the volume shrinks about P.

Hence,

$$\boxed{\operatorname{div} A = \nabla \cdot A = \lim_{\Delta V \to 0} \frac{\oint A \cdot ds}{\Delta V}} \quad (2.22)$$

where ΔV is the volume enclosed by the closed surface S in which P is located.

→ Physically, we may regard the divergence of the vector field A at a given point as a measure of how much the field diverges or emanates from that point. Fig 2.15 (a) shows that the divergence of a vector field at point P is +ve because the vector diverges (or spreads out) at P. In figure 2.15 (b) a vector field has −ve divergence (or convergence) at P, and in Figure 2.15 (c) a vector field has zero divergence at P.

The divergence of a vector field can also be viewed as simply the limit of the field's source strength per unit volume (or source density); it is +ve at the source point in the field, and -ve at a sink point or zero where there is neither sink or source.

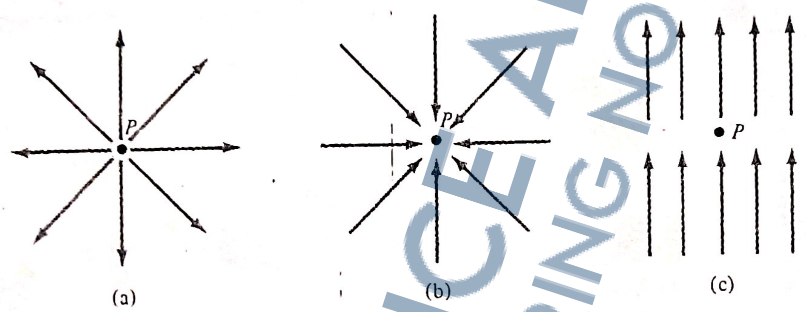


Figure 2.15 Illustration of the divergence of a vector field at P: (a) positive divergence, (b) negative divergence, (c) zero divergence.

Fig 2.15

→ Derivation of eqn (2.27) - Refer Book Sadiku

→ <Not required> <For knowledge you can refer>

Divergence in Coordinate Systems:-

Cartesian

Imp.
$$\boxed{\nabla \cdot A = \frac{\partial}{\partial x} A_x + \frac{\partial}{\partial y} A_y + \frac{\partial}{\partial z} A_z} \quad (2.28)$$

Cylindrical

Imp.
$$\boxed{\nabla \cdot A = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho A_\rho) + \frac{1}{\rho} \frac{\partial}{\partial \phi} A_\phi + \frac{\partial}{\partial z} A_z} \quad (2.29)$$

Spherical Coordinate

$$\boxed{\nabla \cdot A = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}(A_\phi)} \quad \text{— (2.30)}$$

Properties of divergence of a vector field

1. It produces a scalar field (Because dot product / scaler product is involved)

2. $\nabla \cdot (A+B) = \nabla \cdot A + \nabla \cdot B$

3. $\nabla \cdot (VA) = V \nabla \cdot A + A \nabla V$

Divergence Theorem :-

From eqn (2.27), it can be proved that

$$\boxed{\oint_S A \cdot ds = \int_V \nabla \cdot A \, dV} \quad \text{— (2.31)}$$

This is called divergence theorem, otherwise known as Gauss - Ostrogradsky theorem.

Thus, divergence thm states that, the total outward flux of a vector field A through the closed surface 'S' is same as the volume integral of the divergence of A.

Note:- The volume integral of eqn (2.31) is easier to evaluate than the surface integral. To determine the flux of A through a closed surface, we simply find the right-hand side of eqn (2.31) instead of left-hand side of the equation.

Example 2.5) Determine the divergence of these vector fields:

(a) $P = x^2yz\, a_x + xz\, a_z$

(b) $Q = \rho \sin\phi\, a_\rho + \rho^2 z\, a_\phi + z\cos\phi\, a_z$

(c) $T = \frac{1}{r^2}\cos\theta\, a_r + r\sin\theta\cos\phi\, a_\theta + \cos\theta\, a_\phi$

Answer:-

(a) $P = \underbrace{x^2yz}_{(P_x)}\, a_x + \underbrace{xz}_{(P_z)}\, a_z$

$$\nabla \cdot P = \frac{\partial}{\partial x} P_x + \frac{\partial}{\partial y} P_y + \frac{\partial}{\partial z} P_z$$

$$= yz(2x) + 0 + x$$

$$= 2xyz + x$$

(b) $$\nabla \cdot Q = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho Q_\rho) + \frac{1}{\rho}\frac{\partial}{\partial \phi}(Q_\phi) + \frac{\partial}{\partial z}(Q_z)$$

$$\Rightarrow \nabla \cdot Q = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho \cdot \rho \sin\phi) + \frac{1}{\rho}\frac{\partial}{\partial \phi}(\rho^2 z) + \frac{\partial}{\partial z}(z\cos\phi)$$

$$\Rightarrow \nabla \cdot Q = \frac{1}{\cancel{\rho}} \times \sin\phi \times 2\cancel{\rho} + \frac{1}{\rho} \times 0 + \cos\phi$$

$$\Rightarrow \nabla \cdot Q = 2\sin\phi + \cos\phi \quad \text{(Ans)}$$

(c) $$\nabla \cdot T = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(T_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}(T_\phi)$$

