Module-I :- Ch-1- Signal & Spectra

Everyday, in our work and in our leisure time, we come in contact with and use variety of modern communication systems and communication media, the most common being the telephone, radio, T.V and internet. Through these media we are able to communicate instantaneously with people on different continents, transact our daily business and receive information about various developments and events of note that occurs all around the world.

Communication is the process of transferring information from one place to another. The information generated at the source end may need to travel hundred or thousand miles via channel to reach destination.

Channel:-

→ Communication channel is the media by which information is sent. Channel could be wired line (copper wire) or wireless. (Atmosphere)

→ Physical wire → Costlier & less convenient.

→ Consider Block diagram representation of an electronic communication system.

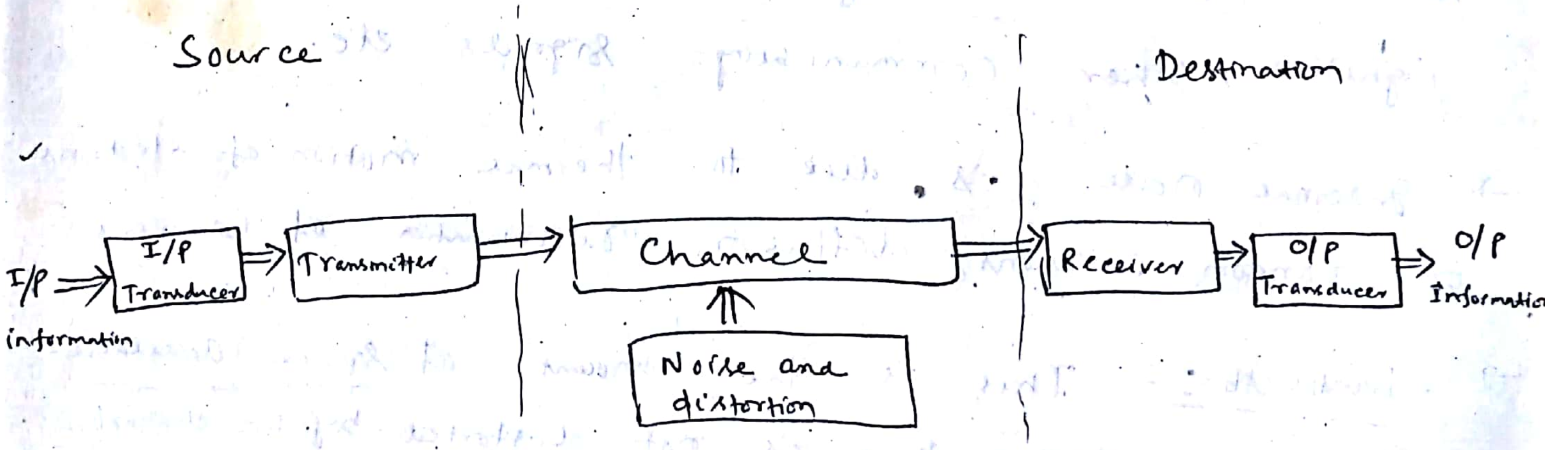


fig:- A block diagram representation of electronics communication system.

→ Noise and distortions are added to the channel. (3)

→ Distortion is the process of changing shape of communicating signal that may mislead the destination about the content of message.

→ Distortion occures due to inability of the channel convey all frequency, phase, amplitude information truthfully from one side to another.

→ If, Distortion parameters vary linearly, it is termed as linear distortion, if vary nonlinearly termed as nonlinear distortion.

→ The loss in amplitude or strength of the signal as it travels through channel is called attenuation.

→ Noise is different from distortion. It can be defined as random, unwanted interference on transmitted signal. Channel is a major source of noise.

→ External noise, are interference from other sources like lightning, electrical switching, automobile ignition, other communicating signals etc.

→ Internal noise is due to thermal motion of electrons or random emissions, diffusion, recombination of carriers.

→ Bandwidth :- This is the amount of space available in frequency domain that is not distorted by the channel.

The more the bandwidth, the better.

→ Co-axial cable or fiber-optic links that are superior to copper lines offering very high bandwidth.

→ Trans-oceanic submarine cables connect continents.

→ Atmosphere as propagating media is used by electromagnetic waves while communication via satellites partly uses free space as communication channel.

Transmitter

→ A transmitter is a device that makes i/p electrical information suitable for efficient transmission over a given channel. In general, the transmitter modulates or changes some parameter say amplitude or frequency of a high frequency carrier signal by original electrical information input which is also known as baseband signal.

→ Transmitter also multiplexes i.e put number of signals in a common pool and effects simultaneous transmission of number of i/p signal.

Receiver:-

→ On the destination side, receiver is the device that receives information from channel and extract intended electrical message from it.

→ Other than demodulation, the reverse of modulation, and demultiplexing the reverse of multiplexing the receiver also amplify and remove noise, distortion from the noise contaminated, distorted, attenuated received signal.

Transducer:-

→ The i/p transducer Converts the information message to be transmitted to it's electrical equivalent signal. [Because electrical signal move with speed of light]

→ Information Can be speech, image, video, text etc.

→ Microphone is an i/p transducer which Converts audio i/p like speech to an electrical signal.

→ The o/p transducer Converts electrical input to a form of message required by user, e.g speech, image, video, text etc. The loud-speaker is an example of output transducer where electrical i/p is Converted to an audio output.

Baseband & Passband Signals:-

The message signal generated from the information source is known as baseband signal. If the baseband signal is transmitted directly, then it is known as baseband transmission. It is preferred at low frequencies and for short distances.

→ Whenever a modulating signal is impressed upon a Carrier, the modulated signal is produced. If the modulated signal is transmitted over the Channel, it is known as bandpass or simply Passband transmission. Therefore, modulated signals are known as

Passband or bandpass signals. Ex:- If 1 KHZ modulating signal is impressed upon a 100 KHZ Carrier signal by using amplitude modulation technique then 2 sidebands will be generated. Lower side band will be at (100-1) i.e 99 KHZ and upper side band will be at (100+1) i.e 101 KHZ

Thus, the amplitude modulated signal will have frequencies from 99 KHZ to 101 KHZ. These frequencies are bandpass type. This means that the modulated signals are bandpass or pass band type.

Analog Signal :-

The signals which vary continuously with time i.e for every value of time they are defined, are known as analog signals.

The name, derives from the fact that such signal is analogous to the physical signal that it represents. The majority of signals in the world around us are analog.

Ex:- Temperature or the atmospheric pressure of certain location with time.

fig:- An analog signal.

Digital Signal:-

→ When a signals represented by sequence of numbers and each number represents the signal magnitude at an instant of time, then the resulting signal is called digital signal.

→ Digital messages are constructed with finite number of symbols.

→ In M-ary communication, M different symbols are used. In binary digital communication, only 2 symbols (0, 1) are used.

→ The numerical data '12' can be represented in '1100' in ~~bi~~ binary.

→ But text like 'A', 'a', 'B', 'b', '!', '%' can be represented in 7 bit. The American Standard Code for Information Interchange (ASCII) has 7 bit representation for each of these character e.g. 'A' can be represented as '1000001' (65) and 'a' as '1100001'. (97)

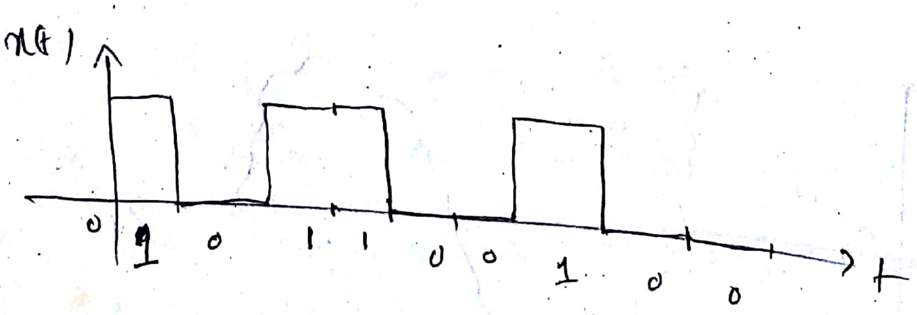


fig:- Variation of a binary digital signal with time.

Here, the waveform is a pulse train with 0V representing a '0' signal or logic '0' and +5V representing logic '1'.

Modulation & multiplexing

Modulation is the process of varying one attribute of carrier signal by message signal. (will be discussed in detail in later chapters)

Why Mod^n required?

1) To have practical antenna length.
2) To avoid interference.
3) To have frequency multiplexing

Multiplexing :- (page-159)

Multiplexing is a technique in which several message signals are combined into a composite signal for transmission over a common channel.

Multiplexing are of 2 types:-

1) Frequency division multiplexing (FDM)
2) Time Division Multiplexing (TDM)

→ Using a carrier to shift the frequency band of a message signal for simultaneous transmission is known as FDM.

→ For digital signal, multiplexing can be achieved by diving the time between 2 samples of signals in various time slots and using each time slot to send one digital signal. Such a method of simultaneous transmission as known as time division multiplexing (TDM).

Transmission Media :-

Different types of transmission media being in use today. The major varieties are

Open wire lines:-

→ The original telephone and telegraph transmission lines.

→ Have low attenuation (0.04 dB/KM) for voice freq. range and useful for long distance communication.

→ Susceptible to environmental, weather and human abuse.

Paired Cables:-

→ Used on telephone n/ws within short distances.

→ Inside building, building to local office etc.

Quad Cables :-

Includes 4 conductors arranged as 2 different pairs for carrying the different signals.

Coaxial Cable :-

It consist of a single wire conductor at the center of a cylindrical cable and outer conductor typically a wire mesh separated by a dielectric.

→ Gives BW in MHZ range, used for T.V connection. Attenuation relatively high (5 dB/Km)

Radio :-

Radio is a wireless propagation where atmosphere or free space is used as transmission media. Radio freq. signals are radiated into them as electromagnetic signal through antenna.

Wave guide :-

This is a hollow conductor of rectangular, circular or elliptical cross section typically feeding or receiving signal from transmitter or receiver antenna in radio propagation.

Optical fiber :-

It is made up of a very fine ~~fibre~~ fibre core that allows light wave to propagate through it with minimum loss. An outer layer called cladding ensures total internal reflection. A final protective layer it from physical damage.

SNR (Signal to Noise ratio)

A quantity that gives relative strength of the signal to Noise is called Signal to Noise Ratio.

$$(SNR)_{in\ dB} = 10 \log_{10} \left(\frac{\text{Signal energy}}{\text{Noise energy}} \right)$$

BW :- (Bandwidth)

BW of a channel can be defined as range of frequencies which can be transmitted through it, within acceptable degradation.

Channel Capacity

The Capacity of a Channel to support a particular rate of information is defined as Channel Capacity and is an important measure in digital Communication.

The upper limit of Channel Capacity C comes from Shanon's equation.

$$C = B \log_2 (1 + SNR) \ \frac{bits}{sec}$$

B ↳ BW

→ It clearly shows that increase of BW reduces SNR when information rate is constant and vice versa.

→ So a trade-off is done betn SNR and channel BW. So the modified (practical) formula for channel capacity

$$\boxed{C = B \log_2 \left(1 + \frac{S}{N_0 B}\right) \ \frac{bits}{sec}}$$

N_0 = ~~Gau~~ White noise or flat noise noise spectrum of strength N_0.

B = BW (Band width).

Signal and its properties:-

→ A signal is a single-valued function of one or more independent variables.

Ex: For speech signal, time is independent variable and

Can be represented by voltage V as V(t) Called function of time 't'.

Signal energy :-

The normalized energy of a signal $x(t)$ is defined as the energy dissipated by a voltage signal applied across $1\,\Omega$ resistor (or by current signal flow through $1\,\Omega$ resistor).

Mathematically

$$\boxed{E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt}$$

$0 < E < \infty$ / E is finite.

unit in Joule.

Signal power :-

Signal Power is the average power spent across $1\,\Omega$ resistor and also called the normalized power referenced to 1 ohm.

Defined as

$$\boxed{P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt}$$

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{+T/2} |x(t)|^2 \, dt$$

Energy signal :-

Signal which has finite energy ($0 < E < \infty$) and Zero power is called energy signal.

Power signal :-

Signal which has finite average power ($0 < P < \infty$) and infinite energy ($E = \infty$) is called power signal.

→ Almost all practical periodic signals are power signals since their avg. power is finite and non-zero.

Ex :- 1) Calculate energy of signal $x(t) = \begin{cases} 2 e^{-3t}, & t \geq 0 \\ 0, & \text{elsewhere} \end{cases}$

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

$$= \int_{0}^{\infty} |2 \cdot e^{-3t}|^2 \, dt$$

$$= 4 \cdot \int_{0}^{\infty} e^{-6t} \, dt$$

$$= 4 \cdot \left. \frac{e^{-6t}}{(-6)} \right]_0^{\infty}$$

$$= -2/3 \cdot \left[e^{-\infty} - e^{0} \right]$$

$$E = -\frac{2}{3} \cdot [0 - 1] = \frac{2}{3} \quad \text{(Ans)}$$

(Joule)

Ex :- 2 : Calculate power of the signal

$$V(t) = 2 \sin 0.5\pi t$$

Ans :

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{+T/2} |x(t)|^2 \, dt .$$

To find time period 'T'.

$$\omega \cancel{t} = 0.5\pi \cancel{t}$$

$$\Rightarrow \omega = \frac{\pi}{2}$$

$$\Rightarrow \frac{2\cancel{\pi}}{T} = \frac{\cancel{\pi}}{2}$$

$$\Rightarrow \boxed{T = 4}$$

$$P = \frac{1}{4} \int_{-4/2}^{4/2} 4 \sin^2 0.5\pi t \, dt$$

$$= \frac{1}{\cancel{4}} \int_{-2}^{+2} \cancel{4} \times \left(\frac{1 - \cos \pi t}{2}\right) dt$$

$$\left(\because \sin^2\theta = \frac{1 - \cos 2\theta}{2}\right)$$

$$= \frac{1}{2} \left[t - \frac{\sin \pi t}{\pi} \right]_{-2}^{2}$$

$$= \frac{1}{2} \left[(2 - 0) - (-2 - 0) \right]$$

$$= \frac{2 + 2}{2}$$

$$P = 2$$

Power = 2 watt

Classification of Signal:-

1) Real & Complex Signal:-

→ Real Signals have only real Components but Complex Signal has both real & imaginary Components.

→ Complex Signal represents both amplitude & Phase i.e delay information, So it has an important place in Commⁿ.

→ Real Signal Ex:- $V(t) = 2e^{-3t}$

→ Complex Signal Ex:- $V(t) = Ae^{j\omega t}$, $j = \sqrt{-1}$

Here magnitude of the signal $|V(t)| = A$.

and Phase information is given by $\angle V(t) = \omega t$.

2) Analog and digital Signal

Discussed earlier.

3) periodic and aperiodic signal:-

→ A periodic signal is repetitive in nature and is defined as, $V(t) = V(t+T)$, where T is the time period of repetition.

→ For discrete periodic signal the relⁿ to be obeyed is $v(n) = v(n+N)$ where N is an integer.

→ For aperiodic signal $V(t) \neq v(t+T)$ i.e it does n't repeat itself.

4) Even and odd Signal :-

→ An even signal is symmetric about $t=0$ axis or Y-axis such that $\boxed{v(t) = v(-t)}$,

→ An odd signal is anti-symmetric about $t=0$ axis such that $v(t) = -v(-t)$ or $v(-t) = -v(t)$

→ Any arbitrary signal $v(t)$ can be written as sum of even and odd parts.

$$v(t) = v_e(t) + v_o(t) \quad \text{where}$$

$$\cancel{v_e(t) = v(t) + v(-t)}$$

$$v_e(t) = \frac{v(t) + v(-t)}{2}$$

$$v_o(t) = \frac{v(t) - v(-t)}{2}$$

Proof :

$$v_e(-t) = \frac{v(-t) + v(t)}{2}$$

$$= \frac{v(t) + v(-t)}{2} = v_e(t)$$

$$v_o(-t) = \frac{v(-t) - v(t)}{2}$$

$$= -\left[\frac{v(t) - v(-t)}{2}\right]$$

$$v_o(-t) = -v_o(t)$$

Since $v_e(-t) = v_e(t)$ It is even signal.

$v_o(-t) = -v_o(t)$ It is odd signal

5) Energy & Power Signal.

Discussed earlier.

6) Deterministic & Random Signal :-

A deterministic signal is completely specified in any form be it mathematical, graphical etc. Ex:- $\boxed{v(t) = A \sin\omega t}$

→ Random signal related to random values and are described in probabilistic term like mean, variance, distribution function etc. The message signal in a communication problem is always random in nature.

7) Causal & non causal signal

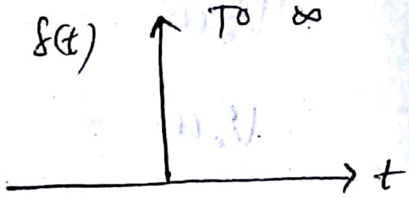
If a signal $x(t)=0$ for $t<0$ then that signal is called causal signal. If $x(t)\neq 0$ for $t<0$, then the signal is non causal.

Singularity functions:-

These functions are not finite or don't have finite derivatives everywhere.

(a) Unit impulse f^n / delta f^n / Dirac-delta f^n

$$\delta(t) = \begin{cases} \infty, & t=0 \\ 0, & t\neq 0 \end{cases}$$



and $\int_{-\infty}^{\infty} \delta(t)\,dt = 1$, $\int_{-\infty}^{\infty} v(t)\,\delta(t)\,dt = v(0)$

$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$ means area under the curve is always unity. $\delta(t)$ is a limiting case of rectangular pulse $v(t)$ whose width (say, ϵ) at origin and its height is $\frac{1}{\epsilon}$, while $\epsilon\to 0$, height becomes ∞.

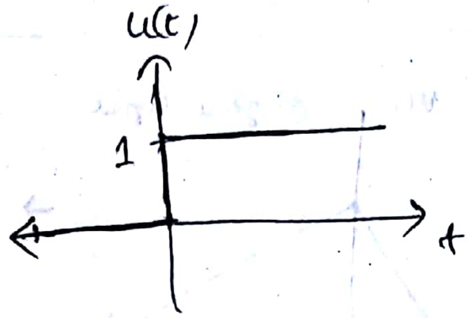
Area $= \epsilon \times \frac{1}{\epsilon} = 1$

$=$ width $\times$ height $= 1$

When width $\to 0$, height $\to \infty$.

(b) Unit-Step f^n

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$



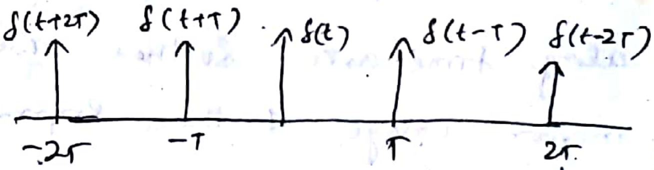
$u(t)$ doesn't have finite derivative at $t=0$.

Rel^n bet^n $\delta(t)$ and $u(t)$

$$\int_{-\infty}^{t} \delta(\tau)\, d\tau = u(t)$$ and

$$\frac{d}{dt} u(t) = \delta(t)$$

→ $\sum_{n=-\infty}^{\infty} \delta(t - nT)$ represents a periodic impulse train of time period T.



Shifting, Inversion, Scaling & Convolution of a Signal

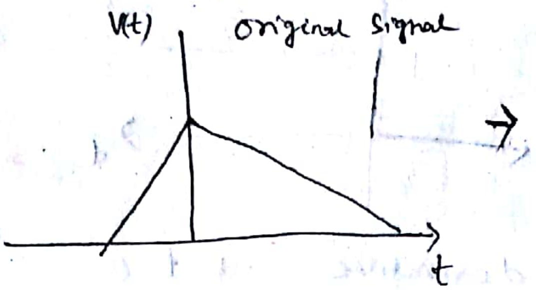
These are some of the useful signal properties or operations related to independent variable, time.