$$\Rightarrow \nabla \cdot T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\cancel{r^2} \times \frac{1}{\cancel{r^2}}\cos\theta\right) + \frac{1}{r\sin\theta}\left\{\frac{\partial}{\partial \theta}(r\sin^2\theta \cdot \cos\phi)\right\} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}(\cos\theta)$$

$$= 0 + \frac{1}{\cancel{r}\cancel{\sin\theta}} \times \cancel{r}\cos\phi \times 2\cancel{\sin\theta} \cdot \cos\theta + 0$$

$$\nabla \cdot T = 2\cos\theta\cos\phi \quad \text{(Ans)}$$

Example: 2.7

If $G(r) = 10e^{2z}(\rho a_\rho + a_z)$, determine the flux of G out of the entire surface of the cylinder $\rho = 1$, $0 \leq z \leq 1$. Confirm the result by using divergence theorem.

Ans:-

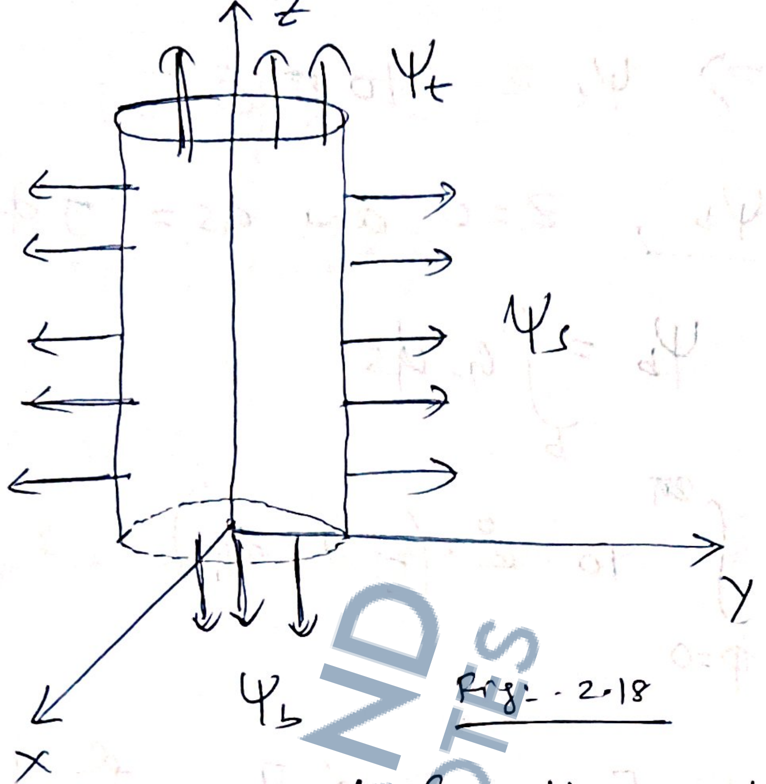


Fig. 2.18

If Ψ is the flux of G through the given surface, shown on figure 2.18, then

$$\Psi = \oint_S G.ds = \Psi_t + \Psi_b + \Psi_s$$ where

Ψ_t, Ψ_b and Ψ_s are the fluxes through top, bottom, and side (curved surface) of the cylinder as shown on figure 2.18.

For Ψ_t, $z=1$, $ds = \rho\, d\rho\, d\phi\, a_z$. (Since normal along z-direction)

Hence, $\Psi_t = \iint G.ds$

$$= \int_{\rho=0}^{1} \int_{\phi=0}^{2\pi} 10 . e^{-2z} (\rho a_\rho - \rho a_z) . (-\rho\, d\rho\, d\phi\, a_z)$$

$$= \int_{\rho=0}^{1} \int_{\phi=0}^{2\pi} \underline{10 . e^{-2}}\, \rho\, d\rho\, d\phi$$ ($\because z=1$ & dot product with a_z component only)

$$= 10 \times e^{-2} \times 2\pi \left.\frac{\rho^2}{2}\right]_0^1 = 10 \times e^{-2} \times 2\pi \times \frac{1}{2} = 10\pi e^{-2}$$

$$\Rightarrow \psi_t = 10\pi e^{-2}$$

For ψ_b, $z=0$ and $ds = \rho\, d\rho\, d\phi\, (-a_z)$

Hence, $\psi_b = \int_b D \cdot ds$

$$= \int_{\rho=0}^{1} \int_{\phi=0}^{2\pi} 10 \cdot e^{0} \cdot (\rho\, d\rho\, d\phi)$$

$$= -\left[10 \times 1 \times 2\pi\right] \left[\frac{\rho^2}{2}\right]_0^1$$

$$= -10\pi$$

For ψ_s, $\rho = 1$, $ds = \rho\, d\phi\, dz\, a_\rho$, Hence

(∵ normal to the curved surface is along 'a_ρ')

$$\therefore \psi_s = \int_s D \cdot ds$$

$$= \int_{z=0}^{1} \int_{\phi=0}^{2\pi} 10 \cdot e^{-2z} (\rho a_\rho + a_z) \cdot (\rho\, d\phi\, dz\, a_\rho)$$

$$= \int_{z=0}^{1} \int_{\phi=0}^{2\pi} (10\, e^{-2z} \rho) \cdot (\rho\, d\phi\, dz)$$

$$= \rho^2 \times 2\pi \times 10 \times \int_{z=0}^{1} e^{-2z} dz$$

$$= 1 \times 20\pi \times \left[\frac{e^{-2z}}{-2}\right]_0^1$$

$$\Rightarrow \quad \psi_s = -10\pi\left[e^{-2} - 1\right]$$

$$\Rightarrow \quad \psi_s = 10\pi\left[1 - e^{-2}\right]$$

Thus

$$\psi = \psi_t + \psi_b + \psi_s$$

$$= 10\pi \cancel{e^{-2}} + \cancel{(-10\pi)} + \cancel{10\pi} - e^{-2}\cancel{10\pi}$$

$$\psi = 0$$

$\therefore$ Total flux of G out of the entire surface of the cylender $= 0$.