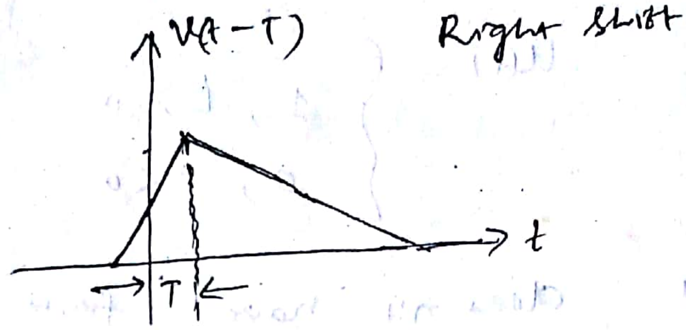
1) Time Shifting :-

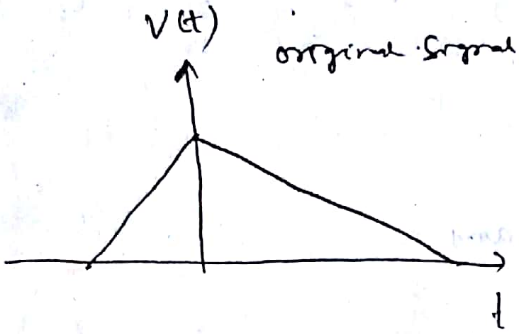
This is an operation by which signal is shifted in time using mathematical change of independent variable time.

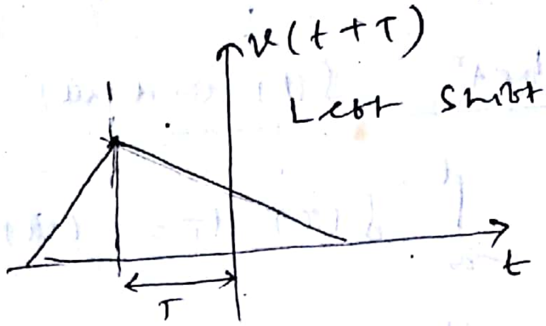
If $v(t)$ is original signal, then $w(t) = v(t+T)$ is a time shifted version of $v(t)$.

→ If T is −ve, then the signal is shifted towards right and if T is +ve it is shifted towards left.









Time inversion :-

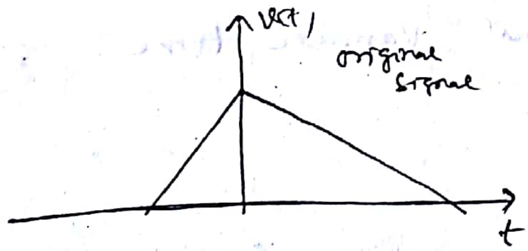
This operation inverts the signal along time axis so that we get a mirror image of the signal with $t=0$ being reflecting line.

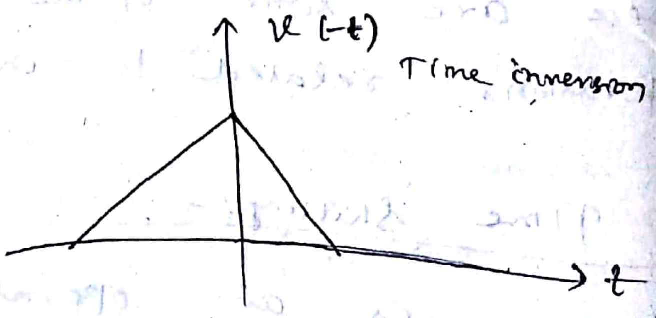
Note :-

$$\int_{-\infty}^{\infty} v(t)\, \delta(t)\, dt = v(0)$$

$$\int_{-\infty}^{\infty} v(\tau)\, \delta(t-\tau)\, d\tau = v(t)$$

If $v(t)$ is original signal then $x(t) = v(-t)$ is a time inverted version of $v(t)$.



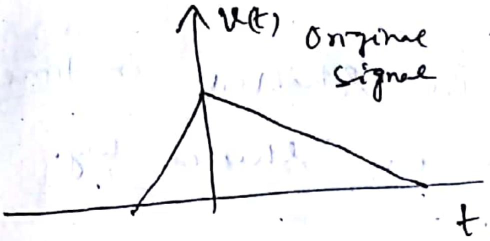


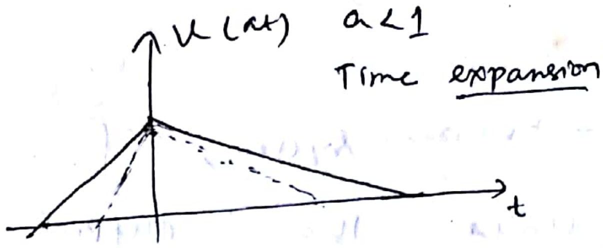
Time Scaling :-

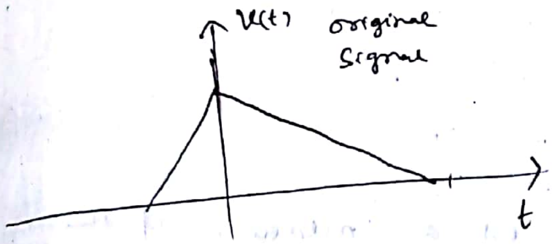
This operation compresses or expands a signal along time axis. If $v(t)$ is original signal then $\boxed{y(t) = v(at)}$ is a time scaled version of $v(t)$

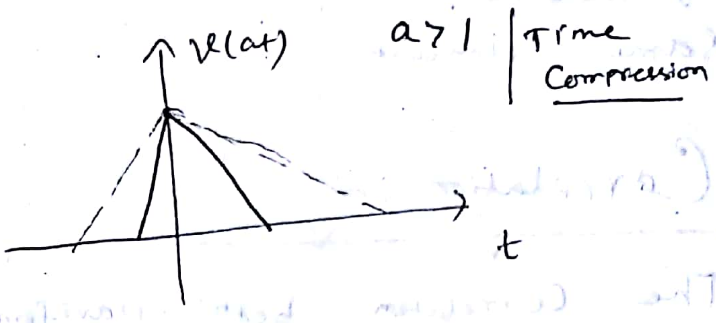
If $a<1$ then $y(t)$ is expanded version of $v(t)$ and if $a>1$ then $y(t)$ is compressed version of $v(t)$.

If $a=-1$, it gives time reversed signal of original.









Convolution :-

This is a process by which one signal is time reversed, shifted, multiplied with another signal and then its integral is calculated to generate a third signal.

→ Mathematically, if $v(t)$ and $h(t)$ are the signals being convolved, then convolved output $w(t)$ is given by

$$\boxed{w(t) = v(t) \otimes h(t) = \int_{-\infty}^{\infty} v(\tau)\, h(t-\tau)\, d\tau}$$

↑ Convolution operator

$$v(t) \otimes h(t) = h(t) \otimes v(t) \quad [\text{Commutative}]$$

The system which obeys this is known as linear-time-invariant system or LTI system.

Linear System are those for which principle of superposition is valid i.e if inputs $v_1(t)$ and $v_2(t)$ generate output $w_1(t)$ and $w_2(t)$ respectively then i/p $[a\, v_1(t) + b\, v_2(t)]$ will generate output $[a\, w_1(t) + b\, w_2(t)]$

→ For Time-invariant system, if i/p is shifted in time by T units then output too is shifted by same amount.

Correlation :-

→ The correlation betⁿ waveforms is a measure of the similarity or relatedness between the waveforms.

→ Suppose $v_1(t)$ and $v_2(t)$ are the waveforms, not necessarily periodic nor confined to finite time interval, then correlation betⁿ them or average Cross Correlation betⁿ $v_1(t)$ and $v_2(t)$ is $R_{12}(\tau)$ is

$\tau \rightarrow$ Shift to left by an amount 'τ'

given as,

$$\boxed{R_{12}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} v_1(t)\, v_2(t+\tau)\, dt}$$

If $v_1(t)$ and $v_2(t)$ are periodic with same fundamental period T_0, then the avg. cross-correlation

$$\boxed{R_{12}(\tau) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} v_1(t) . v_2(t+\tau)\, dt}$$

If $v_1(t)$ and $v_2(t)$ are waveforms of finite energy then

$$\boxed{R_{12}(\tau) = \int_{-\infty}^{\infty} v_1(t) \cdot v_2(t+\tau)\, dt}$$

$\tau \to$ Shifting of V_2 to left by an amount τ.

3) Q/ Prove that $R_{21}(\tau) = R_{12}(-\tau)$

Ans:

$$\boxed{R_{21}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} v_2(t) \cdot v_1(t+\tau)\, dt}$$

L.H.S

$$R_{12}(-\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} v_1(t) \cdot v_2(t-\tau)\, dt$$

put
$t - \tau = x$
$dt = dx$
$t = x + \tau$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} v_1(x+\tau) \cdot v_2(x)\, dx$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} v_2(x) \cdot v_1(x+\tau)\, dx$$

$= R_{21}(\tau)$ (R.H.S)

Power and Cross Correlation:-

Let $V_1(t)$ and $V_2(t)$ be waveforms which are not Periodic nor confined to a finite time interval. Suppose the normalized power of $V_1(t)$ is S_1 and the normalized power of $v_2(t)$ is S_2. Then the normalized power of ~~$V_1(t)$ and~~ $v_1(t) + V_2(t)$

or normalized power of $v_1(t) + v_2(t+\tau)$

$$S_{12} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \left[v_1(t) + v_2(t+\tau) \right]^2 dt$$

$$= \lim_{T\to\infty} \frac{1}{T} \left[\int_{-T/2}^{T/2} v_1^2(t)\, dt + \int_{-T/2}^{T/2} v_2^2(t+\tau)\, dt + 2\int_{-T/2}^{T/2} v_1(t) \cdot v_2(t+\tau)\, dt \right]$$

$$\boxed{S_{12} = S_1 + S_2 + 2R_{12}(\tau)}$$

[Note :– We have taken normalized power of $(v_2(t+\tau))$ is same as normalized power of $v_2(t)$. since the integration extends eventually over entire time axis, a time shift in v_2 will not clearly affect the value of integral] → If 2 waveforms $\boxed{\text{uncorrelated}, R_{12}(\tau) = 0}$

Auto Correlation :–

The Correlation of a function with itself as Called auto correlation. $v_1(t) = v_2(t)$, $R_{12}(\tau) \to R(\tau)$

$$R(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} v(t) \cdot v(t+\tau)\, dt$$

(a) $R(0) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} [v(t)]^2\, dt$ = Avg. Power.

(b) $R(0) \geq R(\tau)$ [∵ Similarity bet^n $v(t)$ & $v(t+\tau)$ is max^m when $\tau = 0$]

(c) $R(\tau) = R(-\tau)$ (d) $R_1(0) = \int_{-\infty}^{\infty}\{v_1(t)\}^2 dt$ [using 2nd formula of cross correlation]

Proof:- L.H.S.

$$R(-\tau) = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} v(t)\cdot v(t-\tau)\,dt$$

$$= \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} v(x+\tau)\cdot v(x)\,dx$$

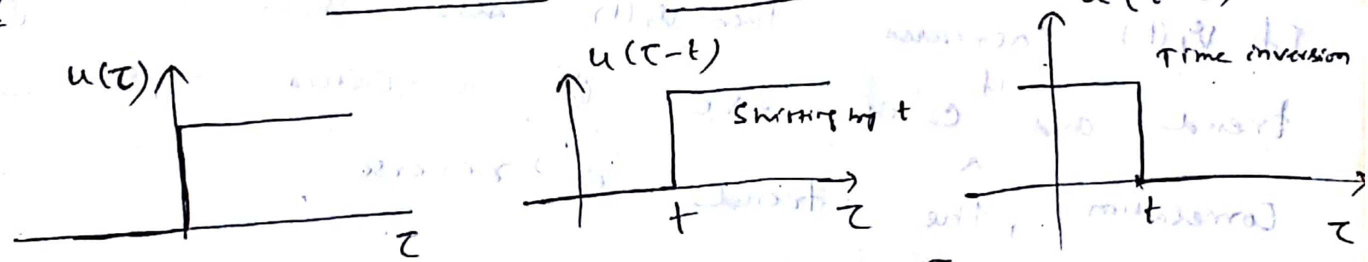
Put $t-\tau = x$, $dt = dx$, $t = x+\tau$

$$= \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} v(x)\cdot v(x+\tau)\,dx$$

$= R(\tau)$ (R.H.S).

Thus Auto correlation function is an even f^n of 'τ'

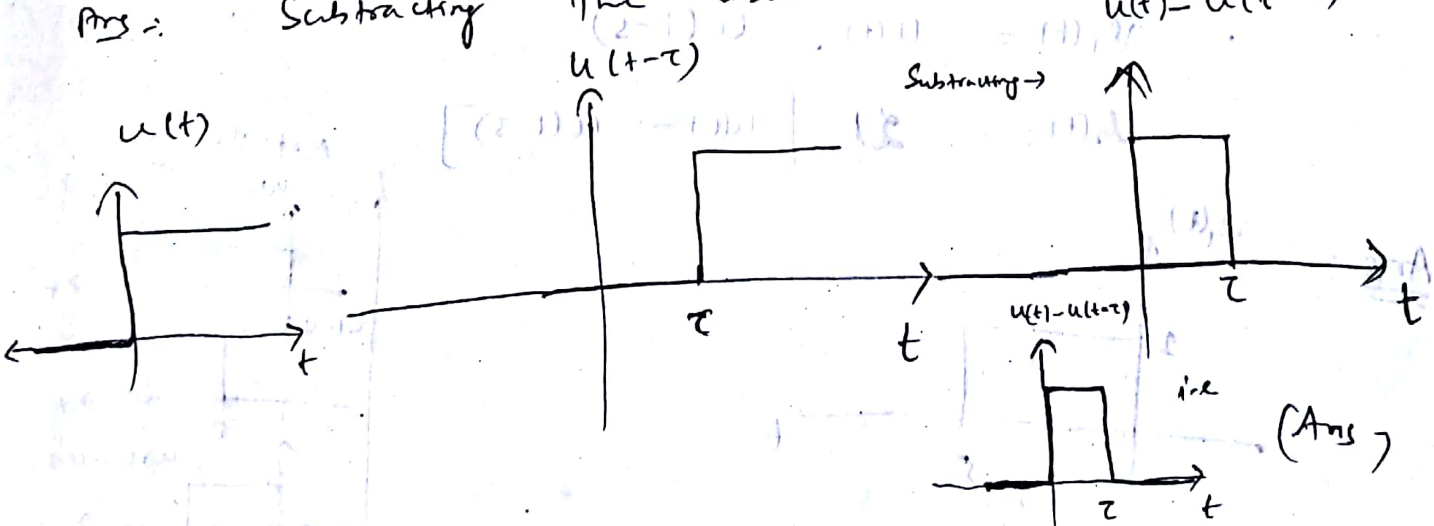
4) Q) Plot $u(t-\tau)$ $\boxed{\text{against } \tau}$, $\boxed{\tau \text{ is variable}}$



(Ans)

5) Plot $u(t) - u(t-\tau)$, where $\boxed{\tau \text{ is +ve const.}}$

Ans:- Subtracting the below waveforms



(Ans)

Correlation Coefficient :-

Correlation Coefficient is also used as a measure of similarity. For 2 signals $v_1(t)$ and $v_2(t)$ with energy E_1 and E_2 respectively, it is defined as

$$C = \frac{1}{\sqrt{E_1 E_2}} \int_{-\infty}^{\infty} v_1(t) \cdot v_2(t)\, dt = \frac{R_{12}(0)}{\sqrt{E_1 E_2}}$$

If $v_1(t) = v_2(t)$, $C = 1$

If $v_1(t) = -v_2(t)$, $C = -1$

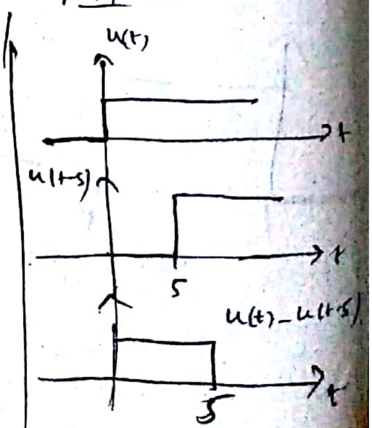
If 'C' is +ve, it is called +ve correlation i.e. if $v_1(t)$ increases then $v_2(t)$ also shows increasing trend and if C is -ve it is called -ve correlation, the trend is reverse.

Ex :- 6) Find Cross-Correlation term $R_{12}(0)$, auto correlation term $R_1(0)$, $R_2(0)$ and energy E_1, E_2 of the 2 signals defined as

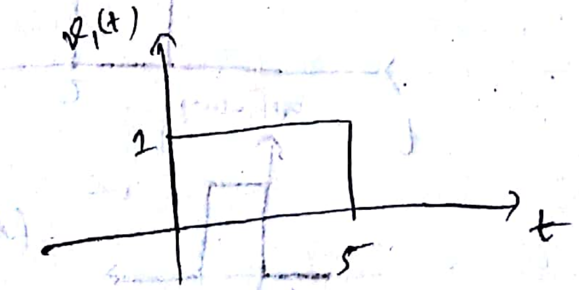
$$v_1(t) = u(t) - u(t-5)$$

$$v_2(t) = 2t\,[u(t) - u(t-3)]$$

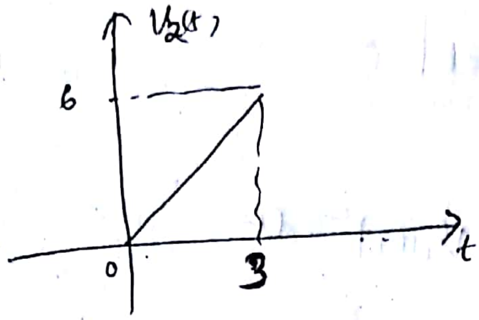
Note :-

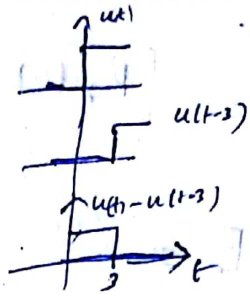


Ans :-



Similarly





$V_2(t) = 2t \times 1 \quad 0<t<3$

$$R_{12}(\tau) = \int_{-\infty}^{\infty} v_1(t)\, v_2(t+\tau)\, dt$$

$$\cancel{R_{12}(0) = \int_{0}^{3} 2t\, dt}$$

$$R_{12}(0) = \int_0^3 v_1(t) \cdot v_2(t)\, dt$$

[∵ Common area of $v_1(t)$ & $v_2(t)$ is ~~...~~ $0<t<3$]

$$= \int_0^3 (1)(2t)\, dt$$

$$= \cancel{2} \cdot \frac{t^2}{\cancel{2}}\Big]_0^3$$

$$\boxed{R_{12}(0) = 9}$$

$$R_1(0) = \int_0^5 [v_1(t)]^2\, dt$$

$$= \int_0^5 1\, dt$$

$$\boxed{R_1(0) = 5}$$

$$R_2(0) = \int_0^3 (2t)^2\, dt$$

$$= 4 \cdot \frac{t^3}{3}\Big]_0^3$$

$$= \frac{4}{\cancel{3}}[\cancel{27}]\, 9 = 36$$

$\boxed{R_2(0) = 36}$

$$E_1 = \int_{-\infty}^{\infty} (v_1(t))^2 \, dt$$

$$= \int_0^5 1 \, dt = 5 \qquad \boxed{E_1 = 5}$$

$$E_2 = \int_{-\infty}^{\infty} [v_2(t)]^2 \, dt$$

$$= \int_0^3 (2t)^2 \, dt = \int_0^3 4t^2 \, dt$$

$$\cancel{= 2 \cdot \frac{t^2}{3} \Big]_0^3}$$

$$E_2 = 4 \cdot \frac{t^3}{3} \Big]_0^3 = \frac{4}{\cancel{3}} \times \cancel{27}^{9} = 36.$$

$$\boxed{E_2 = 36}$$

C = Correlation Coefficient

$$= \frac{R_{12}(0)}{\sqrt{E_1 \times E_2}}$$

$$= \frac{9}{\sqrt{5 \times 36}}$$

$$\boxed{C = 0.6711}$$

7) Find $R_{12}(-1)$ and $R_{12}(1)$

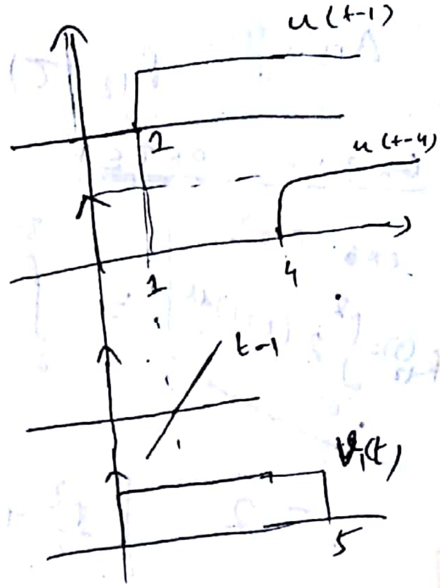
$$R_{12}(\tau) = \int_{-\infty}^{\infty} v_1(t) \, v_2(t+\tau) \, dt$$

$$R_{12}(-1) = \int_{-\infty}^{\infty} v_1(t) \, v_2(t-1) \, dt$$

$V_2(t) = 2t\left[u(t) - u(t-3)\right]$

$V_2(t-1) = 2(t-1)\left[u(t-1) - u(t-4)\right]$

overlapping is $1 \le t \le 4$.