By Divergence them

Since 'S' is a closed surface, we can apply the divergence theorem

$$\psi = \oint_S G \cdot ds = \int_V (\nabla \cdot G)\, dv$$

$$\nabla \cdot G = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho G_\rho) + \frac{1}{\rho}\frac{\partial}{\partial \phi} G_\phi + \frac{\partial}{\partial z} G_z$$

$$= \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \times 10\, e^{-2z} \rho\right) + \frac{1}{\rho}\frac{\partial}{\partial \phi}(0) + \frac{\partial}{\partial z}(10\, e^{-2z})$$

$$= \frac{1}{\cancel{\rho}} \times 10 \times e^{-2z} \times 2\cancel{\rho} + 10 \times e^{-2z}(-2)$$

$$= 20\, e^{-2z} - 20\, e^{-2z}$$

$$= 0$$

∴ Thus G has no outward flux,

Hence, $\Psi = \int_v \nabla \cdot G \, dv = 0$

(Proved)

Curl of a Vector and Stoke's theorem

Earlier we have defined, the circulation of a vector field A around a closed path L as the integral $\oint_L A \cdot dl$.

The curl of A is an axial (or rotational) vector whose magnitude is the maximum circulation of A per unit area as the area tends to zero and whose direction is the normal direction of the area when the area is oriented to make the circulation maximum.

That is,

$$\boxed{\text{Curl } A = \nabla \times A = \left(\lim_{\Delta S \to 0} \frac{\oint_L A \cdot dl}{\Delta S} \right)_{max} a_n} \quad (2.32)$$

Where the area ΔS is bounded by the curve L and a_n is the unit vector normal to the surface ΔS and is determined by using the right-hand rule.

Note :- Proof is not required. You can refer Book for the derivation.

$\nabla \times A$ in Cartesian Coordinates

$$\nabla \times A = \begin{vmatrix} a_x & a_y & a_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \quad \text{—— (2.33)}$$

$\nabla \times A$ in Cylindrical Coordinates

$$\nabla \times A = \frac{1}{\rho} \begin{vmatrix} a_\rho & \rho a_\phi & a_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix} \quad \text{—— (2.34)}$$

$\nabla \times A$ in Spherical Coordinates

$$\nabla \times A = \frac{1}{r^2 \sin\theta} \begin{vmatrix} a_r & r a_\theta & r\sin\theta\, a_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & r A_\theta & r\sin\theta\, A_\phi \end{vmatrix} \quad \text{—— (2.35)}$$

Properties of Curl

1. The Curl of a vector field is another vector field. (∵ Cross Product gives a vector)
2. $\nabla \times (A+B) = \nabla \times A + \nabla \times B$ —— (2.36)
3. $\nabla \times (A \times B) = A(\nabla \cdot B) - B(\nabla \cdot A) + (B \cdot \nabla)A - (A \cdot \nabla)B$ —— (2.37)
4. $\nabla \times (VA) = V \nabla \times A + \nabla V \times A$ —— (2.38)

5. The divergence of the curl of a vector field vanishes; that is

$$\nabla \cdot (\nabla \times A) = 0 \quad -(2.39) \quad (D.C) = 0$$

6. The curl of the gradient of a scalar field vanishes; that is

$$\nabla \times (\nabla V) = 0 \quad -(2.40) \quad (C.G = 0)$$

Physical Significance of Curl

→ The curl provides the maximum value of the circulation of the field per unit area (or circulation density) [Eq (2.32)] and indicates the direction along which this maximum value occurs.

→ The curl of a vector field A at a point P may be regarded as a measure of circulation or how much the field curls around P. For example, Fig 2.20 (a) shows that the curl of a vector field around P is directed out of the page. Fig 2.20(b) shows a vector field with zero curl.

(a) (b)

Fig 2.20 Illustration of a curl (a) Curl at P points out of the page (b) curl at P is zero.

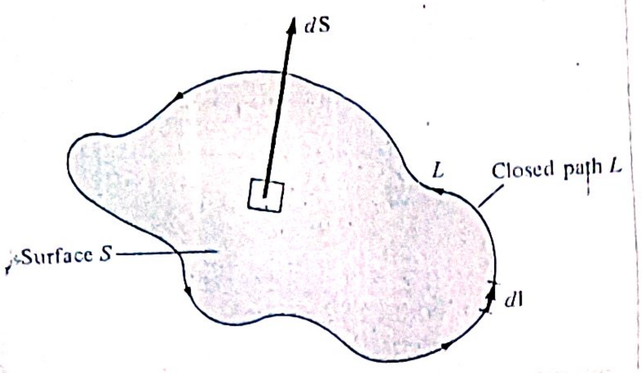


Fig 2.21 Determining the sense of dl and ds involved in Stoke's theorem.