$$R_{12}(-1) = \int_1^4 2(t-1) \cdot 1 \, dt$$

$$= 2\left[\frac{t^2}{2} - t\right]_1^4$$

$$= 2\left[\left(\frac{16}{2} - 4\right) - \left(\frac{1}{2} - 1\right)\right]$$

$$= 2\left[8 - 4 - 0.5 + 1\right]$$

$$R_{12}(-1) = 2 \times 4.5$$

$$\boxed{R_{12}(-1) = 9}$$

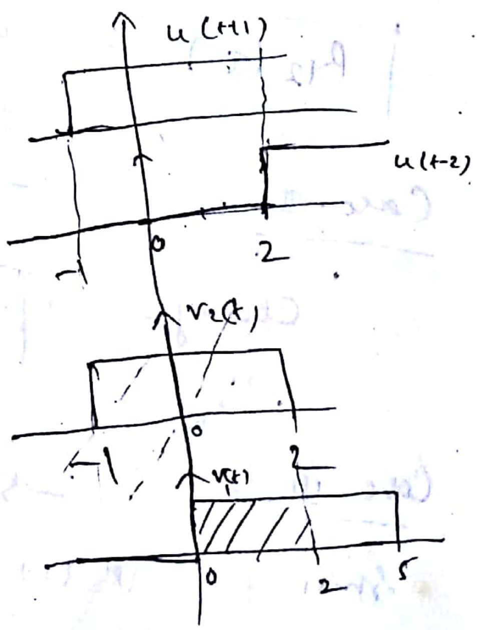
$$R_{12}(1) = \int_{-\infty}^{\infty} v_1(t)\, v_2(t+1)\, dt$$

$V_2(t) = 2t\left[u(t) - u(t-3)\right]$

$V_2(t+1) = 2(t+1)\left[u(t+1) - u(t-2)\right]$

$V_1(t) = u(t) - u(t-5)$

overlapping $v_2(t+1)$ & $v_1(t)$ is
$0 \le t \le 2$.



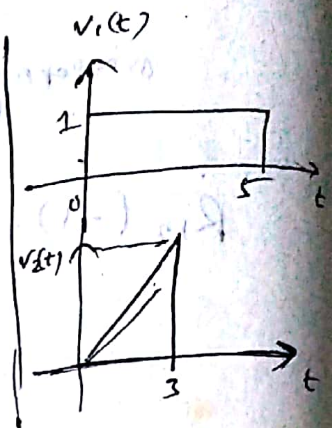
$$R_{12}(1) = \int_0^2 (1)\left[2(t+1)\right] dt$$

$$= 2\left[\frac{t^2}{2} + t\right]_0^2$$

$$= 2\left[2 + 2\right] = 8$$

$R_{12}(1) = 8$

87) Find $R_{12}(\tau)$ for $v_1(t)$ & $v_2(t)$.



$\therefore v_2(t) = 2t$

$v_2(t+\tau) = 2(t+\tau)$

Ans: $R_{12}(\tau) = \int_0^{T} v_1(t) \cdot v_2(t+\tau)\, dt$

Case-I :- $0 \le \tau \le 3$

ex: $\tau = 1$

$R_{12}(1) = \int_0^2 2(t+1)\,dt$

$= \int_0^{3-\tau} 1 \cdot 2(t+\tau)\, dt$

$= 2\left[\frac{t^2}{2} + \tau \cdot t\right]_0^{3-\tau}$

$= \cancel{2\left[\frac{9}{2} + 3\tau\right]}$

$= 2\left[\frac{(3-\tau)^2}{2} + \tau \cdot (3-\tau)\right]$

$= (3-\tau)^2 + 2\tau(3-\tau)$

$= 9 + \tau^2 \cancel{-6\tau} + \cancel{6\tau} - 2\tau^2$

$$\boxed{R_{12}(\tau) = 9 - \tau^2}$$

Case-II $-2 \le \tau \le 0$, overlap does not change $\boxed{R_{12}(\tau) = R_{12}(0) = 9}$

Case-III :- $-5 \le \tau \le -2$, there is greater shift of $v_2(t)$ and overlap can be written in terms of τ.

$$R_{12}(\tau) = \int_{-\tau}^{5} 1 \cdot 2(t+\tau)\, dt$$

$$= 2\left[\frac{t^2}{2} + \tau \cdot t\right]_{-\tau}^{5}$$

$$= t^2 + 2t\tau \Big]_{-\tau}^{5}$$

$$= (5^2 + 2\cdot 5\cdot \tau) - (\tau^2 - 2\tau^2)$$

$$= 25 + 10\tau - \tau^2 + 2\tau^2$$

$$\boxed{R_{12}(\tau) = \tau^2 + 10\tau + 25}$$

Ex → $\tau = -3$

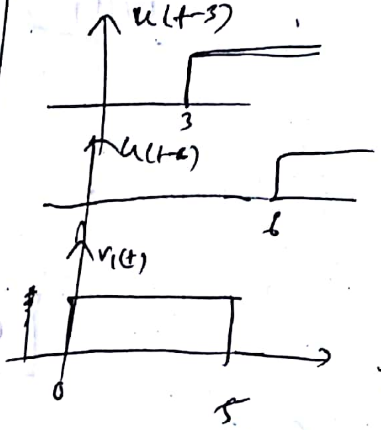
$v_2(t-3)$

~~$= 2(t-3)$~~

~~$[u(t-3) - u(t-6)]$~~

~~$= 2t($~~

$= 2(t-3)\,[u(t-3) - u(t-6)]$



Common overlapping 3 to 5 i.e

$-\tau \le t \le 5$

9) Find Convolution of $v_1(t)$ & $v_2(t)$.

Ans :- $w(t) = v_1(t) \otimes v_2(t)$

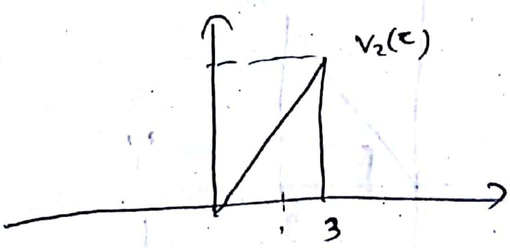
$= v_2(t) \otimes v_1(t)$

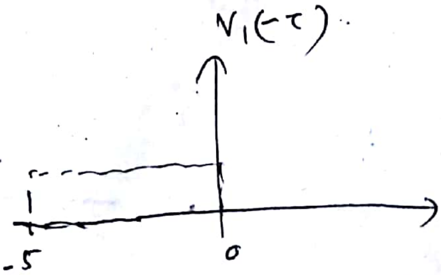
$$= \int_{-\infty}^{\infty} v_2(\tau)\, v_1(t-\tau)\, d\tau$$

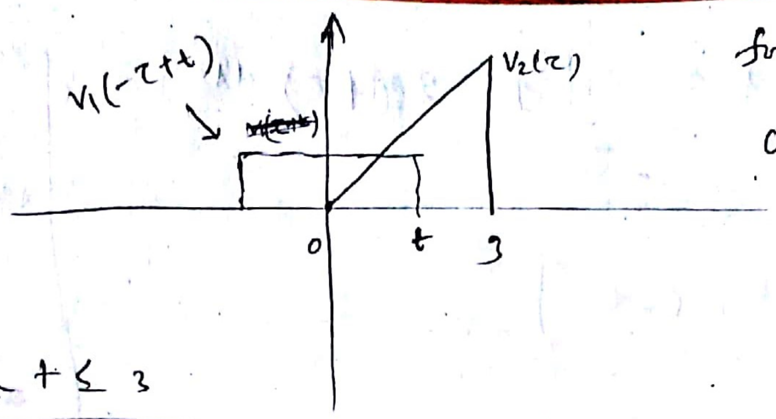
$\tau \rightarrow$ Variable

$t \rightarrow$ constant

Case-I :- $0 \le t \le 3$.







for $0 \le t \le 3$

Overlapping 0 to t

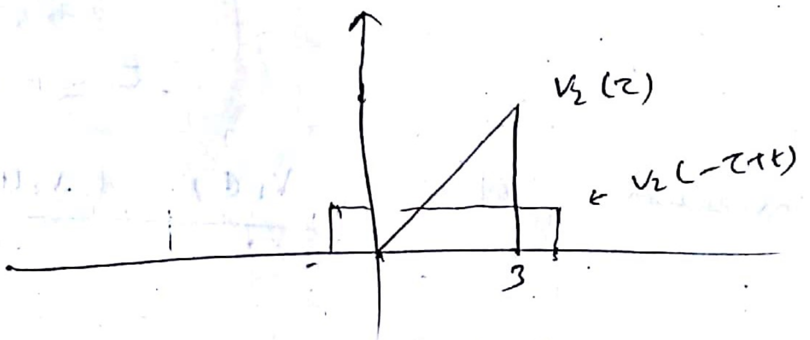
$0 \le t \le 3$

$$w(t) = \int_0^t 2\tau \cdot 1 \cdot d\tau$$

$$= \cancel{2} \times \frac{\tau^2}{\cancel{2}}\Big]_0^t = t^2$$

$$\boxed{w(t) = t^2}$$

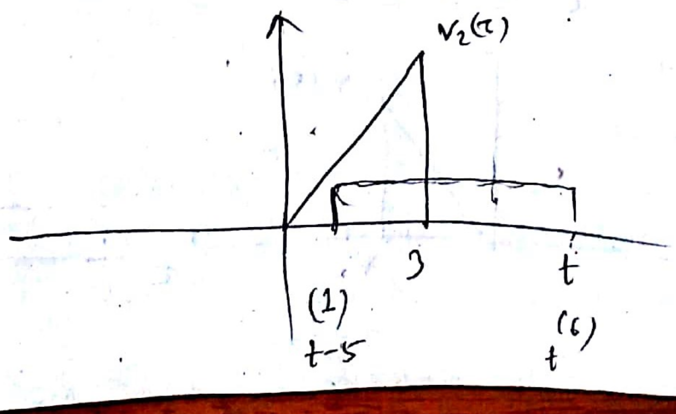
Case-II $3 \le t \le 5$



Overlapping 0 to 3

$$w(t) = \int_0^3 (2\tau)(1)\, d\tau = \cancel{2}\, \frac{\tau^2}{\cancel{2}}\Big]_0^3 = 9$$

Case-III $5 \le t \le 8$ (Say $t = 6$)



Overlapping

$t-5$ to 3

$$W(t) = \int_{t-5}^{3} (2\tau).\,1\,d\tau$$

$$= \cancel{2}.\ \frac{\tau^2}{\cancel{2}}\Big]_{t-5}^{3}$$

$$= 9 - (t-5)^2$$

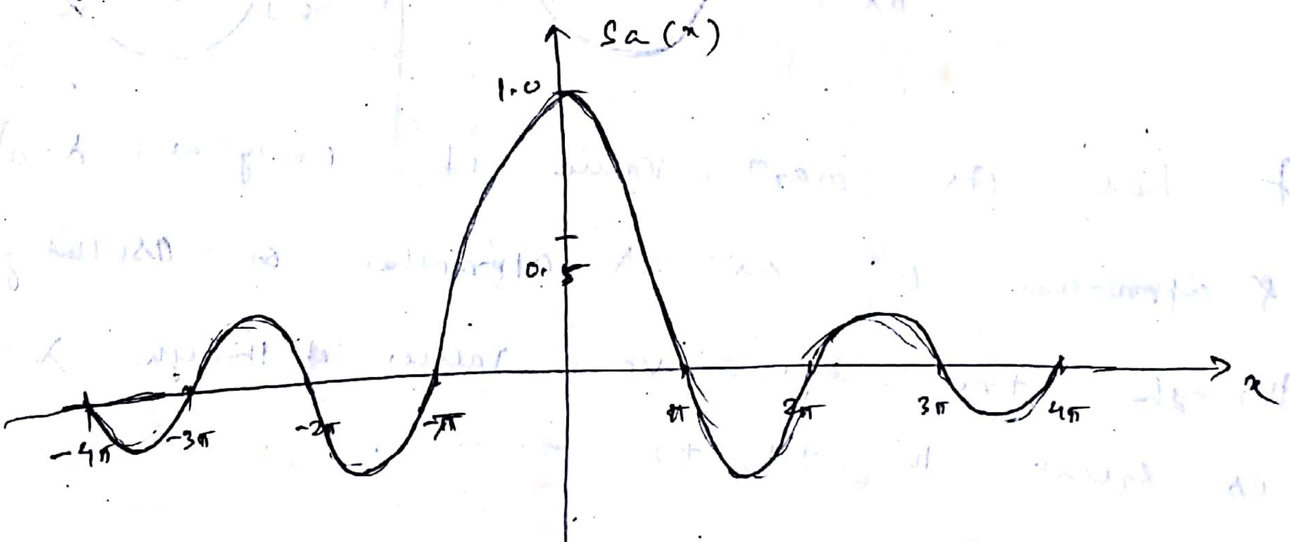
$$= 9 - t^2 - 25 + 10t$$

$$\boxed{W(t) = 10t - t^2 - 16}$$

For further t, $\boxed{W(t) = 0}$

Sampling function:-

$$Sa(x) = \frac{\sin x}{x}$$



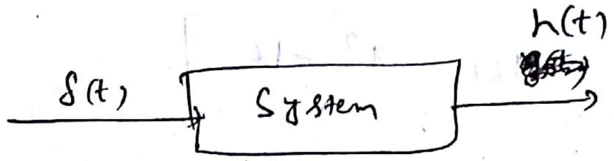
→ It is symmetrical about $x=0$, At $x=0$, has value $Sa(0) = 1$.

→ It oscillates with an amplitude that decreases with increasing x.

→ The function passes through zero at equally spaced intervals at values of $x = \pm n\pi$, where n is an integer other than zero.

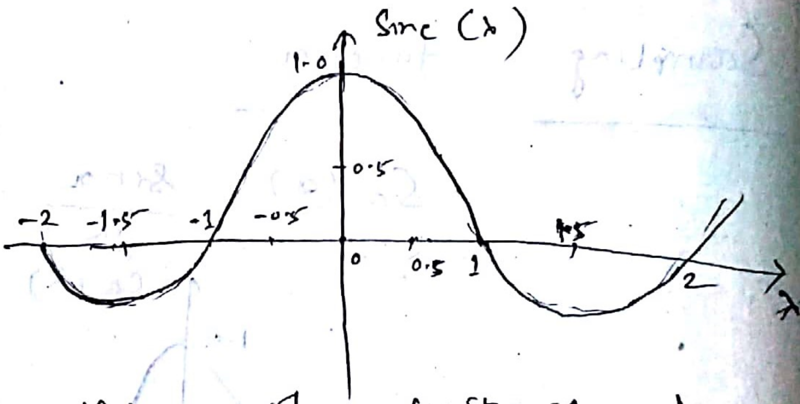
Impulse Response:-

Impulse response is output generated by system when $\delta(t)$ is input.



Here $\delta(t)$ is input & $h(t)$ is the impulse response.

Sinc function:-



$$\text{Sinc}(\lambda) = \frac{\sin(\pi\lambda)}{\pi\lambda}$$

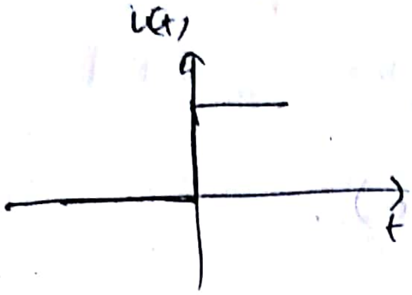
It has its max^m value of unity at $\lambda = 0$ & approaches '0' as λ approaches ∞ oscillating through +ve and -ve values through λ is equal to $\pm 1, \pm 2$ - - -

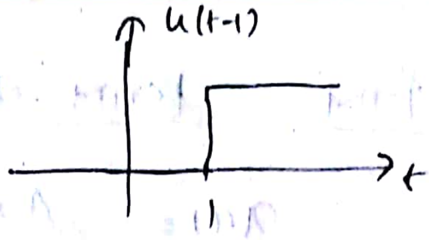
Ex:- 10) Plot

(a) $U(t-1)$ (b) $U(t+1)$ (c) $U(-t-1)$ (d) $U(-t+1)$

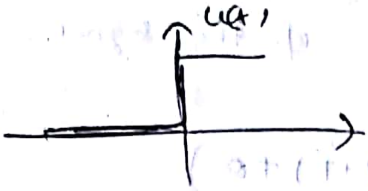
(e) $-2U(t-1)$ (f) $-U(t)$ (g) $-U(t+1)$

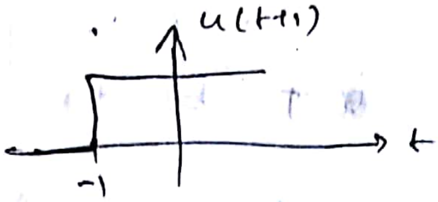
(a)


u(t)
t


u(t-1)
t
1

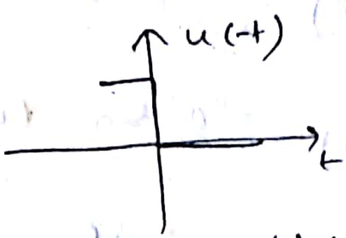
(b)

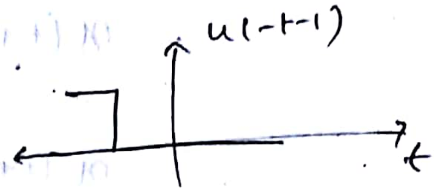

u(t)


u(t+1)
t
-1

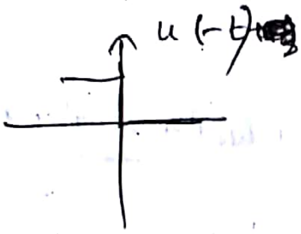
(c)

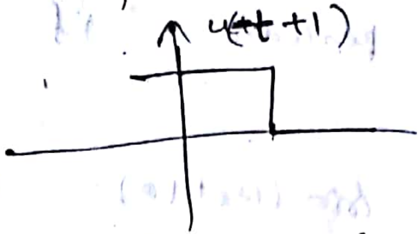

u(t)
t


u(-t)
t

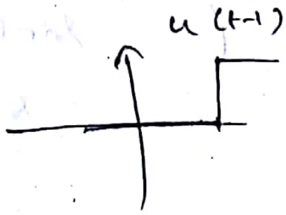

u(-t-1)
t

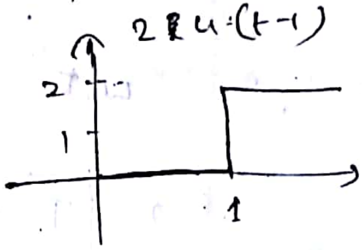
(d)

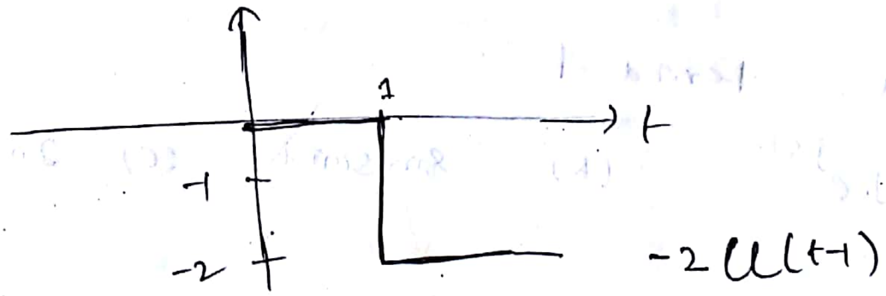

u(-t)


u(-t+1)

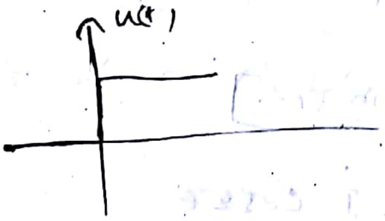
(e)

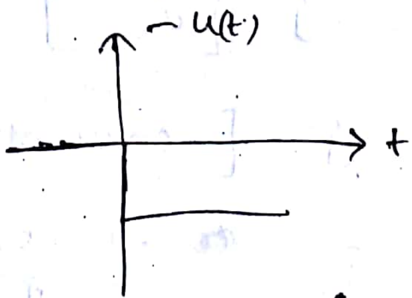

u(t-1)


2u(t-1)
2
1
1

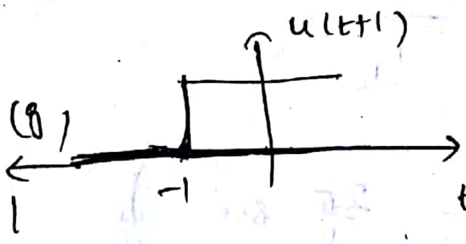

1
t
-1
-2
-2u(t-1)

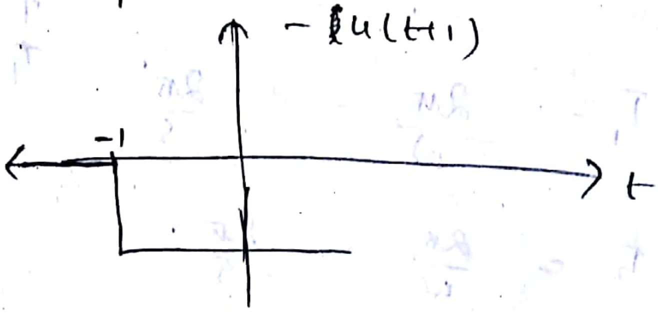
(f)


u(t)


-u(t)
t

(g)


u(t+1)
-1
t


-u(t+1)
-1
t

11) Find period of

$$x(t) = A\sin(\omega_0 t + \theta).$$

Ans: Let T be the period of the signal.

$$x(t+T) = A\sin(\omega_0(t+T) + \theta)$$

$$x(t+T) = A\sin(\omega_0 t + \omega_0 T + \theta)$$

$x(t)$ is periodic if, $x(t) = x(t+T)$

$$\Rightarrow A\sin(\omega_0 t + \theta) = A\sin(\omega_0 t + \underline{\omega_0 T} + \theta)$$

$$\therefore \omega_0 T = 2\pi \qquad (\because \sin(2\pi + \theta) = \sin\theta)$$

$$\Rightarrow \boxed{T = \frac{2\pi}{\omega_0}}$$

12) Find period of

(a) $x(t) = j e^{j5t}$ (b) $\sin 50\pi t$ (c) $20\cos(10\pi t + \frac{\pi}{6})$

Ans: (a)

$$x(t) = j\left[e^{j5t}\right]$$

$$= j\left[\cos 5t + j\sin 5t\right]$$

$$= \underbrace{-\sin 5t}_{T_1} + \underbrace{j\cos 5t}_{T_2}$$

$$T_1 = \frac{2\pi}{\omega} = \frac{2\pi}{5}$$

$$T_2 = \frac{2\pi}{\omega} = \frac{2\pi}{5}$$

$$\boxed{T = \frac{2\pi}{5}\text{ sec.}}$$

(b) $\sin 50\pi t$

$$T = \frac{2\pi}{50\pi} = \frac{1}{25} \text{ sec.}$$

(c) $20\cos(10\pi t + \frac{\pi}{2})$

$$T = \frac{2\pi}{10\pi} = \frac{1}{5} \text{ sec.}$$

13). (a) $4\cos 5\pi t$, (b) $\sin 10\pi t + u(t)$, (c) $e^{|t|}$

(a) $\frac{2\pi}{5\pi} = \frac{2}{5}$ sec

(b) Not a periodic signal.

(c) Not a periodic signal.

14) Imp If $x_1(t)$ & $x_2(t)$ are periodic signals with time period T_1 & T_2 respectively, then $x_1(t) + x_2(t)$ will be periodic, if

$\frac{T_1}{T_2}$ = rational number i.e ratio of 2 integers.

Ex :- 15) (a) $x(t) = 2\cos(10t+1) - \sin(4t-1)$

$$T_1 = \frac{2\pi}{10} = \frac{\pi}{5}, \quad T_2 = \frac{2\pi}{4} = \frac{\pi}{2}.$$

$$\frac{T_1}{T_2} = \frac{\pi/5}{\pi/2} = \frac{\pi}{5} \times \frac{2}{\pi} = \frac{2}{5} = \text{rational number.}$$

$\therefore T = 5T_1 = 2T_2$ i.e $5 \times \frac{\pi}{5} = \pi$

or $2 \times \frac{\pi}{2} = \pi$ sec.

(b) $x(t) = 2u(t) + 2\sin 2t$

A: aperiodic + periodic →

$\therefore$ $x(t)$ is aperiodic.

(c) $x(t) = 3\cos 4t + 2\sin \pi t$

$T_1 = \frac{2\pi}{4}$, $T_2 = \frac{2\pi}{\pi} = 2$

$T_1 = \frac{\pi}{2}$, $T_2 = 2$

$\frac{T_1}{T_2} = \frac{\frac{\pi}{2}}{2} = \frac{\pi}{4}$ is not a rational number

$\therefore$ $x(t)$ is aperiodic.

(d) $\sin^2 t$

A. $\sin^2 t = \frac{1 - \cos 2t}{2}$ $T = \frac{2\pi}{2} = \pi$ sec.

Fourier Series

→ A periodic function of time $x(t)$ having a fundamental period T can be represented as an infinite sum of sinusoidal waveforms. This summation called a Fourier series.

→ Fourier series is used to get frequency spectrum of a time domain signal.

→ Fourier series exist only when the function $x(t)$ satisfies the Dirichlet's condⁿ.

Dirichlet's condⁿ :-

1) $x(t)$ is absolutely integrable over its period

i.e. $\int_0^T |x(t)|\,dt < \infty$

2) The number of maxima and minima of $x(t)$ in each period is finite.

3) The number of discontinuities of $x(t)$ in each period is finite.

Fourier series expansion are of 3 types :-

(i) Trigonometric Fourier series
(ii) Polar Fourier series
(iii) Complex Fourier exponential series.

(i) Trigonometric Fourier series

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) \right]$$

where

$$a_0 = \frac{1}{T} \int_{t_0}^{t_0+T} x(t)\, dt$$

$$a_n = \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \cos(n\omega_0 t)\, dt$$

$$b_n = \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \sin(n\omega_0 t)\, dt.$$

where T = Fundamental time period.

If the interval is $[-\infty, \infty]$

$$a_0 = \frac{1}{T} \int_{-\infty}^{\infty} x(t)\, dt, \quad a_n = \frac{2}{T} \int_{-\infty}^{\infty} x(t) \cos(n\omega_0 t)\, dt$$

$$b_n = \frac{2}{T} \int_{-\infty}^{\infty} x(t) \sin(n\omega_0 t)\, dt;$$

Polar Fourier series: -

$$x(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t - \phi_n)$$

where $\phi_n = \tan^{-1}\left(\frac{b_n}{a_n}\right)$, $C_0 = a_0$

$$C_n = \sqrt{a_n^2 + b_n^2}$$

C_n = Spectral amplitude, ϕ_n = Phase information

Phase spectrum.

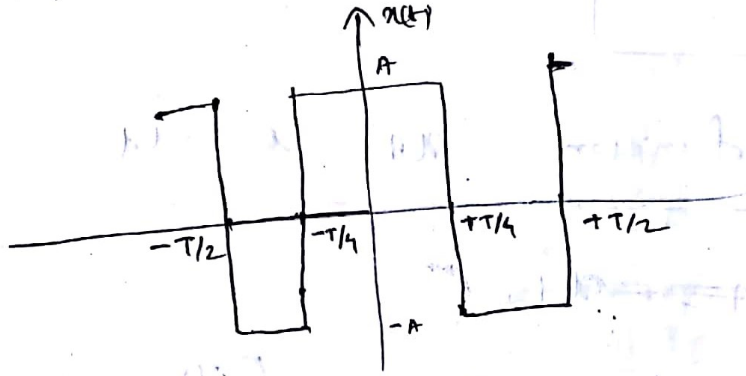
Complex Fourier exponential series: -

$$x(t) = \sum_{n=-\infty}^{\infty} C_n \, e^{jn\omega_0 t}$$

where $$C_n = \frac{1}{T} \int_{-T/2}^{T/2} x(t) \cdot e^{-jn\omega_0 t} \, dt$$

or $$C_n = \frac{a_n - j b_n}{2}$$

<u>Ex:-1)</u> Find the Fourier Series representation of the figure shown below.



<u>Notes:-</u> <u>Symmetry Condn</u>

1) If function $x(t)$ is even

(i) $x(t) = x(-t)$

ex:- t^2, $\cos t$, $t \sin t$

(ii) Symmetric about y-axis.

(iii)

(iv) $a_0 = \frac{1}{T} \int_{-T/2}^{T/2} x(t) \, dt \Rightarrow \boxed{a_0 = \frac{2}{T} \int_{0}^{T/2} x(t) \, dt}$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} \underset{\text{even}}{x(t)}\ \underset{\text{even}}{\cos n\omega_0 t}\, dt$$

$$\boxed{a_n = \frac{4}{T}\int_{0}^{T/2} x(t)\cos n\omega_0 t \, dt}$$

$$b_n = \frac{1}{T}\int_{-T/2}^{T/2} \underbrace{\underset{\text{even}}{x(t)}\cdot \underset{\text{odd}}{\sin n\omega_0 t}}_{\text{odd}}\, dt$$

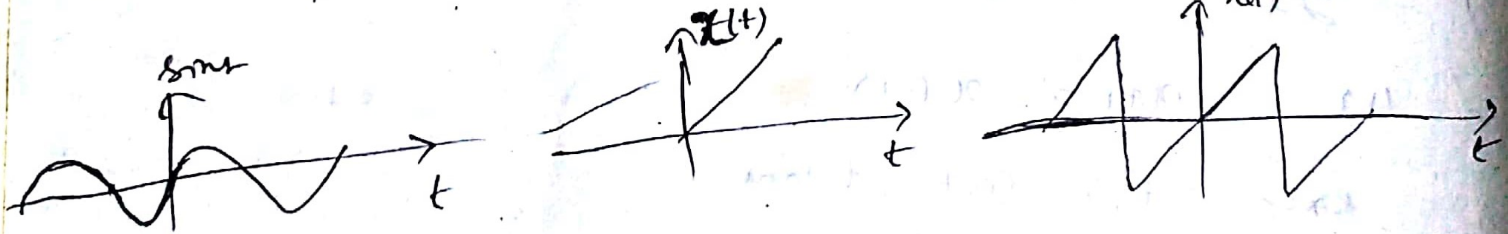
$$\boxed{b_n = 0}$$

2) If function $x(t)$ is odd

(i) ~~$x(t) = dt$~~.

$x(-t) = -x(t)$ (They are symmetric about origin)

(ii) ex:- $\sin t$, t, t^3, $\cos t$



(iii) $a_0 = \frac{1}{T}\int_{-T/2}^{T/2} \underset{\text{odd}}{x(t)}\, dt = 0$

$\boxed{a_0 = 0}$

$$a_n = \frac{1}{T}\int_{-T/2}^{T/2} \underbrace{\underset{\text{odd}}{x(t)}\ \underset{\text{even}}{\cos n\omega_0 t}}_{\text{odd}}\, dt = 0$$

$$\boxed{a_n = 0}$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} \overbrace{\underset{\text{odd}}{x(t)}\ \underset{\text{odd}}{\sin(n\omega_0 t)}}^{\text{even}}\, dt$$

$$\boxed{b_n = \frac{4}{T}\int_{0}^{T/2} x(t)\ \sin(n\omega_0 t)\, dt}$$

<u>Summary :-</u>

$x(t)$ = even ;	$a_0 = \frac{2}{T}\int_0^{T/2} x(t)\,dt$, $a_n = \frac{4}{T}\int_0^{T/2} x(t)\cos n\omega_0 t\,dt$ $b_n = 0$
$x(t)$ = odd ,	$a_0 = 0$, $a_n = 0$ $b_n = \frac{4}{T}\int_0^{T/2} x(t).\sin(n\omega_0 t)\,dt$

3) Sum or product of two or more even function is even.

4) sum of 2 or more odd f^n is odd.

5) Product of 2 odds function is even.

Ans :- to EX-1 Since the function is symmetrical about y-axis, it is even function.

$$a_0 = \frac{2}{T}\int_0^{T/2} x(t)\, dt$$

$$= \frac{2}{T}\left[\int_0^{+T/4} x(t)\,dt + \int_{+T/4}^{T/2} x(t)\,dt\right]$$

$$a_0 = \frac{2}{T}\left[\int_0^{T/4} A\,dt + \int_{T/4}^{T/2} -A\,dt\right]$$

$$= \frac{2}{T}\left[A\,(t)\Big]_0^{T/4} - A\,(t)\Big]_{T/4}^{T/2}\right]$$

$$= \frac{2A}{T}\left[\left(\frac{T}{4}-0\right) - \left(\frac{T}{2}-\frac{T}{4}\right)\right]$$

$$a_0 = \frac{2A}{T}\left(\cancel{\frac{T}{4}} - \cancel{\frac{T}{2}} + \cancel{\frac{T}{4}}\right]$$

$$\Rightarrow \boxed{a_0 = 0}$$

$$a_n = \frac{4}{T}\int_0^{T/2} x(t)\cos n\omega_0 t\,dt$$

$$= \frac{4}{T}\left[\int_0^{T/4} A\cos n\omega_0 t\,dt + \int_{T/4}^{T/2} -A\cos n\omega_0 t\,dt\right]$$

$$= \frac{4A}{T}\left[\frac{\sin n\omega_0 t}{n\omega_0}\Big]_0^{T/4} - \frac{\sin n\omega_0 t}{n\omega_0}\Big]_{T/4}^{T/2}\right]$$

$$= \frac{4A}{Tn\omega_0}\left[\left(\sin n\cdot\frac{2\pi}{\cancel{T}}\cdot\frac{\cancel{T}}{\cancel{4}_2} - 0\right) - \left(\sin n\cdot\frac{2\pi}{\cancel{T}}\cdot\frac{\cancel{T}}{\cancel{2}} - \sin n\cdot\frac{2\pi}{\cancel{T}}\cdot\frac{\cancel{T}}{\cancel{4}_2}\right)\right]$$

$$\left(\because \omega_0 = \frac{2\pi}{T}\right)$$

$$= \frac{4A}{\cancel{T}\cdot n\times\frac{2\pi}{\cancel{T}}}\left[\sin\frac{n\pi}{2} - \cancel{\sin n\pi}^{\,0} + \sin\frac{n\pi}{2}\right]$$

$$\Rightarrow \quad a_n = \frac{4A}{\cancel{2}n\pi}\left[\cancel{2}\sin\frac{n\pi}{2}\right]$$

$$\boxed{a_n = \frac{4A}{n\pi}\sin\left(\frac{n\pi}{2}\right)}$$

Since even function, $\boxed{b_n = 0}$

$$\therefore \quad x(t) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos n\omega_0 t + b_n \sin n\omega_0 t\right]$$

$$= 0 + \sum_{n=1}^{\infty} \frac{4A}{n\pi}\sin\left(\frac{n\pi}{2}\right)\cos n\omega_0 t + 0$$

$$\Rightarrow \quad \boxed{x(t) = \sum_{n=1}^{\infty} \frac{4A}{n\pi}\sin\left(\frac{n\pi}{2}\right)\cos n\omega_0 t}$$

$$\therefore \quad x(t) = \frac{4A}{\pi}\left(\frac{1}{1}\cdot\sin\frac{\pi}{2}\cdot\cos\omega_0 t + \frac{1}{2}\sin\pi\cdot\cos 2\omega_0 t + \frac{1}{3}\sin\frac{3\pi}{2}\cdot\cos 3\omega_0 t + \frac{1}{4}\cdot\sin 2\pi\cdot\cos 4\omega_0 t + \frac{1}{5}\cdot\sin\left(\frac{5\pi}{2}\right)\cdot\cos 5\omega_0 t + \cdots\right)$$

$$\Rightarrow \quad \boxed{x(t) = \frac{4A}{\pi}\left[\cos\omega_0 t - \frac{1}{3}\cos 3\omega_0 t + \frac{1}{5}\cos 5\omega_0 t - \cdots\right]}$$

(b) Polar form :-

$$x(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t - \phi_n)$$

$C_0 = a_0 = 0$, $C_n = \sqrt{a_n^2 + b_n^2} = \sqrt{a_n^2 + 0}$

$b_n = 0$ $\quad$ $C_n = a_n$

$$x(t) = 0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t - \phi_n)$$

$$\phi_n = \tan^{-1}\left(\frac{b_n}{a_n}\right) = \tan^{-1}\left(\frac{0}{a_n}\right) = 0$$

$$\therefore \boxed{x(t) = \sum_{n=1}^{\infty} \frac{4A}{n\pi} \sin\frac{n\pi}{2} \cdot \cos(n\omega_0 t)}$$

(c) Exponential Fourier Series :-

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

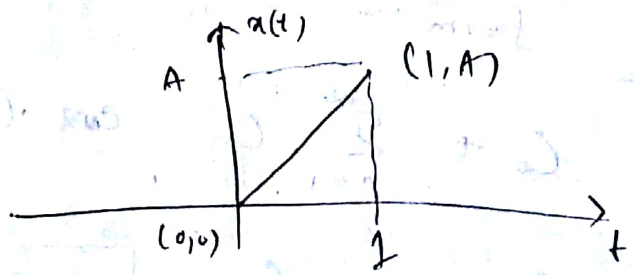
$$C_n = \frac{a_n - j b_n}{2} = \frac{a_n - 0}{2} = \frac{a_n}{2},$$

$$C_n = \frac{4A}{2n\pi} \sin\left(\frac{n\pi}{2}\right). \quad \left[\because a_n = \frac{4A}{n\pi}\sin\left(\frac{n\pi}{2}\right)\right]$$

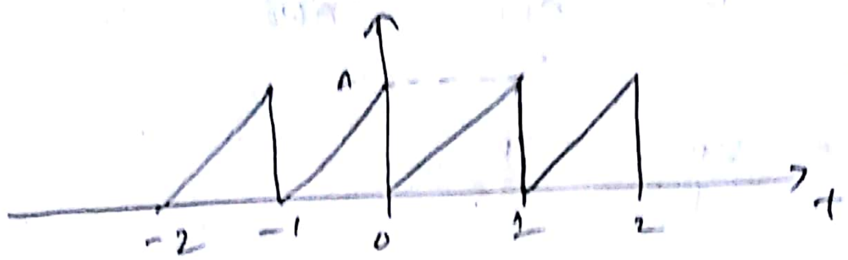
$$\boxed{x(t) = \sum_{n=-\infty}^{\infty} \frac{4A}{2n\pi} \sin\left(\frac{n\pi}{2}\right) \cdot e^{jn\omega_0 t}}$$

Ex:-2 Expand a function $x(t)$ shown in figure, over the interval $(0,1)$ by using

(a) Trigonometric (b) Polar (c) Exponential F. Series.