From the defination of the curl of A in eqn (2.32), we may expect that

$$\boxed{\oint_L A \cdot dl = \int_S (\nabla \times A) \cdot ds} \quad \text{—(2.41)}$$

This is called Stoke's theorem.

Stoke's theorem states that the circulation of a vector field A around a (closed) path L is equal to the surface integral of the curl of A over the open surface S bounded by L (See fig 2.21), provided A and $\nabla \times A$ are continuous on S.

→ Proof of Stoke's theorem is not required. For knowledge you can refer the Sadiku Book.

Example 2.8

Determine the curl of each of the vector field

(a) $P = x^2yz\, a_x + xz\, a_z$

(b) $Q = \rho \sin\phi\, a_\rho + \rho^2 z\, a_\phi + z\cos\phi\, a_z$

(c) $T = \dfrac{1}{r^2}\cos\theta\, a_r + r\sin\theta\cos\phi\, a_\theta + \cos\theta\, a_\phi$

Ans:-

(a)

$$\nabla \times P = \begin{vmatrix} a_x & a_y & a_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P_x & P_y & P_z \end{vmatrix}$$

$$= \begin{vmatrix} a_x & a_y & a_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2yz & 0 & xz \end{vmatrix}$$

$$= a_x(0-0) - a_y\left(\frac{\partial}{\partial x}(xz) - \frac{\partial}{\partial z}(x^2yz)\right) + a_z(0 - x^2z)$$

$$= 0 - a_y(z - x^2y) + a_z(-x^2z)$$

$$= (x^2y - z)\,a_y - x^2z\,a_z \quad \text{(Ans)}$$

(b)

$$\nabla \times Q = \frac{1}{\rho}\begin{vmatrix} a_\rho & \rho a_\phi & a_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix}$$

$$= \frac{1}{\rho} \begin{vmatrix} a_\rho & \rho a_\phi & a_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ \rho \sin\phi & \rho . \rho^2 z & z\cos\phi \end{vmatrix}$$

$$= \frac{1}{\rho}\left[a_\rho \left(-z\sin\phi - \rho^3\right) - \rho a_\phi \left[(0) - 0\right] + a_z \left[z \times 3\rho^2 - \rho\cos\phi\right] \right]$$

$$\Rightarrow \nabla \times Q = \frac{1}{\rho}\left[-(z\sin\phi + \rho^3) a_\rho + (3\rho^2 z - \rho\cos\phi) a_z \right]$$

$$\Rightarrow \nabla \times \phi = -\frac{1}{\rho}(z\sin\phi + \rho^3) a_\rho + (3\rho z - \cos\phi) a_z$$ (Ans)

(c) $$\nabla \times T = \frac{1}{r^2 \sin\theta} \begin{vmatrix} a_r & r a_\theta & r\sin\theta\, a_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & r A_\theta & r\sin\theta\, A_\phi \end{vmatrix}$$

$$= \frac{1}{r^2 \sin\theta} \begin{vmatrix} a_r & r a_\theta & r\sin\theta\, a_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ \frac{1}{r^2}\cos\theta & r . r\sin\theta\cos\phi & r\sin\theta\, \cos\theta \end{vmatrix}$$

$$\Rightarrow \nabla \times T = \frac{1}{r^2 \sin\theta} \Big(a_r \left[r \frac{\partial}{\partial \theta} (\sin\theta \cdot \cos\theta) - \frac{\partial}{\partial \phi} (r^2 \sin\theta \cos\phi) \right]$$

$$- r a_\theta \left[0 - \frac{\partial}{\partial r} (r \sin\theta \cos\theta) \right]$$

$$+ r \sin\theta \, a_\phi \left[\frac{\partial}{\partial r} (r^2 \sin\theta \cos\phi) - \frac{\partial}{\partial \theta} \left(\frac{1}{r^2} \cos\theta \right) \right] \Big)$$

$$= \frac{1}{r^2 \sin\theta} a_r \left[\frac{r}{2} \times \cos 2\theta \times 2 + r^2 \sin\theta \cdot \sin\phi \right]$$

$$- \frac{1}{r \, r^2 \sin\theta} r a_\theta \left(\sin\theta \cos\theta \right)$$

$$+ \frac{1}{r \, r^2 \sin\theta} \times r \sin\theta \, a_\phi \left(\sin\theta \cos\phi \cdot 2r + \frac{1}{r^2} \sin\theta \right)$$

$$\Rightarrow \nabla \times T = \left(\frac{\cos 2\theta}{r \sin\theta} + \sin\phi \right) a_r - \frac{\cos\theta \, a_\theta}{r}$$

$$\left(2\cos\phi + \frac{1}{r^3} \right) \sin\theta \, a_\phi$$

(Ans)

Example 2.9

If $A = \rho \cos\phi \, a_\rho + \sin\phi \, a_\phi$, evaluate $\oint A \cdot dl$ around the path shown on Figure 2.23. ~~Consider~~ Confirm this by using Stoke's theorem, Solution :- Let

$$\oint A \cdot dl = \left[\int_a^b + \int_b^c + \int_c^d + \int_d^a \right] A \cdot dl$$

Where Path L has been divided into segments ab, bc, cd, and da as on Figure 2.23.