~~or~~ Find the Fourier series of the wave-form shown below.



Ans:- Defn of function

$$y = mx$$

$$\Rightarrow x(t) = \left(\frac{A-0}{1-0}\right).t$$

$$\Rightarrow x(t) = At$$

$$a_0 = \frac{1}{T}\int_0^T x(t)\,dt$$

$$= \frac{1}{T}\int_0^T At\,dt = \frac{1}{T}.A\frac{t^2}{2}\Big]_0^T$$

$$= \frac{A}{2\cancel{T}}\left[T^{\cancel{2}}\right] = \frac{AT}{2}$$

Here $T = 1$, $\boxed{a_0 = \frac{A}{2}}$

$$a_n = \frac{2}{T}\int_0^T x(t)\cos n\omega_0 t\,dt$$

$$= \frac{2}{T}\int_0^T At\cos n\omega_0 t\,dt$$

$$= \frac{2A}{T}.\int_0^T t.\cos n\omega_0 t\,dt$$

$$\Rightarrow a_n = \frac{2A}{T}\left[t.\frac{\sin n\omega_0 t}{n\omega_0} - 1.\frac{1}{n\omega_0}\left(-\frac{\cos n\omega_0 t}{n\omega_0}\right)\right]_0^T$$

$$\Rightarrow a_n = \frac{2A}{T}\left[t\frac{\sin n\omega_0 t}{n\omega_0} + \frac{1}{n^2\omega_0^2}\cos n\omega_0 t \right]_0^T$$

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{1} = 2\pi$$

$$\Rightarrow a_n = 2A\left[t\frac{\sin 2n\pi t}{2n\pi} + \frac{1}{(2n\pi)^2}\cos 2n\pi t \right]_0^1$$

$$\Rightarrow a_n = 2A\left[\cancel{\frac{\sin 2n\pi}{2n\pi}}^{\,0} + \frac{1}{(2n\pi)^2}\cdot\cos 2n\pi - 0 - \frac{1}{(2n\pi)^2} \right]$$

$$a_n = 2A\left[\frac{1}{(2n\pi)^2} - \frac{1}{(2n\pi)^2} \right] \quad \left(\because \begin{array}{l} \sin 2n\pi = 0 \\ \cos 2n\pi = 1 \end{array} \right)$$

$$\Rightarrow \boxed{a_n = 0.}$$

$$b_n = \frac{2}{T}\int_0^T x(t)\sin n\omega_0 t \, dt$$

$$T = 1, \quad \sin n\omega_0 t = \sin n\cdot\frac{2\pi}{T}\cdot t = \sin 2n\pi t$$

$$b_n = 2\int_0^1 At \sin 2n\pi t \, dt$$

$$b_n = 2A\int_0^1 t \sin 2n\pi t \, dt$$

$$= 2A\left[t\cdot\left(-\frac{\cos 2n\pi t}{2n\pi}\right) - 1\cdot\left(-\frac{1}{2n\pi}\right)\cdot\frac{\sin 2n\pi t}{2n\pi} \right]_0^1$$

$$\left(\because \int U\cdot V = UV_1 - U'\cdot V_2 + U''V_3 - U'''V_4 + \cdots \right)$$

$$\left(\begin{aligned} V_1 &= \int v_1\, dt \\ U' &= \frac{du}{dt} \end{aligned} \right)$$

$$\Rightarrow b_n = 2A \left[\frac{-t}{2n\pi} \cos 2n\pi t + \frac{1}{4n^2\pi^2} \sin 2n\pi t \right]_0^1$$

$$= 2A \left[\frac{-1}{2n\pi} \cos 2n\pi + \frac{1}{4n^2\pi^2} \cancel{\sin 2n\pi}^{\,0} - 0 - 0 \right]$$

$$= -\cancel{2}A \cdot \frac{1}{\cancel{2}n\pi} \cos 2n\pi$$

$$(\because \cos 2n\pi = 1)$$

$$\boxed{b_n = \frac{-A}{n\pi}}$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t$$

$$x(t) = \frac{A}{2} + \sum_{n=1}^{\infty} \left(\frac{-A}{n\pi} \right) \sin n\omega_0 t$$

$$(\because a_n = 0)$$

$$\boxed{x(t) = \frac{A}{2} - \frac{A}{\pi} \sum_{n=1}^{\infty} \frac{\sin 2n\pi t}{n}}$$

$$(\because n\omega_0 t = n \cdot \frac{2\pi}{1} \cdot t = 2n\pi t)$$

(b) Polar form.

$$C_0 = a_0 = \frac{A}{2}, \quad a_n = 0, \quad b_n = -\frac{A}{n\pi}.$$

$$C_n = \sqrt{a_n^2 + b_n^2} = \sqrt{0 + b_n^2} = b_n = -\frac{A}{n\pi}.$$

$$\phi_n = \tan^{-1} \left(\frac{b_n}{a_n} \right) = \tan^{-1}(\infty) = \frac{\pi}{2}.$$

$$\boxed{x(t) = \frac{A}{2} + \sum_{n=1}^{\infty} \left(-\frac{A}{n\pi} \right) \cdot \cos \left(2n\pi t - \frac{\pi}{2} \right)}$$

(C) Exponential form

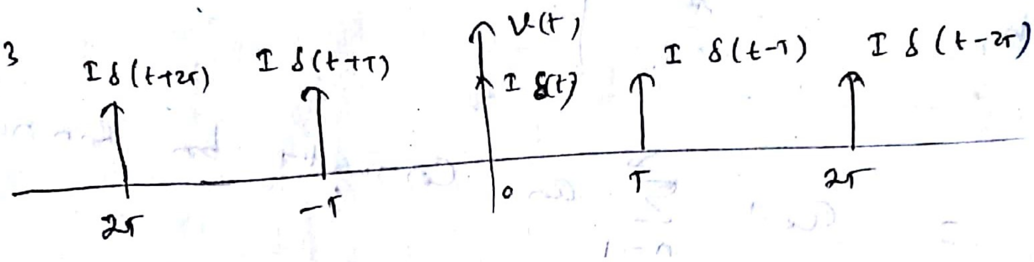
$$C_n = \frac{a_n - jb_n}{2} = \frac{0 - jb_n}{2} = -j\left(\frac{-A}{2n\pi}\right)$$

$$\therefore C_n = \frac{jA}{2n\pi}$$

$$x(t) = \sum_{n=-\infty}^{\infty} \left(\frac{jA}{2n\pi}\right) \cdot e^{jn\omega_0 t}$$

$$\boxed{x(t) = \sum_{n=-\infty}^{\infty} \left(\frac{jA}{2n\pi}\right) \cdot e^{j2n\pi t}}$$

Ex:-3



Find F. Series:-

Ans: $a_0 = \frac{1}{T}\int_{-T/2}^{T/2} I\,\delta(t)\,dt$

$$= \frac{I}{T}\int_{-T/2}^{T/2} \delta(t)\,dt$$

$$\boxed{a_0 = \frac{I}{T}} \qquad \left(\because \int_{-T/2}^{+T/2} \delta(t)\,dt = 1\right)$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} I\,\delta(t)\cos n\omega_0 t\,dt$$

$$= \frac{2I}{T}\cos n\omega_0 t\Big]_{t=0} \qquad \left(\because \int_{-T/2}^{T/2} y(t)\,\delta(t)\,dt = y(0)\right)$$

$$\boxed{a_n = \frac{2I}{T}}$$

$$b_n = \frac{2I}{T}\int_{-T/2}^{T/2} \delta(t)\, \sin n\omega_0 t\, dt$$

$$= \frac{2I}{T} \sin n\omega_0 t \Big]_{t=0}$$

$$\boxed{b_n = 0}$$

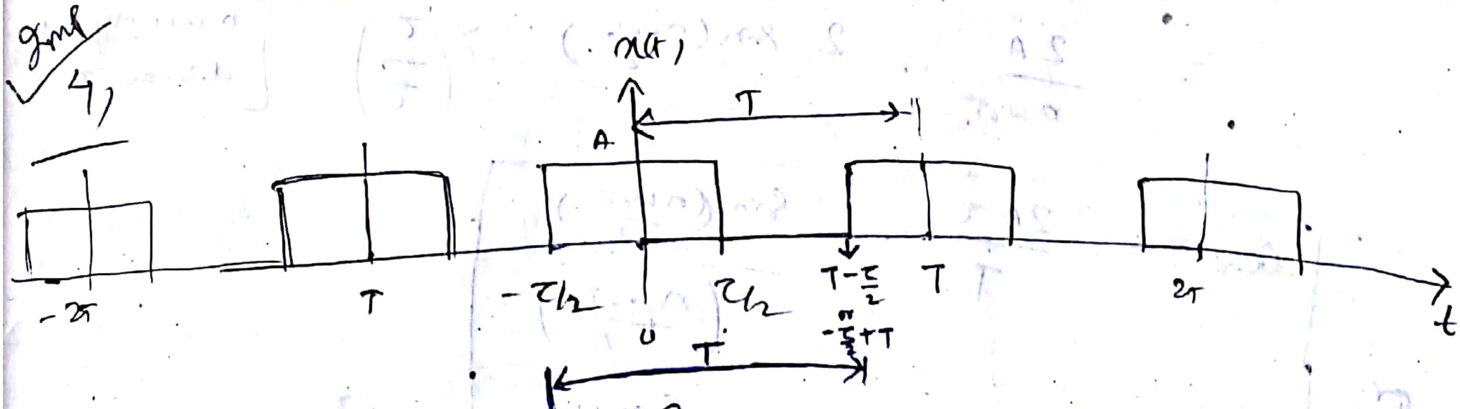
$$C_n = \frac{a_n - j b_n}{2} = \frac{a_n - 0}{2} = \frac{a_n}{2}.$$

$$C_n = \frac{\cancel{2}I}{T} \times \frac{1}{\cancel{2}} = \frac{I}{T}$$

$$x(t) = \sum_{n=-\infty}^{\infty} \frac{I}{T} \cdot e^{jn\omega_0 t} \quad \text{(exponential Form)}$$

$$\boxed{x(t) = \frac{I}{T}\sum_{n=-\infty}^{\infty} e^{jn\omega_0 t}}$$

Imp
4)



$$x(t) = \begin{cases} A, & -\frac{\tau}{2} < t < \frac{\tau}{2} \\ 0, & \frac{\tau}{2} < t < -\frac{\tau}{2} + T \end{cases}$$

$$a_0 = \frac{1}{T}\int_{-\tau/2}^{-\tau/2+T} x(t)\, dt$$

$$= \frac{1}{T}\left[\int_{-\tau/2}^{+\tau/2} x(t)\, dt + \int_{+\tau/2}^{-\tau/2+T} 0 \cdot dt\right]$$

$$\Rightarrow \quad a_0 = \frac{1}{T} \int_{-\tau/2}^{+\tau/2} A \, dt$$

$$a_0 = \frac{A}{T} \left[t \right]_{-\tau/2}^{+\tau/2}$$

$$\boxed{a_0 = \frac{A\tau}{T}}$$

$$a_n = \frac{2}{T} \int_{-\tau/2}^{+\tau/2} A \cos n\omega_0 t \, dt + \int_{+\tau/2}^{-\tau/2+T} 0 \cdot dt$$

$$a_n = \frac{2A}{T} \left. \frac{\sin n\omega_0 t}{n\omega_0} \right]_{-\tau/2}^{+\tau/2}$$

$$\Rightarrow a_n = \frac{2A}{n\omega_0 T} \left[\sin \frac{n\omega_0 \tau}{2} - \sin\left(-\frac{n\omega_0\tau}{2}\right) \right]$$

$$= \frac{2A}{n\omega_0 T} \cdot 2 \sin\left(\frac{n\omega_0\tau}{2}\right) \times \left(\frac{\tau}{\tau}\right) \quad \left[\text{multiply and divide } \tau\right]$$

$$\boxed{a_n = \frac{2A\tau}{T} \cdot \frac{\sin\left(\frac{n\omega_0\tau}{2}\right)}{\left(\frac{n\omega_0\tau}{2}\right)}}$$

or

$$\boxed{a_n = \frac{2A\tau}{T} \, Sa\left(\frac{n\omega_0\tau}{2}\right)}$$

Since even f^n, $\boxed{b_n = 0}$

$$\boxed{x(t) = \frac{A\tau}{T} + \sum_{n=1}^{\infty} \frac{2A\tau}{T} \frac{\sin\left(\frac{n\omega_0\tau}{2}\right)}{\left(\frac{n\omega_0\tau}{2}\right)} \cdot \cos n\omega_0 t}$$

Exponential form :-

$$x(t) = \sum_{n=-\infty}^{\infty} C_n \, e^{jn\omega_0 t}$$

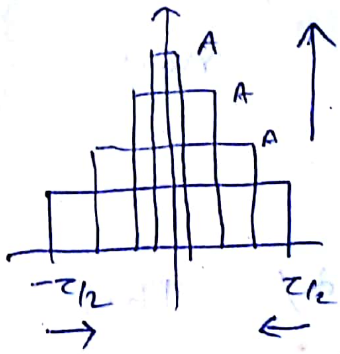
$$C_n = \frac{a_n - jb_n}{2} = \frac{a_n - 0}{2} = \frac{a_n}{2}$$

$$C_n = \frac{A\tau}{T} \cdot \frac{\sin\left(\frac{n\omega_0\tau}{2}\right)}{\left(\frac{n\omega_0\tau}{2}\right)}$$

$$\boxed{x(t) = \sum_{n=-\infty}^{\infty} \frac{A\tau}{T} \frac{\sin\left(\frac{n\omega_0\tau}{2}\right)}{\left(\frac{n\omega_0\tau}{2}\right)} \cdot e^{jn\omega_0 t}}$$

Note :- We can derive F. series of pulse train from the rectangular pulse train, assuming $A\tau = \text{Const} = I$.

when $\tau \to 0$, $A \to \infty$

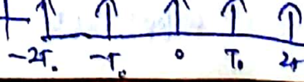


$$\lim_{\tau \to 0} \frac{\sin\left(\frac{n\omega_0\tau}{2}\right)}{\left(\frac{n\omega_0\tau}{2}\right)} = 1.$$

$\therefore$ The above eqn,

$$x(t) = \frac{I}{T} \sum_{n=-\infty}^{\infty} 1 \cdot e^{jn\omega_0 t} \quad (\because A\tau = I)$$

$$\Rightarrow \boxed{x(t) = \frac{I}{T} \sum_{n=-\infty}^{\infty} e^{jn\omega_0 t}}$$ for pulse train.



Fourier Transform (F.T)

Fourier transform of $x(t)$ is defined as

$$\boxed{X(\omega) = F[x(t)] = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t}\, dt}$$

$X(\omega)$ = is called Fourier transform of $x(t)$

$X(\omega)$ is frequency domain representation.

$x(t)$ is time - domain representation.

Fourier transform of $x(t)$ exist only when Dirichlet's Cond^ns are satisfied i.e

1) $x(t)$ is absolutely integrable on the real line i.e

$$\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$$

2) The number of maxima and minima of $x(t)$ in any finite interval on the real line is finite

3) The number of discontinuities of $x(t)$ on the real line in any finite interval is finite.

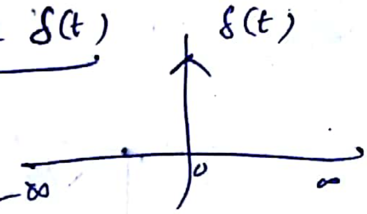
→ Fourier Series is applicable for periodic signal only but Fourier transform is applicable for both periodic & aperiodic signals.

Inverse Fourier transform of $\underline{X(\omega)}$ will generate original signal $\underline{x(t)}$.

$$\boxed{x(t) = F^{-1}[X(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega}$$

→ $x(t)$ and $X(\omega)$ are called <u>Fourier Transform</u> pair.

<u>Ex: 1)</u> <u>Find F.T of $x(t) = \delta(t)$</u>



$x(t) = \delta(t)$

$$X(\omega) = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} e^{-j\omega t}\, \delta(t)\, dt$$

<u>Note</u> :-

$$\int_{-\infty}^{\infty} y(t)\,\delta(t)\,dt = y(0)$$

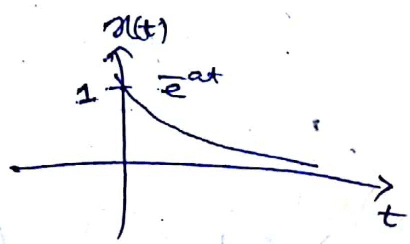
Here $y(t) = e^{-j\omega t}$, $\cancel{y(0) = 1}$.

$$= e^{-j\omega t}\Big|_{t=0}$$

$$= e^{0}$$

$$\boxed{X(\omega) = 1}$$

2) <u>F.T of $x(t) = e^{-at}\,u(t)$</u>



$$X(\omega) = \int_{-\infty}^{0} e^{-at} \cdot 0 \cdot e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-at} \cdot 1 \cdot e^{-j\omega t}\, dt$$

$\because u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$

$$\Rightarrow X(\omega) = \int_0^{\infty} e^{-(a+j\omega)t} \, dt$$

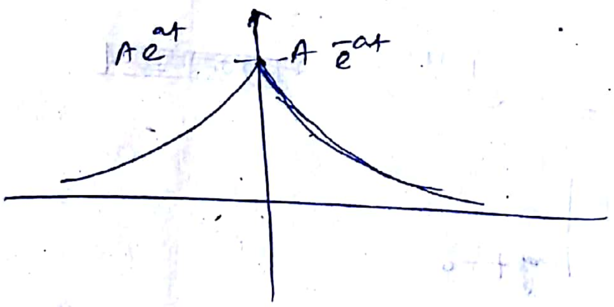
$$X = \left[\frac{e^{-(a+j\omega)}}{-(a+j\omega)} \right]_0^{\infty}$$

$$= \frac{1}{-(a+j\omega)} \left[e^{-\infty} - e^{0} \right]_0^{\infty}$$

$$= \frac{1}{-(a+j\omega)} \left[0 - 1 \right]$$

$$\boxed{X(\omega) = \frac{1}{a+j\omega}} = \boxed{\frac{a-j\omega}{a^2+\omega^2}}$$

3) F.T of $\underline{A e^{-a|t|}}$



Ans ÷ $x(t) = A \cdot e^{-a|t|}$ $\quad \because |t| = \begin{cases} t, & t>0 \\ -t, & t<0 \end{cases}$

$$= \begin{cases} A e^{-at}, & t>0 \\ A e^{at}, & t<0 \end{cases}$$

$$X(\omega) = \int_{-\infty}^{\infty} A \cdot e^{-a|t|} \cdot e^{-j\omega t} \, dt$$

$$= \int_{-\infty}^{0} A \cdot e^{at} \cdot e^{-j\omega t} \, dt + \int_0^{\infty} A \cdot e^{-at} \cdot e^{-j\omega t} \, dt$$

$$\Rightarrow X(\omega) = A\int_{-\infty}^{0} e^{(a-j\omega)t}\,dt + \int_{0}^{\infty} A\, e^{-(a+j\omega)t}\,dt$$

$$= A.\left[\frac{e^{(a-j\omega)}}{a-j\omega}\right]_{-\infty}^{0} + A\left[\frac{e^{-(a+j\omega)}}{-(a+j\omega)}\right]_{0}^{\infty}$$

$$= \frac{A}{a-j\omega}\left[1-0\right] + \frac{A}{-(a+j\omega)}\left[0-1\right]$$

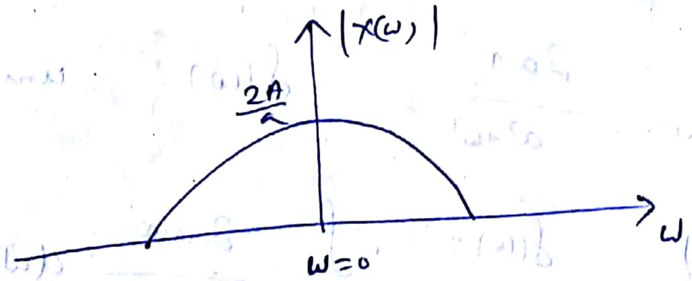
$$= A\left[\frac{1}{a-j\omega}+\frac{1}{a+j\omega}\right]$$

$$= A\left[\frac{a+\cancel{j\omega}+a-\cancel{j\omega}}{(a-j\omega)(a+j\omega)}\right]$$

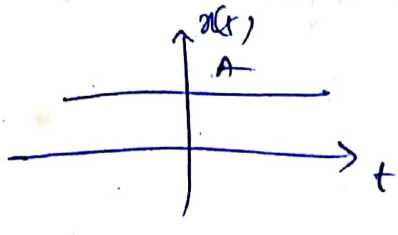
$$\boxed{X(\omega) = \frac{2Aa}{a^2+\omega^2}}$$

$\omega \to 0$, $X(\omega) = \frac{2A}{a}$

$\omega \to \infty$, $X(\omega) = 0$



4) F.T of $\underline{x(t) = A}$



$$X(\omega) = \int_{-\infty}^{\infty} A.\, e^{-j\omega t}\,dt$$

$\Rightarrow X(\omega) = A \cdot \left.\frac{e^{-j\omega t}}{-j\omega}\right]_{-\infty}^{\infty}$

$= \frac{A}{-j\omega}\left[0 - \infty\right]$

$X(\omega) = \infty$ (X)

So $X(\omega)$ does n't exist as $X(\omega) \to \infty$

But F.T can be found out under certain limit.

$x(t) = A = \lim_{a\to 0} A \cdot e^{-a|t|}$

$X(\omega) = \lim_{a\to 0} \frac{2aA}{a^2+\omega^2}$.

If $\omega = 0$, $\lim_{a\to 0} \frac{2\cancel{a}A}{a\cancel{^2}} = \infty$

$\omega \neq 0$, $\lim_{a\to 0} \frac{2aA}{a^2+\omega^2} = 0$

i.e $\lim_{a\to 0} \frac{2aA}{a^2+\omega^2} = \delta(\omega)$ = unit impulse fn.

magnitude of $\delta(\omega) = \int_{-\infty}^{\infty} \frac{2aA}{a^2+\omega^2} d\omega$

$= 2aA \int_{-\infty}^{\infty} \frac{d\omega}{a^2+\omega^2}$

$= 2\cancel{a}A \cdot \frac{1}{\cancel{a}}\left[\tan^{-1}\left(\frac{\omega}{a}\right)\right]_{-\infty}^{\infty}$

$\Rightarrow$ magnitude of $\delta(\omega) = 2\cancel{a}A\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right]$

$= 2A[\pi] = 2\pi A.$

$$\boxed{X(\omega) = 2\pi A \ \delta(\omega)}$$

5) F.T of 1.

Let $A=1$,

$X(\omega) = \cancel{2\pi\delta} \ 2\pi.(1).\delta(\omega)$

$$\boxed{X(\omega) = 2\pi\,\delta(\omega)}$$

6) F.T of $e^{j\omega_0 t}$

$X(\omega) = \int_{-\infty}^{\infty} e^{j\omega_0 t} \cdot e^{-j\omega t}\,dt$

$= \int_{-\infty}^{\infty} 1.\, e^{-j(\omega-\omega_0)t}\,dt$

$$\boxed{X(\omega) = 2\pi\,\delta(\omega-\omega_0)},$$

$\int_{-\infty}^{\infty} A \cdot e^{-j\omega t}\,dt$

$= 2\pi A\delta(\omega)$

Here $A=1$,

$\delta(\omega) \rightarrow \delta(\omega-\omega_0)$

7) Find inverse Fourier transform of $\delta(\omega)$.

$F^{-1}[X(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega).\, e^{j\omega t}\,d\omega$

$$\Rightarrow F^{-1}[\delta(\omega)] = \frac{1}{2\pi} \cdot e^{j\omega t}\Big|_{\omega=0}$$

$$= \frac{1}{2\pi} \cdot e^{0}$$

$$\Rightarrow \boxed{F^{-1}[\delta(\omega)] = \frac{1}{2\pi}}$$

or $\quad F[1] = 2\pi\,\delta(\omega)$

8) Find inverse transform of $\delta(\omega-\omega_0)$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega-\omega_0) \cdot e^{+j\omega t}\, d\omega$$

$$= \frac{1}{2\pi} \cdot e^{j\omega t}\Big]_{\omega=\omega_0}$$

$$\int_{-\infty}^{\infty} \delta(t-t_0) \cdot V(t)\, dt = V(t_0)$$

$$\int_{-\infty}^{\infty} V(t) \cdot \delta(t-t_0)\, dt = V(t_0)$$

$$\boxed{F^{-1}[\delta(\omega-\omega_0)] = \frac{1}{2\pi} \cdot e^{j\omega_0 t}}$$

$$\Rightarrow F[e^{j\omega_0 t}] = 2\pi\,\delta(\omega-\omega_0)$$

Note:-

$$F[e^{-j\omega_0 t}] = 2\pi\,\delta(\omega+\omega_0)$$

9) F.T of $\cos\omega_0 t$, $\sin\omega_0 t$

$$\cos\omega_0 t = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$$

$$\cancel{F[\cos\omega_0 t] = \frac{1}{2}\int_{-\infty}^{\infty}[e^{j\omega_0 t}}$$

$$F.T[\cos\omega_0 t] = \frac{1}{2}\left[F.T[e^{j\omega_0 t}] + F.T[e^{-j\omega_0 t}]\right]$$

$$= \frac{1}{2}\left[2\pi\delta(\omega-\omega_0) + 2\pi\delta(\omega+\omega_0)\right]$$

$$\boxed{F[\cos\omega_0 t] = \pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)}$$

$$\sin\omega_0 t = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}$$

$$F.[\sin\omega_0 t] = \frac{1}{2j}\left[F.T[e^{j\omega_0 t}] - F.T[e^{-j\omega_0 t}]\right]$$

$$= \frac{1}{\cancel{2}j}\left[\cancel{2}\pi\delta(\omega-\omega_0) - \cancel{2}\pi\delta(\omega+\omega_0)\right]$$

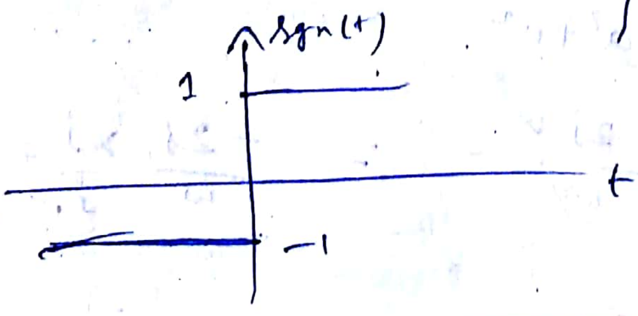
$$= \frac{j}{j\times j}\left[\pi\delta(\omega-\omega_0) - \pi\delta(\omega+\omega_0)\right]$$

$$\boxed{F.T[\sin\omega_0 t] = j\pi\left[\delta(\omega+\omega_0) - \delta(\omega-\omega_0)\right]}$$

$[\because j^2 = -1]$

10) imp Find F.T of sgn(t)

$$sgn(t) = x(t) = \begin{cases} -1, & -\infty < t < 0 \\ 0, & t = 0 \\ 1, & t > 0 \end{cases}$$



$$X(\omega) = \int_{-\infty}^{0} (-1)\cdot e^{-j\omega t}\,dt + \int_{0}^{\infty} 1\cdot e^{-j\omega t}\,dt$$

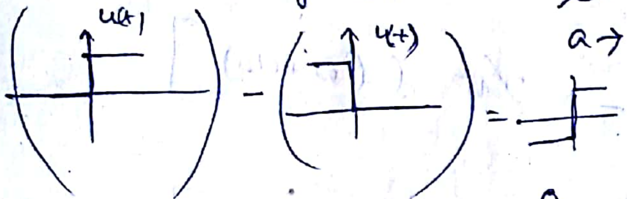
$$= (-1)\cdot \left.\frac{e^{-j\omega t}}{-j\omega}\right]_{-\infty}^{0} + \left.\frac{e^{-j\omega t}}{-j\omega}\right]_{0}^{\infty}$$

$$= \infty \quad (X)$$

In this method we can n't find $X(\omega)$

But F.T can be found out under certain limit.

$$\mathrm{sgn}(t) = \lim_{a \to 0} \left[e^{-at} u(t) - e^{at} u(-t) \right]$$



$$X(\omega) = \lim_{a \to 0} \left\{ F.T\left[e^{-at} u(t) \right] - F.T\left(e^{+at} u(-t) \right) \right\}$$

$$= \lim_{a \to 0} \left[\frac{1}{a + j\omega} - \frac{1}{a - j\omega} \right]$$

$$= \lim_{a \to 0} \left[\frac{\cancel{a} - j\omega - \cancel{a} - j\omega}{(a + j\omega)(a - j\omega)} \right]$$

$$= \lim_{a \to 0} \frac{-2j\omega}{a^2 + \omega^2}$$

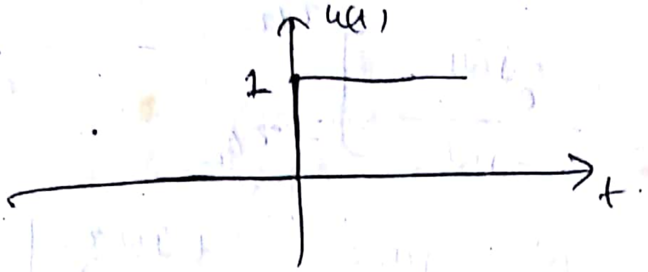
$$= \frac{-2j\cancel{\omega}}{\omega^{\cancel{2}}} = \frac{-2j}{\omega} \times \frac{j}{j} = \frac{2}{j\omega}$$

$$\therefore \boxed{X(\omega) = \frac{2}{j\omega}}$$

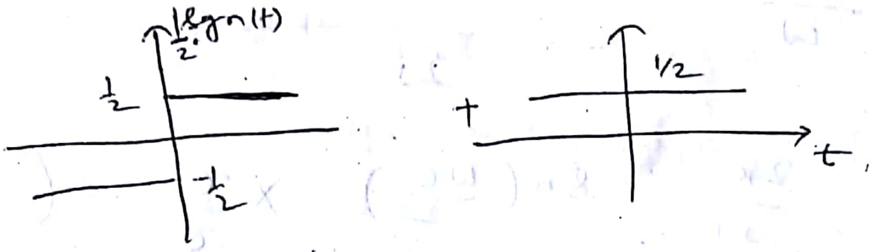
$$\boxed{F[sgn(t)] = \frac{2}{j\omega}}$$

11) Imp

$x(t) = u(t)$, Find $X(\omega) = ?$



$$u(t) = \frac{1}{2} sgn(t) + \frac{1}{2}$$

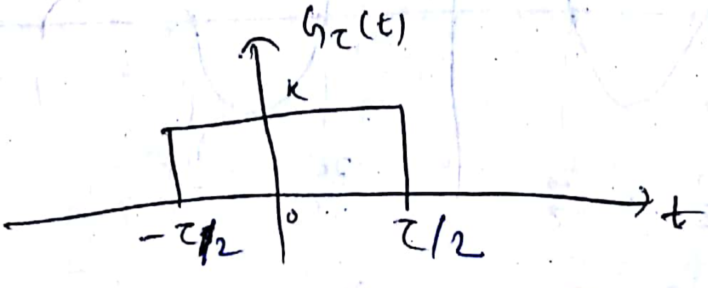


$$F[u(t)] = \frac{1}{2} F[sgn(t)] + \frac{1}{2} F[1]$$

$$= \cancel{\frac{1}{2}} \cdot \frac{\cancel{2}}{j\omega} + \frac{1}{\cancel{2}} \cdot \cancel{2}\pi\,\delta(\omega)$$

$$\boxed{F[u(t)] = \frac{1}{j\omega} + \pi\,\delta(\omega)}$$

12) Find the F.T of gate f^n, shown below.



Ans :

$$x(t) = \begin{cases} K, & -\tau < t < \tau \\ 0, & \text{elsewhere} \end{cases}$$

$$X(\omega) = \int_{-\tau/2}^{+\tau/2} K \cdot e^{-j\omega t}\, dt$$

$$= K \cdot \left.\frac{e^{-j\omega t}}{-j\omega}\right]_{-\tau/2}^{\tau/2}$$

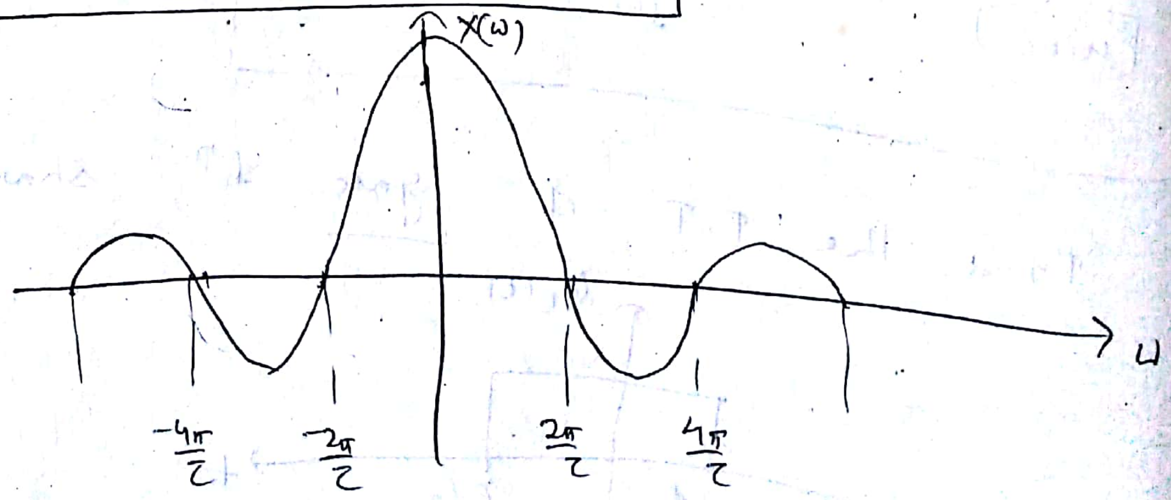
$$= \frac{K}{-j\omega}\left[e^{-j\omega \cdot \frac{\tau}{2}} - e^{+j\omega\frac{\tau}{2}}\right] \times \frac{2}{2}$$

$$= \frac{2K}{\omega} \cdot \frac{e^{j\omega\frac{\tau}{2}} - e^{-j\omega\frac{\tau}{2}}}{2j}$$ (Dividing & multiplying 2)

$$= \frac{2K}{\omega} \cdot \sin\left(\frac{\omega\tau}{2}\right) \times \frac{\tau}{\tau}$$ (Dividing & multiplying τ)

$$= K\tau \, \frac{\sin\left(\frac{\omega\tau}{2}\right)}{\left(\frac{\omega\tau}{2}\right)}$$

$$\boxed{X(\omega) = K\tau \; Sa\left(\frac{\omega\tau}{2}\right)}$$



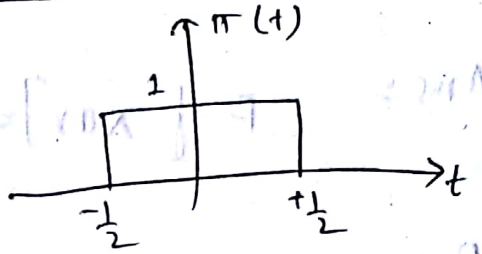
($\because$ $\sin\theta = 0$, when $\theta = \pi, 2\pi, 3\pi$ - - -

$\sin\frac{w\tau}{2} = 0 \Rightarrow \frac{w\tau}{2} = \pi, 2\pi$ - - - -

$\Rightarrow w\tau = 2\pi, 4\pi$ - - -

$\Rightarrow w = \frac{2\pi}{\tau}, \frac{4\pi}{\tau}$ - - -)

13) Find F.T of $\pi(t) \rightarrow$ Rectangular f^n



$$\pi(t) = \begin{cases} 1, & |t| < \frac{1}{2} \\ \frac{1}{2}, & t = \pm\frac{1}{2} \\ 0, & \text{otherwise.} \end{cases}$$

$$F[\pi(t)] = \int_{-\infty}^{\infty} \pi(t) \cdot e^{-j2\pi ft}\, dt$$

Note:-

In some book, F.T is defined as,

$F[x(t)] = \int_{-\infty}^{\infty} x(t) \cdot e^{-j2\pi ft} dt$

I.F.T

$F^{-1}[X(f)] = \int_{-\infty}^{\infty} X(f) \cdot e^{+j2\pi ft} df$

$\rightarrow$ Instead of $w \rightarrow 2\pi f$

$\rightarrow$ In IFT, No $\left(\frac{1}{2\pi}\right)$ term.

because in w, 2π term will come

$$= \int_{-1/2}^{1/2} 1 \cdot e^{-j2\pi ft}\, dt$$

$$= \left.\frac{e^{-j2\pi ft}}{-j2\pi f}\right]_{-1/2}^{1/2}$$

$$= \frac{-1}{j2\pi f}\left[e^{-j2\pi f \cdot \frac{1}{2}} - e^{-j2\pi f \cdot \left(-\frac{1}{2}\right)}\right]$$

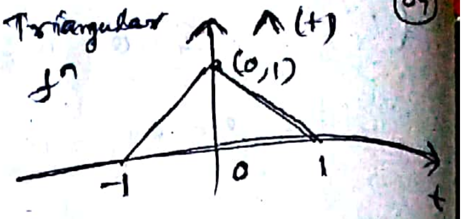
$$= \frac{1}{\pi f} \cdot \left(\frac{e^{j\pi f} - e^{-j\pi f}}{2j}\right)$$

$$= \frac{\sin(\pi f)}{\pi f}$$

$$\boxed{F[\pi(t)] = \cancel{\sin(\pi f)}\ \mathrm{sinc}(f)}$$

14) H.W

Find F.T of $\Lambda(t) \to$ Triangular f^n



$$\Lambda(t) = \begin{cases} t+1, & -1<t<0 \\ -t+1, & 0<t<1 \\ 0, & \text{Otherwise.} \end{cases}$$

Ans :- $F[\Lambda(t)] = \left(\frac{\sin \pi f}{\pi f}\right)^2 = \text{sinc}^2(f)$

Properties of Fourier Transform :-

1) Linearity Property :-

If $\quad x_1(t) \longleftrightarrow X_1(\omega)$

$\quad\quad x_2(t) \longleftrightarrow X_2(\omega)$

Then

$$F[a_1 x_1(t) + a_2 x_2(t)] = a_1 X_1(\omega) + a_2 X_2(\omega)$$

i.e A linear combination of 2 functions $x_1(t)$ and $x_2(t)$ in time domain is also linear combination in frequency domain.