Along ab, $\rho = 2$, $dl = \rho \, d\phi \, a_\phi$, Hence

$$\int_a^b A \cdot dl = \int_{\phi=60}^{\phi=30^\circ} (\rho \cos\phi \, a_\rho + \sin\phi \, a_\phi) \cdot \rho \, d\phi \, a_\phi$$

$$= \int_{\phi=60^\circ}^{\phi=30^\circ} \rho \sin\phi \, d\phi$$

$$= \rho \left[-\cos\phi \right]_{\phi=60^\circ}^{\phi=30^\circ}$$

$$\Rightarrow \int_a^b A \cdot dl = -\rho \left[\frac{\sqrt{3}}{2} - \frac{1}{2} \right]$$

$$\Rightarrow \int_a^b A \cdot dl = -\cancel{2} \times \frac{(\sqrt{3}-1)}{\cancel{2}}$$

$$\Rightarrow \int_a^b A \cdot dl = -(\sqrt{3}-1) \quad \text{———— (7')}$$

Along bc, $\phi = 30^\circ$ and $dl = d\rho \, a_\rho$. Hence

$$\int_b^c A \cdot dl = \int_{\rho=2}^{5} \rho \cos\phi \, d\rho = \cos\phi \left(\frac{\rho^2}{2} \right]_{\rho=2}^{\rho=5}$$

$$\Rightarrow \int_b^c A \cdot dl = \frac{\cos\phi}{2}[25-4]$$

$$= \frac{\cos 30^\circ}{2}[21]$$

$$= \frac{\sqrt{3}}{2} \times \frac{1}{2} \times 21$$

$$\Rightarrow \int_b^c A \cdot dl = \frac{21\sqrt{3}}{4} \quad \text{——— (ii)}$$

<u>Along cd</u>, $\rho = 5$, $dl = \rho\, d\phi\, a_\phi$ Hence,

$$\int_c^d A \cdot dl = \int_{\phi=30^\circ}^{\phi=60^\circ} (\rho\cos\phi\, a_\rho + \sin\phi\, a_\phi) \cdot \rho\, d\phi\, a_\phi$$

$$= \int_{\phi=30^\circ}^{\phi=60^\circ} \rho \sin\phi\, d\phi = \rho \int_{\phi=30^\circ}^{\phi=60^\circ} \sin\phi\, d\phi$$

$$= 5 \left[-\cos\phi\right]_{\phi=30^\circ}^{\phi=60^\circ}$$

$$= -5\left[\frac{1}{2} - \frac{\sqrt{3}}{2}\right]$$

$$= \frac{5(\sqrt{3}-1)}{2} \quad \text{——— (iii)}$$

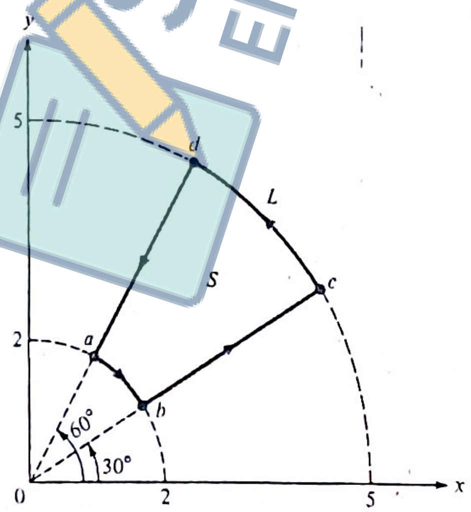


Fig 2.23 for example 2.9

Along da $\quad \phi = 60^\circ$, $dl = d\rho\, a_\rho$.

Hence $A \cdot dl = \int_{\rho=5}^{2} (\rho \cos\phi\, a_\rho + \sin\phi\, a_\phi) \cdot d\rho\, a_\rho$

$$= \int_{\rho=5}^{2} \rho \cos\phi \, d\rho$$

$$= \cos 60^\circ \times \left.\frac{\rho^2}{2}\right]_5^2$$

$$= \frac{1}{2} \times \frac{1}{2} \times (4-25)$$

$$= -\frac{21}{4} \quad \text{———— (iv)}$$

Putting all these together in,

$$\oint A \cdot dl = (i) + (ii) + (iii) + (iv)$$

$$= -\sqrt{3}+1 + \frac{21\sqrt{3}}{4} + \frac{5(\sqrt{3}+1)}{2} + \left(\frac{-21}{4}\right)$$

$$= \frac{-4\sqrt{3} + 4 + 21\sqrt{3} + 10\sqrt{3} - 10 - 21}{4}$$

$$= \frac{-27 + 27\sqrt{3}}{4}$$

$$= +\frac{27}{4}(\sqrt{3}-1)$$

$\oint A \cdot dl = 4.941$ (Ans). ———— (v)

To verify using Stoke's thm

According to Stoke's theorem (L Here is a closed Path)

$$\oint_L A \cdot dl = \int_S (\nabla \times A) \cdot ds$$

Here the curve is along xy plane.