Proof :- $F[a_1 x_1(t) + a_2 x_2(t)]$

$$= \int_{-\infty}^{\infty} [a_1 x_1(t) + a_2 x_2(t)] e^{-j\omega t} dt$$

$$= a_1 \int_{-\infty}^{\infty} x_1(t) . e^{-j\omega t} + a_2 \int_{-\infty}^{\infty} x_2(t) . e^{-j\omega t} dt$$

$$= a_1 X_1(\omega) + a_2 X_2(\omega)$$

$$\therefore \boxed{F\left[a_1 x_1(t) + a_2 x_2(t)\right] = a_1 X_1(\omega) + a_2 X_2(\omega)} \quad \text{(fourier)}$$

2) Time delay / Time shifting property :-

If a signal $x(t)$ is delayed (shifted) in time by 't_0' sec, then the spectrum is modified by a phase shift of $-\omega t_0$.

or A shift in time domain by an amount 't_0' is equivalent to multiplication $e^{-j\omega t_0}$ in frequency domain

i.e $F\left[x(t-t_0)\right] = X(\omega) \cdot e^{-j\omega t_0}$

Proof :- $F\left[x(t-t_0)\right] = \int_{-\infty}^{\infty} x(t-t_0) \cdot e^{-j\omega t} \, dt$

put $t - t_0 = \theta$

$\Rightarrow t = \theta + t_0$

$dt = d\theta$

$t \to \infty, \theta \to \infty$

$t \to -\infty, \theta \to -\infty$

$$= \int_{-\infty}^{\infty} x(\theta) \cdot e^{-j\omega(\theta + t_0)} \, d\theta$$

$$= e^{-j\omega t_0} \cdot \int_{-\infty}^{\infty} x(\theta) \cdot e^{-j\omega\theta} \, d\theta$$

$$\Rightarrow \boxed{F\left[x(t-t_0)\right] = e^{-j\omega t_0} \, X(\omega)}$$

$$x(t-t_0) \longleftrightarrow e^{-j\omega t_0} X(\omega)$$

3) Frequency Shifting Property :-

It states that multiplication of function $x(t)$ by $e^{j\omega_0 t}$ is equivalent to shifting its fourier transform $X(\omega)$ in the +ve direction by an amount $\underline{\omega_0}$.

i.e

$$x(t) \leftrightarrow X(\omega)$$

$$e^{j\omega_0 t} x(t) \leftrightarrow X(\omega-\omega_0).$$

Proof :-

$$F\left[e^{j\omega_0 t} x(t)\right] = \int_{-\infty}^{\infty} e^{j\omega_0 t} x(t) . e^{-j\omega t} dt$$

$$X = \int_{-\infty}^{\infty} x(t) . e^{-j(\omega-\omega_0)t} dt$$

$$\therefore \boxed{F\left[e^{j\omega_0 t} x(t)\right] = X(\omega-\omega_0)}$$

4) Time differentiation Property :-

It states that the differentiation of function $x(t)$ in the time domain is equivalent to multiplication of its Fourier transform by a factor $\underline{j\omega}$.

(i)

$$x(t) \leftrightarrow X(\omega)$$

$$\frac{d}{dt} x(t) \leftrightarrow j\omega X(\omega)$$

Proof :-

$$x(t) = F^{-1}[X(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) . e^{j\omega t} d\omega$$

$$\therefore \cancel{F\left[\frac{d}{dt} x(t)\right]}$$

$$\frac{d}{dt} x(t) = \frac{d}{dt}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\cdot e^{j\omega t}\, d\omega\right]$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\cdot \frac{d}{dt}\left(e^{j\omega t}\right) d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\cdot (j\omega)\cdot e^{j\omega t}\, d\omega$$

$$= j\omega \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\cdot e^{j\omega t}\, d\omega\right]$$

$$\frac{d}{dt}[x(t)] = j\omega \; F^{-1}[X(\omega)]$$

$$\cancel{\Rightarrow X(\omega)}$$

Taking Fourier transform both the order.

$$\boxed{F\left[\frac{d}{dt} x(t)\right] = j\omega \; X(\omega)}$$

Note: $F\left[\frac{d^n}{dt^n} x(t)\right] = (j\omega)^n X(\omega).$

(ii) Differentiation in frequency domain

If $x(t) \leftrightarrow X(f)$

$$(-j2\pi t)\, x(t) \leftrightarrow \frac{d}{df} X(f)$$

Proof: $X(f) = \int_{-\infty}^{\infty} x(t)\cdot e^{-j2\pi f t}\, dt$

$$\frac{d}{df} X(f) = \int_{-\infty}^{\infty} x(t)\cdot e^{-j2\pi f t}\cdot(-j2\pi t)\, dt$$

$$\Rightarrow \frac{d}{df} X(f) = (-j2\pi t) \cdot \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$$

$$\Rightarrow \frac{d}{df} X(f) = \int_{-\infty}^{\infty} \left\{ x(t) \cdot (-j2\pi t) \right\} e^{-j2\pi f t}\, dt$$

$$\Rightarrow \frac{d}{df} X(f) = F\left[x(t) \cdot (-j2\pi t) \right]$$

$$\boxed{\frac{d}{df} X(f) \longleftrightarrow (-j2\pi t) \cdot x(t)}$$

5) **Duality or Symmetric Property :-**

$$x(t) \longleftrightarrow X(\omega)$$

$$X(t) \longleftrightarrow 2\pi\, x(-\omega)$$

Proof :-

$$F^{-1}\left[X(\omega)\right] = x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \cdot e^{j\omega t}\, d\omega$$

$$x(-t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \cdot e^{-j\omega t}\, d\omega$$

$$\Rightarrow 2\pi\, x(-t) = \int_{-\infty}^{\infty} X(\omega) \cdot e^{-j\omega t}\, d\omega$$

Since ω is dummy variable, interchanging t & ω, we get

$$\Rightarrow 2\pi\, x(-\omega) = \int_{-\infty}^{\infty} X(t) \cdot e^{-j\omega t}\, dt = F\left[X(t)\right]$$

$$\Rightarrow \boxed{F\left[X(t)\right] = 2\pi\, x(-\omega)}$$

(6) Scale change property :-

$$x(at) \longleftrightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

Proof :-

$$F[x(at)] = \int_{-\infty}^{\infty} x(at)\, e^{-j\omega t}\, dt$$

Let $at = \alpha$, $t = \frac{\alpha}{a}$, $dt = \frac{1}{a} \cdot d\alpha$.

$t \to -\infty$, $\alpha \to -\infty$

$t \to \infty$, $\alpha \to \infty$

$$F[x(at)] = \frac{1}{a} \int_{-\infty}^{\infty} x(\alpha)\, e^{-j\omega \cdot \frac{\alpha}{a}}\, d\alpha$$

$$= \frac{1}{a} \int_{-\infty}^{\infty} x(\alpha) \cdot e^{-j\left(\frac{\omega}{a}\right) \cdot \alpha}\, d\alpha$$

$$= \frac{1}{a} X\left(\frac{\omega}{a}\right)$$

when $a > 0$,

$$F[x(at)] = \frac{1}{a} X\left(\frac{\omega}{a}\right)$$

when $a < 0$

$$F[x(at)] = -\frac{1}{a} X\left(\frac{\omega}{a}\right).$$

In general ,

$$\boxed{F[x(at)] = \frac{1}{|a|} X\left(\frac{\omega}{a}\right)}$$

7) Modulation Property

$$x(t) \cdot \cos w_c t \leftrightarrow \frac{1}{2}\left[X(w-w_c) + X(w+w_c)\right]$$

Proof :-

$$x(t)\cos w_c t = x(t) \cdot \left[\frac{e^{jw_c t} + e^{-jw_c t}}{2}\right]$$

$$F\left[x(t)\cos w_c t\right] = \int_{-\infty}^{\infty}\left(\frac{e^{jw_c t} + e^{-jw_c t}}{2}\right) \cdot x(t) \cdot e^{-jwt}\, dt$$

$$= \frac{1}{2}\left[\int_{-\infty}^{\infty} x(t) \cdot e^{-j(w-w_c)t}\, dt + \int_{-\infty}^{\infty} x(t) \cdot e^{-j(w+w_c)t}\, dt\right]$$

$$= \frac{1}{2}\left[X(w-w_c) + X(w+w_c)\right]$$

$$\therefore F\left[x(t)\cos w_c t\right] = \frac{1}{2}\left[X(w-w_c) + X(w+w_c)\right]$$

8) Convolution thm

(i) Time Convolution thm

It states that convolution in time domain is equivalent to multiplication of their spectra in frequency domain.

If $x_1(t) \leftrightarrow X_1(w)$

$x_2(t) \leftrightarrow X_2(w)$

$$x_1(t) * x_2(t) \leftrightarrow X_1(w) \cdot X_2(w)$$

↳ Convolution

Proof :- $F\left[x_1(t) \otimes \overset{\text{Convolution}}{x_2(t)}\right]$

$$= \int_{-\infty}^{\infty} x_1(t) \otimes x_2(t) \, e^{-j\omega t} \, dt$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x_1(\tau) \cdot x_2(t-\tau) \, d\tau\right] e^{-j\omega t} \, dt$$

Changing the order of integration

$$= \int_{-\infty}^{\infty} x_1(\tau) \left[\int_{-\infty}^{\infty} x_2(t-\tau) \, e^{-j\omega t} \, dt\right] d\tau$$

By using time shifting property,

$$F\left[x(t-t_0)\right] = e^{-j\omega t_0} X(\omega)$$

$$= \int_{-\infty}^{\infty} x_1(\tau) \cdot e^{-j\omega\tau} X_2(\omega) \, d\tau$$

$$= X_2(\omega) \int_{-\infty}^{\infty} x_1(\tau) \cdot e^{-j\omega\tau} \, d\tau$$

$$= X_2(\omega) \cdot X_1(\omega)$$

$$\therefore \quad x_1(t) \otimes x_2(t) \longleftrightarrow X_1(\omega) \cdot X_2(\omega)$$

(ii) **Frequency Convolution thm:-**

The multiplication of 2 functions in time domain is equivalent to convolution of their spectra in frequency domain.

$$x_1(t) \longleftrightarrow X_1(\omega)$$

$$x_2(t) \longleftrightarrow X_2(\omega)$$

Then $F[x_1(t) \cdot x_2(t)] = \frac{1}{2\pi}[X_1(\omega) \circledast X_2(\omega)]$

9) **Time Reversal :-**

$$F[x(-t)] = X(-\omega)$$

Ans :-

$$F[x(-t)] = \int_{-\infty}^{\infty} x(-t) \cdot e^{-j\omega t}\, dt$$

Put
$-t = \theta$
$dt = -d\theta$
$t \to -\infty, \theta \to \infty$
$t \to \infty, \theta \to -\infty$

$$= \int_{+\infty}^{-\infty} x(\theta) \cdot e^{j\omega\theta}\, (-d\theta)$$

$$= \int_{-\infty}^{\infty} x(\theta) \cdot e^{-j(-\omega)\cdot\theta}\, d\theta$$

Changing limit of integration with change in -ve sign.

$$= X(-\omega)$$

$$\boxed{x(-t) \leftrightarrow X(-\omega)}$$

10) **Integration**

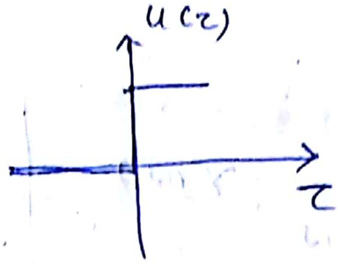
$$F.T\left[\int_{-\infty}^{t} x(\tau)\, d\tau\right] = \frac{1}{j\omega} X(\omega) + \pi X(0)\, \delta(\omega)$$

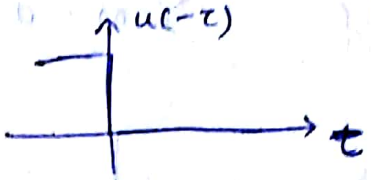
$$= \frac{1}{j\omega} X(\omega) \quad \text{if } X(0) = 0$$

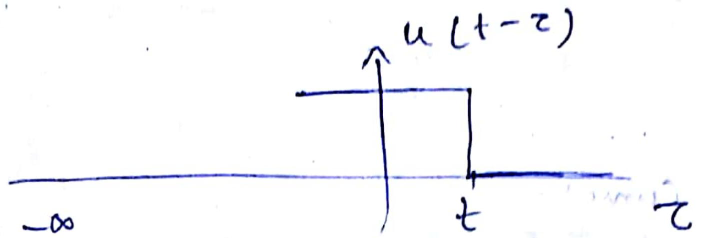
Proof :-

Consider

$$x(t) \otimes u(t) = \int_{-\infty}^{\infty} x(\tau)\, u(t-\tau)\, d\tau$$







$$x(t) \otimes u(t) = \int_{-\infty}^{t} x(\tau)\, u(t-\tau)\, d\tau$$

$$x(t) \otimes u(t) = \int_{-\infty}^{t} x(\tau)\, d\tau \quad \text{———} \quad (1)$$

$(\because u(t-\tau) = 1)$

From eqn (1).

$$F\left[\int_{-\infty}^{t} x(\tau)\, d\tau\right] = F\left[x(t) \otimes u(t)\right]$$

$$= \cancel{X(\omega) \cdot U(\omega)}$$

$$= F[x(t)] \cdot F[u(t)]$$

$$= X(\omega) \cdot \left[\frac{1}{j\omega} + \pi\, \delta(\omega)\right]$$

$$= \frac{1}{j\omega} X(\omega) + \pi X(0)\, \delta(\omega)$$

(Proved)

Since $\delta(\omega)$ exist at $\omega = 0$

$$\text{If } \left\{ \frac{1}{2\pi} \cdot X(0) \right\} = \int_{-\infty}^{\infty} x(\tau)\, d\tau$$ i.e. area under $x(t)$

For zero d.c signal $\boxed{X(0) = 0}$

If $X(0) = 0$,

$$\boxed{F\left[\int_{-\infty}^{t} x(\tau)\, d\tau\right] = \frac{1}{j\omega} X(\omega)}$$

11) Area under Curve :-

(i) The Area under function $x(t)$ is equal to the value of its Fourier transform $X(\omega)$ at $\omega = 0$,

if $x(t) \leftrightarrow X(\omega)$

$$\int_{-\infty}^{\infty} x(t)\, dt = \frac{1}{2\pi} X(0)$$

(ii) Similarly, area under Fourier transform $X(\omega)$ of a function $x(t)$ is equal to the value of function $x(t)$ at $t = 0$.

i.e.

$$\int_{-\infty}^{\infty} X(\omega)\, d\omega = x(0)$$

Properties of Fourier Transform :-

Properties	$x(t)$	$X(\omega)$
1) Linearity	$a_1 x_1(t) + a_2 x_2(t)$	$a_1 X_1(\omega) + a_2 X_2(\omega)$
2) imp Time Shifting	$x(t-t_0)$	$X(\omega) \cdot e^{-j\omega t_0}$
3) Time reversal	$x(-t)$	$X(-\omega)$
4) Duality or Symmetric	$X(t)$	$2\pi\, x(-\omega)$
5) imp Freq. Shifting	$e^{j\omega_0 t} x(t)$	$X(\omega-\omega_0)$
6) imp Modulation	$x(t)\cos\omega_c t$	$\frac{1}{2}\left[X(\omega-\omega_c)+X(\omega+\omega_c)\right]$
7) Scaling	$x(at)$	$\frac{1}{\lvert a\rvert} X\left(\frac{\omega}{a}\right)$
8) Time Differentiation	$\frac{d}{dt} x(t)$	$j\omega\, X(\omega)$
9) Freq Differentiation	$-j2\pi t\, x(t)$	$\frac{d}{df} X(f)$
10) Integration	$\int_{-\infty}^{t} x(\tau)\, d\tau$	$\frac{1}{j\omega} X(\omega) + \pi X(0)\,\delta(\omega)$; $\frac{1}{j\omega} X(\omega)$, if $X(0)=0$
11) imp Convolution	$x_1(t) \otimes x_2(t)$	$X_1(\omega) \cdot X_2(\omega)$
12) Multiplication	$x_1(t) \cdot x_2(t)$	$\frac{1}{2\pi} \left[X_1(\omega) \otimes X_2(\omega)\right]$

13) Perseval's Property

If Fourier transform of the signal $x(t)$ and $y(t)$ are denoted by $X(f)$ and $Y(f)$ respectively, then

$$\int_{-\infty}^{\infty} x(t)\, y^*(t)\, dt = \int_{-\infty}^{\infty} X(f)\, Y^*(f)\, df$$

14) <u>Rayleigh's Property</u>

$$\int_{-\infty}^{\infty} |x(t)|^2\, dt = \int_{-\infty}^{\infty} |X(f)|^2\, df$$

15) <u>Auto correlation Property :-</u>

The time auto correlation function of the signal $x(t)$ is denoted by $R_x(\tau)$ and is defined by

$$R_x(\tau) = \int_{-\infty}^{\infty} x(t)\, x^*(t-\tau)\, dt .$$

Then

$$F\left[R_x(\tau)\right] = |X(f)|^2 = PSD$$

16) <u>Moment Property</u>

If $F[x(t)] = X(f)$, then $\int_{-\infty}^{\infty} t^n x(t)\, dt$,

n^{th} moment of $x(t)$ can be obtained from the rel^n,

$$\int_{-\infty}^{\infty} t^n x(t)\, dt = \left(\frac{j}{2\pi}\right)^n \frac{d^n}{df^n} X(f) \Big|_{f=0}$$

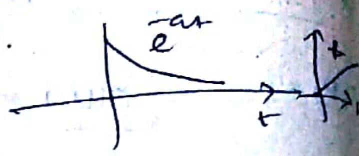
Fourier Transform of different f^ns

	$x(t)$		$X(\omega)$
1)	$\delta(t)$	$\longrightarrow$	1
2)	$\delta(t-t_0)$	$\longrightarrow$	$e^{-j\omega t_0}$
3)	1	$\longrightarrow$	$2\pi\,\delta(\omega)$
4)	$u(t)$	$\longrightarrow$	$\pi\,\delta(\omega) + \frac{1}{j\omega}$
5)	$sgn(t)$	$\longrightarrow$	$\frac{2}{j\omega}$
6)	$e^{j\omega_0 t}$	$\longrightarrow$	$2\pi\,\delta(\omega-\omega_0)$
7)	$\cos\omega_0 t$	$\longrightarrow$	$\pi\left[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)\right]$
8)	$\sin\omega_0 t$	$\longrightarrow$	$j\pi\left[\delta(\omega+\omega_0) - \delta(\omega-\omega_0)\right]$
9)	$e^{-at}u(t)$, $a>0$	$\longrightarrow$	$\frac{1}{a+j\omega}$
10)	$t\,e^{-at}u(t)$, $a>0$	$\longrightarrow$	$\frac{1}{(a+j\omega)^2}$
11)	$e^{-a\lvert t\rvert}$	$\longrightarrow$	$\frac{2a}{a^2+\omega^2}$
12)	$e^{-b^2t^2}$	$\longrightarrow$	$\frac{\sqrt{\pi}}{b}\, e^{-\left(\frac{\omega}{2b}\right)^2}$

13) $\Pi(t)$ [Rectangular f^n] $= \begin{cases} 1, & \lvert t\rvert < \frac{1}{2} \\ \frac{1}{2}, & t = \pm\frac{1}{2} \\ 0, & \text{otherwise} \end{cases} \longrightarrow \operatorname{sinc}(f)$

14) $\Lambda(t)$ [Triangular f^n] $= \begin{cases} t+1, & -1 < t < 0 \\ -t+1, & 0 \le t \le 1 \\ 0, & \text{otherwise} \end{cases} \longrightarrow \operatorname{sinc}^2(f)$