So the $ds = \rho \, d\phi \, d\rho \, a_z$, $\left(\because a_z \text{ is normal to the plane} \right)$

$$\nabla \times A = \frac{1}{\rho} \begin{vmatrix} a_\rho & \rho a_\phi & a_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix}$$

Here $A = \rho \cos\phi \, a_\rho + \sin\phi \, a_\phi$

$$\nabla \times A = \frac{1}{\rho} \begin{vmatrix} a_\rho & \rho a_\phi & a_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ \rho \cos\phi & \rho \sin\phi & 0 \end{vmatrix}$$

$$= \frac{1}{\rho} \left[a_\rho (0 - 0) - \rho a_\phi (0 - 0) + a_z (\sin\phi + \rho \sin\phi) \right]$$

$$= \frac{1}{\rho} (1 + \rho) \sin\phi \, a_z$$

Hence, $\int_S (\nabla \times A) \cdot ds$ will be

$$\int_S \left(\frac{1}{\cancel{\rho}} \times (1+\rho) \sin\phi \; a_z\right) \cdot \left(\cancel{\rho} \; d\phi \; d\rho \; a_z\right)$$

$$= \int_{\phi=30^\circ}^{60^\circ} \int_{\rho=2}^{5} \left(\frac{1+\rho}{\;}\right) \sin\phi \; d\phi \; d\rho$$

$$= \int_{\phi=30^\circ}^{60^\circ} \sin\phi \; d\phi \times \int_{2}^{5} (1+\rho) \; d\rho$$

$$= -\cos\phi \Big]_{30^\circ}^{60^\circ} \times \left(\rho + \frac{\rho^2}{2}\right)\Big]_2^5$$

$$= -\left(\frac{1}{2} - \frac{\sqrt{3}}{2}\right) \times \left(5 + \frac{25}{2} - 2 - 2\right)$$

$$= \left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right)\left(\frac{25}{2} + 1\right)$$

$$= \frac{\sqrt{3}-1}{2} \times \frac{27}{2}$$

$$= \frac{27}{4}(\sqrt{3}-1)$$

$$= 4.941 \quad \text{———} \quad (vi)$$

From eqn (v) & (vi)

$$\therefore \oint A \cdot dl = \int_S (\nabla \times A) \cdot ds$$

(Proved)

∴ Stoke's them is verified.

Example 2.10

For a vector field A, show that $\nabla \cdot \nabla \times A = 0$

i.e divergence of the curl of any vector field is zero.

Proof:- For simplicity, assume that A is in Cartesian coordinates.

$$\nabla \cdot \nabla \times A = \left(\frac{\partial}{\partial x} a_x + \frac{\partial}{\partial y} a_y + \frac{\partial}{\partial z} a_z\right) \cdot \begin{vmatrix} a_x & a_y & a_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial x} a_x + \frac{\partial}{\partial y} a_y + \frac{\partial}{\partial z} a_z\right) \cdot$$

$$\left[a_x\left(\frac{\partial}{\partial y} A_z - \frac{\partial}{\partial z} A_y\right) - a_y\left(\frac{\partial}{\partial x} A_z - \frac{\partial}{\partial z} A_x\right) + a_z\left(\frac{\partial}{\partial x} A_y - \frac{\partial}{\partial y} A_x\right)\right]$$

Making the dot product

$$= \cancel{\frac{\partial^2 A_z}{\partial x \partial y}} - \cancel{\frac{\partial^2 A_y}{\partial x \partial z}} - \cancel{\frac{\partial^2 A_z}{\partial y \partial x}} + \cancel{\frac{\partial^2 A_x}{\partial y \partial z}} + \cancel{\frac{\partial^2 A_y}{\partial z \partial x}} - \cancel{\frac{\partial^2 A_x}{\partial z \partial y}}$$

$$= 0$$

$$\left(\because \frac{\partial^2 A_z}{\partial x \partial y} = \frac{\partial^2 A_z}{\partial y \partial x} \text{ and so on}\right)$$

$\therefore \quad \nabla \cdot (\nabla \times A) = 0$

Assignment - 6

For a scalar field V, show that $\nabla \times \nabla V = 0$; i.e the curl of the gradient of any scalar field vanishes.

Ans:
$$\nabla \times \nabla V$$

$$= \nabla \times \left(\underbrace{\frac{\partial V}{\partial x}}_{(A_x)} a_x + \underbrace{\frac{\partial V}{\partial y}}_{(A_y)} a_y + \underbrace{\frac{\partial V}{\partial z}}_{(A_z)} a_z \right)$$

Now →

$$= \begin{vmatrix} a_x & a_y & a_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} & \frac{\partial V}{\partial z} \end{vmatrix}$$

$$= a_x \left(\frac{\partial^2 V}{\partial y \partial z} - \frac{\partial^2 V}{\partial z \cdot \partial y} \right) - a_y \left(\frac{\partial^2 V}{\partial x \partial z} - \frac{\partial^2 V}{\partial z \partial x} \right)$$

$$+ a_z \left(\frac{\partial^2 V}{\partial x \partial y} - \frac{\partial^2 V}{\partial y \partial x} \right)$$

$$= a_x \cdot 0 - a_y (0) + a_z (0)$$

$$= 0$$

Laplacian of a Scalar

The Laplacian of a scalar field V, written as $\nabla^2 V$, is the divergence of the gradient of V.

Thus, in Cartesian coordinates,

Laplacian $V = \nabla \cdot \nabla V = \nabla^2 V$

$$= \left(\frac{\partial}{\partial x} a_x + \frac{\partial}{\partial y} a_y + \frac{\partial}{\partial z} a_z\right) \cdot \left(\frac{\partial V}{\partial x} a_x + \frac{\partial V}{\partial y} a_y + \frac{\partial V}{\partial z} a_z\right)$$

$$\therefore \boxed{\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}} \quad \text{—(2.42)}$$

Notice that Laplacian of a scalar field is another scalar field.