Ex :- 157 Find F.T of $t e^{-at} u(t)$



Ans :- Traditional method :-

$$\int_0^\infty t\, e^{-at} \cdot e^{-j\omega t}\, dt$$

$$= \int_0^\infty t \cdot e^{-(a+j\omega)t}\, dt$$

$$= \left[t \cdot \frac{e^{-(a+j\omega)t}}{-(a+j\omega)} - 1 \cdot \left(\frac{-1}{a+j\omega}\right) \cdot \frac{e^{-(a+j\omega)t}}{-(a+j\omega)} \right]_0^\infty$$

$$= (0-0) - \left(0 - \frac{1}{(a+j\omega)^2}\right)$$

$$= \frac{1}{(a+j\omega)^2}$$

or Using Differentiation Property

$$\frac{d}{df} X(f) = F\left[-j2\pi t\, x(t)\right] \quad \text{——} (1)$$

Let $x(t) = t\, e^{-at} u(t)$

$$X(f) = \frac{1}{a+j2\pi f} \qquad \frac{1}{a+j\omega}$$

$$\frac{d}{df} X(f) = \frac{-1}{(a+j2\pi f)^2} \times j2\pi \quad \text{——} (2)$$

From eqn (1) & (2),

$$\frac{\cancel{-}\cancel{j2\pi}}{(a+j2\pi f)^2} = F\left[\cancel{-}\cancel{j2\pi} t\, x(t)\right]$$

$$\Rightarrow F\left[t\, x(t)\right] = \frac{1}{(a+j2\pi f)^2} = \frac{1}{(a+j\omega)^2}$$

16) Find F.T of $\frac{1}{t}$:-

Using Duality Property

$$F[X(t)] = x(-f). \quad \text{—— (1)}$$

Let $x(t) = sgn(t)$.

$$X(f) = \frac{2}{j2\pi f}.$$

$$\Rightarrow X(t) = \frac{2}{j2\pi t}. \quad \text{—— (2)}$$

$$x(-f) = sgn(-f) \quad \text{—— (3)}$$

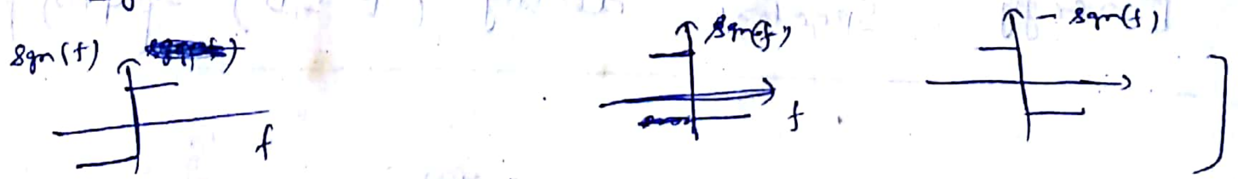
i.e $F[X(t)] = x(-f)$

From (1), (2) & (3), $F\left[\frac{\cancel{2}}{j\cancel{2}\pi t}\right] = sgn(-f)$

$$\Rightarrow F\left[\frac{1}{t}\right] = j\pi \, sgn(-f)$$

$$= -j\pi \, sgn(f)$$

(∵ ~~sgn(-f) = -sgn~~ $sgn(-f) = -sgn(f)$)



$$\therefore F\left[\frac{1}{t}\right] = -j\pi \, sgn(f)$$

$$\boxed{F\left[\frac{1}{t}\right] = -j\pi \, sgn(f)}$$

Ex 1.22, Taub Schilling

17) Find $W(f)$, Fourier transform output $w(t)$ which is produced when signal $v(t) = e^{-at}u(t)$ passes through a quadrature mirror filter having impulse response $\boxed{h(t) = \frac{1}{\pi t}}$

$$e^{-at}u(t) = v(t) \longrightarrow \boxed{h(t) = \frac{1}{\pi t}} \longrightarrow w(t)$$

$$w(t) = v(t) \otimes h(t)$$

$$\Rightarrow W(f) = V(f) \cdot H(f)$$ [using Convolution thm]

$$= F.T\left[e^{-at}u(t)\right] \cdot F\left[\frac{1}{\pi t}\right]$$

$$= \frac{1}{a + j2\pi f} \cdot \frac{1}{\cancel{\pi}} \cdot \left[-j\cancel{\pi}\, sgn(f)\right]$$

(Last Problem We have done)

$$\boxed{W(f) = \frac{-j\, sgn(f)}{a + j2\pi f}}$$

Energy Spectral density (ESD) - [$\psi(\omega)$]

$$\text{i/p} \quad \underset{X(\omega)}{\overset{x(t)}{\longrightarrow}} \boxed{H(j\omega)} \underset{Y(\omega)}{\overset{y(t)}{\longrightarrow}} \quad \text{o/p}$$

Lets Consider a signal $x(t)$, which is applied to an ideal low pass filter.

$$Y(\omega) = X(\omega) \cdot H(\omega)$$

Then, Energy per unit bandwidth is known as Energy Spectral Density ($\psi(\omega)$)

$$\psi(\omega) = \frac{E_0}{\Delta f} = |X(\omega)|^2$$

↳ Energy per unit BW.

$$\boxed{ESD = |X(\omega)|^2}$$

Properties of ESD

1) The total area under ESD function is equal to the total energy of that signal.

$$E = \int_{-\infty}^{\infty} \psi(f)\, df$$

2) If $x(t)$ is i/p to linear time invariant (LTI) system with Transfer function $H(\omega)$, then i/p and o/p ESD fn are related as

$$\psi_o(\omega) = |H(\omega)|^2 \, \psi_i(\omega)$$

where $\psi_o(\omega)$ = o/p ESD function.

$\psi_i(\omega)$ = i/p ESD function.

$|H(\omega)|^2$ = Energy gain at freq ω.

3) imp The Auto Correlation function ACF and ESD form a Fourier Transform Pair.

$$F[R(\tau)] \rightarrow \psi(\omega)$$

$$F^{-1}[\psi(\omega)] \rightarrow R(\tau)$$

i.e. $$\boxed{R(\tau) \leftrightarrow \psi(\omega)}$$

Perseval's them for energy signal (Rayleigh's Them)

This them defines ~~as~~ in terms of CTS

Energy of the signal

Fourier transform.

$$E = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega$$

or

$$E = \int_{-\infty}^{\infty} |X(f)|^2 \, df$$

Proof :-

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

$$= \int_{-\infty}^{\infty} x(t) \cdot x^*(t) \, dt$$

$$= \int_{-\infty}^{\infty} x^*(t) \cdot \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \cdot e^{j\omega t} \, d\omega \right] dt$$

changing the order of integration.

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \left[\int_{-\infty}^{\infty} x^*(t) \, e^{j\omega t} \, dt \right] d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \cdot X^*(\omega) \, d\omega$$

$$\left(\because F[x(t)] = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} \, dt \right.$$

$$\left. F[x^*(t)] = \int_{-\infty}^{\infty} x^*(t) \, e^{+j\omega t} \, dt \right)$$

$$E = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega$$

(Proved)

$$\left(\because \omega = 2\pi f \Rightarrow d\omega = 2\pi \, df \right)$$

$$\text{or } E = \int_{-\infty}^{\infty} |X(f)|^2 \, df$$

Ex 1.7) Verify Perseval's Thm for energy signal

$$x(t) = e^{-at}u(t), \quad a > 0$$

Ans :

Energy of the signal

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$= \int_{-\infty}^{\infty} \left[e^{-at}u(t)\right]^2 dt$$

$$= \int_{0}^{\infty} \left[e^{-at}\right]^2 dt \qquad \Big| \because u(t) \text{ exist } t > 0$$

$$= \int_{0}^{\infty} e^{-2at} dt$$

$$= \left.\frac{e^{-2at}}{-2a}\right]_0^{\infty} = \frac{-1}{2a}\left[0 - 1\right] = \frac{1}{2a}$$

$$\boxed{E = \frac{1}{2a}} \quad \text{——— (1)}$$

Using Perseval's Thm

$$E = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

$$X(\omega) = \int_{-\infty}^{\infty} e^{-at}u(t)\, \cancel{dt}\, e^{-j\omega t} dt$$

[Derived earlier].

$$X(\omega) = \frac{1}{a + j\omega}$$

$$|X(\omega)| = \frac{1}{\sqrt{a^2 + \omega^2}}, \quad |X(\omega)|^2 = \frac{1}{a^2 + \omega^2}$$

$$E = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{d\omega}{a^2 + \omega^2}$$

$$E = \frac{1}{2\pi}\left[\frac{1}{a}\tan^{-1}\left(\frac{\omega}{a}\right)\right]_{-\infty}^{\infty}$$

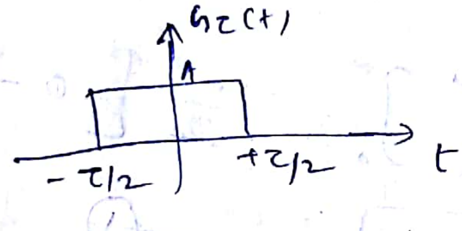
$$= \frac{1}{2\pi a} \times \left[\frac{\pi}{2}+\frac{\pi}{2}\right]$$

$$= \frac{1}{2\cancel{\pi} a} \times \cancel{\pi}$$

$$E = \frac{1}{2a} \text{ ——— (ii)}$$

From (i) & ②, Perseval's Th^m is verified.

Ex:-18) Find ESD of Gate f^n, having width τ

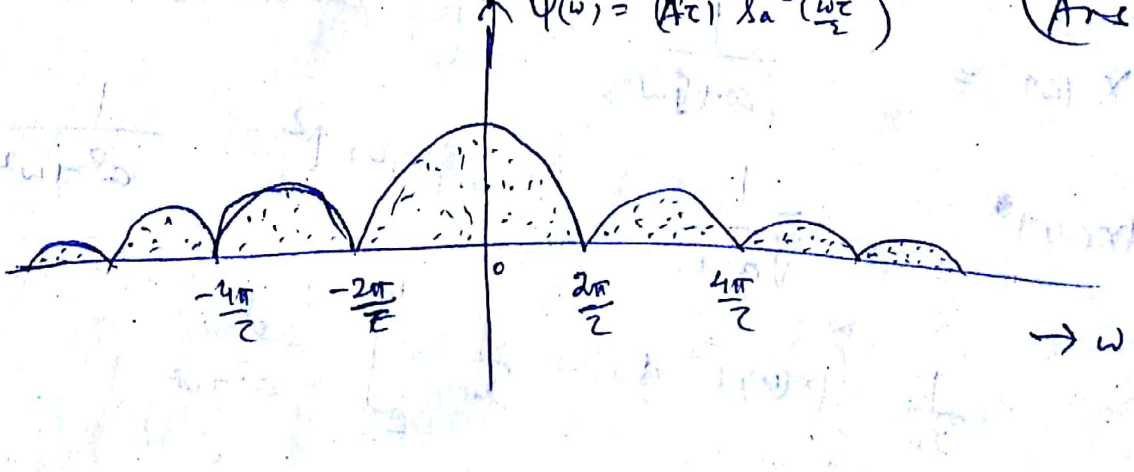


Ans :- $ESD = |X(\omega)|^2$.

$$X(\omega) = A\tau \; Sa\left(\frac{\omega\tau}{2}\right) \quad \left[\begin{array}{l}\text{Derived} \\ \text{earlier}\end{array}\right]$$

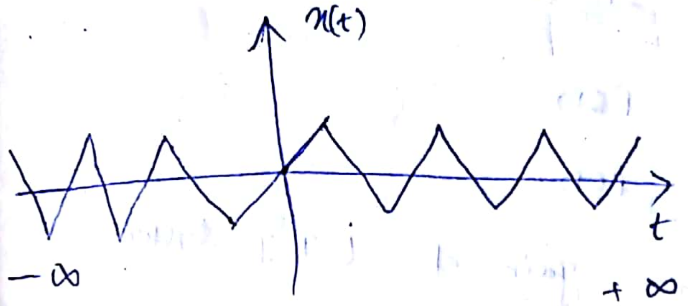
$$ESD = |X(\omega)|^2 = (A\tau)^2 \, Sa^2\left(\frac{\omega\tau}{2}\right)$$

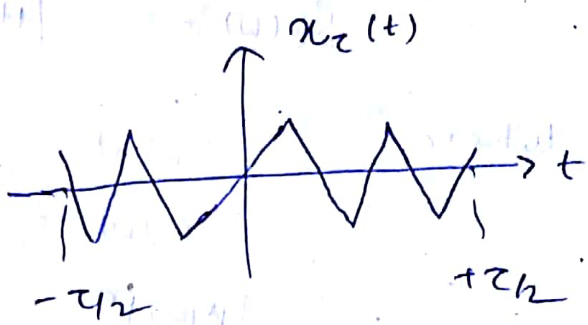
$$\psi(\omega) = (A\tau)^2 Sa^2\left(\frac{\omega\tau}{2}\right) \quad \text{(Ans)}$$



Power Spectral Density (PSD)

The expression for PSD may be derived by assuming the power signal as a limiting case of energy signal.





Lets consider a power signal $x(t)$ which is extended to infinity. Lets terminate the signal outside the interval, $\pm\frac{\tau}{2}$, such that it is zero

i.e $x_\tau(t) = \begin{cases} x(t), & |t| < \tau \\ 0, & \text{elsewhere} \end{cases}$

Then the Avg. Power per unit bandwidth is called the Power Spectral Density (PSD).

$$\boxed{PSD = S(\omega) = \lim_{\tau \to \infty} \frac{|X_\tau(\omega)|^2}{\tau}}$$

Properties of PSD:-

1) The area under PSD f^n is equal to the avg. power of that signal

$$P = \int_{-\infty}^{\infty} S(f)\, df$$

2) If $x(t)$ is i/p to LTI (Linear time invariant) system with transfer function $H(\omega)$ then i/p and o/p PSD are related as

$$S_o(\omega) = |H(\omega)|^2 S_i(\omega)$$

where $S_o(\omega) =$ o/p PSD

$S_i(\omega) =$ i/p PSD

$|H(\omega)|^2 =$ Power gain of LTI system

3) V. Imp

The Auto Correlation function $R(\tau)$ and PSD $(S(\omega))$ form Fourier Transform Pair.

$$F[R(\tau)] = S(\omega)$$

$$F^{-1}[S(\omega)] = R(\tau)$$

$$R(\tau) \longleftrightarrow S(\omega)$$

ESD $(\psi(\omega))$	PSD $[S(\omega)]$
1) Energy per unit BW (Band width)	1) Avg power per unit Bandwidth.
1) ESD gives distribution of energy of a signal in freq domain.	1) PSD gives distribution of power of a signal in freq domain.
2) $\psi(\omega) = |X(\omega)|^2$	2) $S(\omega) = \lim_{T \to \infty} \frac{|X_T(\omega)|^2}{T}$
3) Total energy $E = \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi(\omega)\, d\omega = \int_{-\infty}^{\infty} \psi(f)\, df$	3) Total Power $P = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\, d\omega = \int_{-\infty}^{\infty} S(f)\, df$

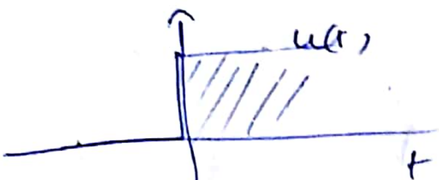
4) Auto correlation f^n and ESD form a Fourier Transform Pair. $R(\tau) \leftrightarrow \psi(\omega)$	4) Auto correlation f^n and PSD form a Fourier transform pair. $R(\tau) \leftrightarrow S(\omega)$

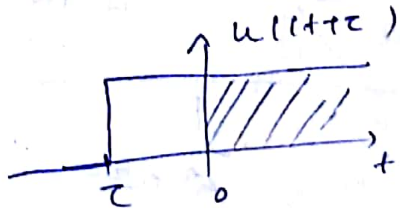
Q:(9) Find the time auto correlation function of the signal $g(t) = e^{-at}u(t)$. and ~~that~~ from thus obtain ESD of $g(t)$,

Ans :- $$R(\tau) = \int_{-\infty}^{\infty} g(t).\ g(t+\tau)\, dt$$

$$g(t) = e^{-at}u(t)$$

$$g(t+\tau) = e^{-a(t+\tau)}.\ u(t+\tau)$$





Common area betn $u(t)$ and $u(t+\tau)$ is

$$0 < t < \infty$$

$$\Rightarrow R(\tau) = \int_{0}^{\infty} e^{-at}.(1).e^{-a(t+\tau)}.(1)\ dt$$

$$= \int_{0}^{\infty} e^{-a\tau}.\ e^{-2at}\ dt$$

$$= e^{-a\tau}.\ \frac{e^{-2at}}{-2a}\Big]_{0}^{\infty} = \frac{e^{-a\tau}}{-2a}\left[0-1\right]$$

$$R(\tau) = \frac{1}{2a}.e^{-a\tau}$$

Since $R(\tau) = R(-\tau)$

[Note:- Since $R(-\tau)$ will be same as $R(\tau)$, if $|\tau|$ is there, it will make $|\tau| = \tau$ i.e $R(-\tau) = e^{-a\tau}$]

$$\boxed{R(\tau) = \frac{1}{2a} \cdot e^{-a|\tau|}}$$

ACF and ESD are Fourier Transform pair.

$$ESD = F[R(\tau)]$$

$$= F\left[\frac{1}{2a} \cdot e^{-a|\tau|}\right]$$

$$= \frac{1}{2a} \cdot F\left[e^{-a|\tau|}\right]$$

$$= \frac{1}{\cancel{2a}} \cdot \frac{\cancel{2a}}{a^2+\omega^2} \quad \text{(Derived page No. 54)}$$

$$\boxed{ESD = \frac{1}{a^2+\omega^2}}$$

Or

$$ESD = |X(\omega)|^2$$

$$x(t) = e^{-at} u(t)$$

$$X(\omega) = \frac{1}{a+j\omega}$$

$$|X(\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}$$

$$|X(\omega)|^2 = \frac{1}{a^2+\omega^2}$$

$$\Rightarrow \boxed{ESD = \frac{1}{a^2+\omega^2}}$$

Perseval's Power th^m:-

Perseval's power th^m defines the power of a signal in terms of its Fourier Series coefficients i.e. in terms of harmonics components present in the signal.

Mathematically,

$$\boxed{P = \sum_{n=-\infty}^{\infty} |C_n|^2}$$

Proof: Consider function $x(t)$

We know that $|x(t)|^2 = x(t) \cdot x^*(t)$

Where $x^*(t)$ complex conjugate of $x(t)$.

Avg. Power of $x(t)$ for one cycle is,

$$P = \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2 dt = \frac{1}{T}\int_{-T/2}^{T/2} x(t) \cdot x^*(t)\, dt$$

From the exponential Fourier series,

$$x(t) = \sum_{n=-\infty}^{\infty} C_n\, e^{jn\omega_0 t}$$

$$P = \frac{1}{T}\int_{-T/2}^{T/2} \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} \cdot x^*(t)\, dt$$

$$= \frac{1}{T}\sum_{n=-\infty}^{\infty} C_n \int_{-T/2}^{T/2} x^*(t) \cdot e^{jn\omega_0 t} dt = \sum_{n=-\infty}^{\infty} C_n \left[\frac{1}{T}\int_{-T/2}^{T/2} x^*(t) \cdot e^{jn\omega_0 t} dt\right]$$

$$= \sum_{n=-\infty}^{\infty} C_n\, C_n^*$$

$$\left[\because C_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jn\omega_0 t} dt \qquad C_n^* = \frac{1}{T}\int_{-T/2}^{T/2} x^*(t)\, e^{+jn\omega_0 t} dt\right]$$

$$\boxed{P = \sum_{n=-\infty}^{\infty} |C_n|^2}$$

(Proved)