Laplacian in Cylindrical Coordinate System

$$\nabla^2 V = \frac{1}{\rho} \frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2} \frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} \quad \text{—(2.43)}$$

in Spherical Coordinate System

$$\nabla^2 V = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 V}{\partial \phi^2} \quad \text{—(2.44)}$$

→ A scalar field V is said to be harmonic on a given region if its Laplacian vanishes on that region. In other words, if

$$\boxed{\nabla^2 V = 0} \quad (2.45)$$

is satisfied on the region, the solution for V in eqn (2.45) is harmonic (i.e it is of the form of sine or cosine). Eqn (2.45) is called Laplace's equation.

→ Laplacian of Vector A

$$\boxed{\nabla^2 A = \nabla(\nabla \cdot A) - \nabla \times \nabla \times A} \quad (2.46)$$

$\nabla^2 A$ = Gradient of divergence − Curl of Curl of A

Note:-

Prove that in Cartesian Coordinate System

$$\nabla^2 A = \nabla^2 A_x a_x + \nabla^2 A_y a_y + \nabla^2 A_z a_z \quad (2.47)$$

Proof: We know

$$a \times b \times c = b(a \cdot c) - c(a \cdot b) \quad \text{(vector triple product)}$$

$$\nabla \times \nabla \times A = \nabla(\nabla \cdot A) - A(\nabla \cdot \nabla)$$

$$\Rightarrow \nabla \times \nabla \times A = \nabla(\nabla \cdot A) - A\nabla^2$$

$$\Rightarrow \nabla^2 A = \nabla(\nabla \cdot A) - \nabla \times \nabla \times A$$

$\because \nabla^2$ is a scalar $A\nabla^2 = \nabla^2 A$

Proof

$$\nabla^2 A = \nabla^2(A_x a_x + A_y a_y + A_z a_z) = \nabla^2 A_x a_x + \nabla^2 A_y a_y + \nabla^2 A_z a_z$$

(∵ ∇^2 is scalar, it is multiplied with every component of $\vec{A}$ vector)

Example 2.11

Find the Laplacian of the scalar fields

(a) $V = e^{-z} \sin 2x \cosh y$

(b) $U = \rho^2 z \cos 2\phi$

(c) $W = 10 r \sin^2\theta \cos\phi$

Solution:-

The Laplacian in the Cartesian system can be found by taking the first derivative and later the second derivative.

(a) $\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$

$= \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial V}{\partial z}\right)$

$= \frac{\partial}{\partial x}\left(e^{-z} \cosh y \cdot 2\cos 2x\right) + \frac{\partial}{\partial y}\left(e^{-z} \sin 2x \sinh y\right)$

$+ \frac{\partial}{\partial z}\left(-\sin 2x \cosh y \cdot e^{-z}\right)$

$= 2e^{-z} \cosh y \cdot (-\sin 2x)(2) + e^{-z} \sin 2x \cdot \cosh y$

$+ -(\sin 2x) \cosh y \cdot \left(e^{-z}\right)(-)$

$\Rightarrow \nabla^2 V = -9\, e^{-z} \sin 2x \cosh y$ (Ans) $\left(\because \text{Adding 3 components}\right)$

(b) $\nabla^2 U$

$$= \frac{1}{\rho} \frac{\partial}{\partial \rho}\left(\rho \frac{\partial U}{\partial \rho}\right) + \frac{1}{\rho^2} \frac{\partial}{\partial \phi}\left(\frac{\partial U}{\partial \phi}\right) + \frac{\partial}{\partial z}\left(\frac{\partial U}{\partial z}\right)$$

$$= \frac{1}{\rho} \frac{\partial}{\partial \rho}\left(\rho \times 2\rho \times z \cos 2\phi\right) + \frac{1}{\rho^2}\frac{\partial}{\partial \phi}\left(\rho^2 z \left(-2 \sin 2\phi\right)\right) + \frac{\partial}{\partial z}\left(\rho^2 \cos 2\phi\right)$$

$$= \frac{1}{\cancel{\rho}} \cdot z \cos 2\phi \times 2 \times 2\cancel{\rho} + \frac{-2}{\cancel{\rho^2}} \times \cancel{\rho^2}\, z \times \cos 2\phi \times 2 + 0$$

$$= 4z \cos 2\phi - 4z \cos 2\phi$$

$$= 0$$ (Ans)

(c) $\nabla^2 W$

$$= \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial W}{\partial r}\right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial W}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\left(\frac{\partial}{\partial \phi}\left(\frac{\partial W}{\partial \phi}\right)\right)$$

$$\Rightarrow \nabla^2 W = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \times 10 \sin^2\theta \cos\phi\right)$$

$$+ \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \times 10r \times \cos\phi \times \sin 2\theta\right)$$

$$+ \frac{1}{r^2 \sin^2\theta}\left(\frac{\partial}{\partial \phi}\left(-10r \sin^2\theta \sin\phi\right)\right)$$

$$= \frac{1}{\cancel{r^2}_{\,r}} \times 10 \sin^2\theta \cos\phi \times 2\cancel{r}$$

$$+ \frac{1}{{}_{r}\,\cancel{r^2} \sin\theta} \times 10\cancel{r} \cos\phi \left[2\sin\theta \cdot \cos 2\theta + \sin 2\theta \cdot \cos\theta\right]$$

$$+ \frac{1}{{}_{r}\,\cancel{r^2} \cancel{\sin^2\theta}} \times -10\cancel{r} \cancel{\sin^2\theta} \times \cos\phi$$

$$= \frac{10\cos\phi}{r}\left[2\sin^2\theta\right] + \frac{10\cos\phi}{r}\left[2\cos 2\theta + 2\cos^2\theta\right]$$

$$- \frac{10\cos\phi}{r}\left[1\right]$$

$$= \frac{10\cos\phi}{r}\left[\underline{2\sin^2\theta} + 2\cos 2\theta + \underline{2\cos^2\theta} - 1\right]$$

$$= \frac{10\cos\phi}{r}\left(2 + 2\cos 2\theta - 1\right)$$

$$\Rightarrow \nabla^2 W = \frac{10\cos\phi}{r}\left(1 + 2\cos 2\theta\right) \quad \text{(Ans)}$$

Classification of Vector fields

A vector field is uniquely characterized by its divergence and curl. Neither the divergence nor curl of a vector field is sufficient to completely describe the field. All vector fields can be classified in terms of their vanishing or non vanishing divergence or curl as follows

(a) $\nabla \cdot A = 0$, $\nabla \times A = 0$

(b) $\nabla \cdot A \neq 0$, $\nabla \times A = 0$

(c) $\nabla \cdot A = 0$, $\nabla \times A \neq 0$

(d) $\nabla \cdot A \neq 0$, $\nabla \times A \neq 0$

Fig 2.24 illustrates typical fields in these four categories

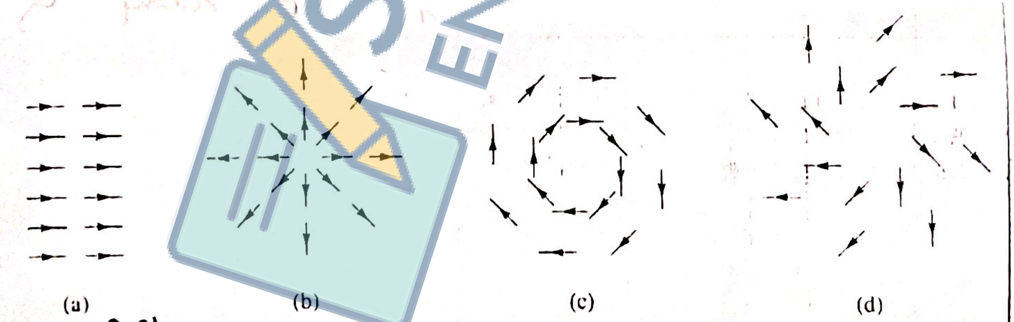


Figure 2.24 Typical fields with vanishing and nonvanishing divergence or curl.

A vector field A is said to be solenoidal (or divergenceless) if $\nabla \cdot A = 0$.

Such a field has neither source or sink of flux.

A vector field $\mathbf{A}$ is said to be **solenoidal** (or **divergenceless**) if $\nabla \cdot \mathbf{A} = 0$.

Such a field has neither source nor sink of flux. From the divergence theorem,

$$\oint_S \mathbf{A} \cdot d\mathbf{S} = \int_v \nabla \cdot \mathbf{A} \, dv = 0$$

In general, the field of curl $\mathbf{F}$ (for any $\mathbf{F}$) is purely solenoidal because $\nabla \cdot (\nabla \times \mathbf{F}) = 0$, Thus, a solenoidal field $\mathbf{A}$ can always be expressed in terms of another vector $\mathbf{F}$; that is,

if $\quad \nabla \cdot \mathbf{A} = 0$

then $\quad \oint_S \mathbf{A} \cdot d\mathbf{S} = 0 \quad$ and $\quad \mathbf{A} = \nabla \times \mathbf{F}$

→ A vector field A is said to be irrotational if $\nabla \times A = 0$

That is, a curl-free vector is irrotational.

From Stoke's theorem

$$\int_S (\nabla \times A)\, ds = \oint_L A \cdot dl = 0 \quad \text{——— (2.49)}$$

Thus, in an irrotational field A, the circulation of A around a closed path is identically zero.

This implies that the line integral of A is independent of the closed path.

Therefore, an irrotational field is also known as conservative field. Example of irrotational fields include the electrostatic field and gravitational field. In general, the field of gradient V (for any scalar V) is purely irrotational, since

$$\nabla \times (\nabla V) = 0 \quad \text{— (2.50)}$$

∵ curl of gradient = 0

Thus, an irrotational field A can always be expressed in terms of a scalar field V;

that is

$$\boxed{\begin{array}{l} \nabla \times A = 0 \\ \text{then} \quad \oint_L A \cdot dl = 0 \quad \text{and} \quad A = -\nabla V \end{array}} \quad -(2.51)$$

$$\left(\because \quad \begin{array}{l} \nabla \times (\nabla V) = 0 \quad \& \\ \nabla \times A = 0 \end{array} \right)$$

For this reason, A may be called a potential field and V is the scalar potential of A. The -ve sign in eqn (2.51) has been inserted for physical reasons that will become evident in the Chapter of electrostatic fields.

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